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The Fundamental Gap and One-Dimensional Collapse

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Abstract. The *fundamental gap* of a bounded, connected domain in $\mathbb{R}^n$ is the difference between the first two (positive) eigenvalues of the Euclidean Laplacian with Dirichlet boundary condition. The *one dimensional collapse* considered here is the degeneration of a family of convex, bounded domains in $\mathbb{R}^n$ to a domain in $\mathbb{R}^{n-1}$. The boundary of the domains need not be smooth, merely Lipschitz continuous. Our results show that the fundamental gap detects the geometry of one-dimensional collapse, and that depending upon the geometry of the collapse, the gap can either diverge or remain bounded.

Contents

1. Motivation and results

Our sign convention and boundary conditions for the Laplace equation on a domain $\Omega \subset \mathbb{R}^n$ shall be

$$\Delta u = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = -\lambda u, \quad \text{Dirichlet boundary condition: } u = 0 \text{ on } \partial\Omega.$$

This equation arises from physics by separating the time and space variables in the wave equation, and the eigenvalues correspond to the frequencies of standing waves on Ω. The set of all eigenvalues is known as *the spectrum.* Based on the physical interpretation Marc Kac posed the now well-known question, "Can one hear the shape of a drum?" [**21**]. The mathematical formulation is: if two Euclidean domains have the same spectrum, do they have the same shape? For arbitrary bounded, connected domains with piecewise smooth boundary Gordon, Webb, and Wolpert proved that the answer is "no" [**14, 15**]. For convex domains, however,

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Kac's question is an open problem. In some cases one *can* "hear the shape of a drum" by restricting to "drums" with specific geometric properties. Durso used the heat and wave traces to prove that isospectral triangular domains are congruent [**8**]. Grieser and Maronna gave a proof using only the heat trace [**20**].

For both proofs one must know that the entire spectra of two triangles coincide to prove that they are congruent. Since three independent parameters determine a triangle (up to congruence), it is natural to conjecture that the first three eigenvalues suffice to determine if two triangles are congruent. Antunes and Freitas demonstrated strong numerical evidence in support of this conjecture in [**3**].

In some cases a finite set of eigenvalues can indeed detect geometric features such as symmetry. Pólya and Szegő proved that the first eigenvalue detects the regular n-gon among all convex n-gons with fixed area for $n = 3$ and 4 [**29**]. For $n \geq 5$, the analogous result has not been proven and is known as the Pólya-Szegő Conjecture.

The next natural object to study after the first eigenvalue is the difference between it and the rest of the spectrum, known as the *fundamental gap.* If one scales a bounded domain $\Omega \subset \mathbb{R}^n$ by a constant factor c, then the eigenvalues change according to:

$$\lambda_k(c\Omega) = c^{-2}\lambda_k(\Omega).$$

This motivates the definition of the scale-invariant *gap function*

$$\xi(\Omega) := d^2(\Omega)\left(\lambda_2(\Omega) - \lambda_1(\Omega)\right),$$

where d is the diameter of the domain.

In [**26**] we demonstrated that the gap function detects the equilateral triangle among all Euclidean triangles.

THEOREM 1 ([**26**]). *Let T be a Euclidean triangle. Then*

$$\xi(T) \geq \frac{64\pi^2}{9}$$

with equality if and only if T is equilateral.

A fundamental result concerning the gap function is due to Andrews and Clutterbuck [**2**] who proved that for any convex domain $\Omega \subset \mathbb{R}^n$,

$$\xi(\Omega) \geq 3\pi^2.$$

It is a straightforward exercise to compute that the right side of the above inequality is equal to the gap function on a one dimensional bounded domain (a segment). Andrews's and Clutterbuck's Theorem shows that the gap function in n dimensions is bounded below by the gap function in 1 dimension. This motivated us to investigate the behavior of the gap function on families of collapsing domains and to determine if and when it converges to the gap function on a lower dimensional domain. Further motivation to study eigenvalues of thin and collapsing domains comes from physics; see for example [**16**], [**17**], and [**18**].

1.1. Examples. Although the following examples may be well-known, we include them for the sake of completeness and to give intuition for that which follows.

1.1.1. *Rectangles.* There are very few domains for which one may explicitly compute the spectrum. The most elementary example of a planar domain with computable spectrum is a rectangle. For a rectangle $R \cong [0, L] \times [0, W]$ the set of eigenvalues and eigenfunctions may be computed by separation of variables. The eigenvalues are thus:

$$\lambda_{j,k} = \frac{k^2\pi^2}{L^2} + \frac{j^2\pi^2}{W^2}, \quad j, k \in \mathbb{N}.$$

Without loss of generality we assume $W \leq L$. Then, the gap function

$$\xi(R) = (L^2 + W^2)\left(\frac{4\pi^2}{L^2} + \frac{\pi^2}{W^2} - \frac{\pi^2}{L^2} - \frac{\pi^2}{W^2}\right) = \frac{3\pi^2(L^2 + W^2)}{L^2}.$$

The gap function detects the shape of a square since for any rectangle R

$$\xi(R) \leq 6\pi^2,$$

with equality if and only if R is a square. For a family of rectangles undergoing one-dimensional collapse by letting $W \to 0$ the gap function converges to $3\pi^2$ which is the gap function for a segment.

1.1.2. *Circular sectors.* Circular sectors may be the most natural example to consider after rectangular domains because in polar coordinates a circular sector looks like a rectangle.

The Laplacian in polar coordinates (r, θ) is

$$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2.$$

Separating variables in the Laplace equation leads to the equations

$$g''(\theta) = -\mu g(\theta), \quad r^2 f''(r) + r f'(r) + \lambda r^2 f(r) = \mu f(r), \qquad \mu > 0. \tag{1.1}$$

For a circular sector of opening angle $\alpha\pi$ and radius 1, the boundary conditions are

$$g(0) = g(\alpha\pi) = 0, \quad f(0) = f(1) = 0.$$

The solutions to the first equation in (1.1) are

$$g(\theta) = \sin(k\theta/\alpha), \quad k \in \mathbb{N}, \quad \mu = \mu_k = \frac{k^2}{\alpha^2}.$$

The second equation in (1.1) can be re-arranged to

$$r^2 f''(r) + r f'(r) + (\lambda r^2 - \mu_k) f = 0, \quad f(0) = f(1) = 0,$$

which one recognizes as a Bessel equation. The solutions are

$$J_{k/\alpha}(\sqrt{\lambda} r), \quad J_{-k/\alpha}(\sqrt{\lambda} r),$$

where J_x denotes the Bessel function of order x. To satisfy the boundary conditions the second solution is not allowed since it does not vanish at $r = 0$. Consequently,

$$\sqrt{\lambda} = \sqrt{\lambda_{j,k}} = Z_{j,k},$$

is the j^{th} zero of the Bessel function of order k/α. The eigenvalues of the sector are therefore

$$\lambda_{j,k} = Z_{j,k}^2.$$

The first and second zeros of the Bessel function of order ν are

$$Z_{\nu,i} = \nu - \frac{a_i}{2^{1/3}}\nu^{1/3} + O(\nu^{-1/3}), \quad i = 1, 2, \tag{1.2}$$

where a_i is the i^{th} zero of the Airy function of the first kind. These formulae can be found in [**23**, **25**], and [**30**] for Bessel functions of real order and were previously demonstrated in [**34**] and [**28**] for Bessel functions of integer order. Since

$$a_1 \approx -2.34, \text{ and } a_2 \approx -4.09, \tag{1.3}$$

the first two Dirichlet eigenvalues of the circular sector of opening angle $\alpha\pi$ and radius one are approximately

$$\lambda_i(S_\alpha) = \frac{1}{\alpha^2} + \frac{c_i}{\alpha^{4/3}} + O(\alpha^{-1}), \quad i = 1, 2, \tag{1.4}$$

where

$$c_1 = -a_1 2^{2/3} \approx 3.71 \text{ and } c_2 = -a_2 2^{2/3} \approx 6.49. \tag{1.5}$$

Consequently the gap function of a circular sector S_α of opening angle $\alpha\pi$ is asymptotic to

$$\xi(S_\alpha) = \frac{c_2 - c_1}{\alpha^{4/3}} + O(\alpha^{-1}), \quad \alpha \to 0. \tag{1.6}$$

In this case the gap function is unbounded as the sector collapses to the segment.

These examples show that the gap function is sensitive to the geometry of convex planar domains which collapse to a segment. Motivated by the strikingly different asymptotic behavior of the gap function on rectangles and circular sectors, in [**26**] we investigated the behavior of the gap function on simplicial domains and proved the following.

THEOREM 2 ([**26**]). *Let Y be an $n-1$ simplex for some $n \geq 2$. Let $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of n-simplices each of which is a graph over Y. Assume the height of X_j over Y vanishes as $j \to \infty$. Then $\xi(X_j) \to \infty$ as $j \to \infty$. More precisely, there is a constant $C > 0$ depending only on n and Y such that $\xi(X_j) \geq Ch(X_j)^{-4/3}$, where $h(X_j)$ is the height of X_j.*

Any triangle with unit diameter is a 2-simplex with base

$$Y = \{(x, 0) : 0 \leq x \leq 1\}.$$

By Theorem 2, the gap function on a family of triangles which collapses to a segment diverges with at least the same asymptotic rate as a collapsing sector (1.6). In §2, we explore different ways in which triangles may collapse to the segment and present an elementary proof that the gap function on any sequence of collapsing triangles diverges. In §3, we prove that the gap function is sensitive to the geometry of convex planar domains which collapse to a segment. The proof of our main result comprises §4.

To state this result we shall identify $\mathbb{R}^{n-1}$ with the subset

$$\{(x_1, \cdots, x_{n-1}, 0) \mid (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}\} \subset \mathbb{R}^n.$$

Let $E \subset \mathbb{R}^{n-1}$ be a convex domain, and let h_ε^- and h_ε^+ be respectively one-parameter families of convex, non-positive and concave non-negative Lipschitz continuous functions. Assume that $h_\varepsilon^\pm$ are uniformly bounded from above and below, and that their moduli of continuity are uniformly bounded from above for all ε. Define

$$h_\varepsilon(x) := |h_\varepsilon^+(x) - h_\varepsilon^-(x)|, \quad \sigma_\varepsilon := \max\{h_\varepsilon(x) \mid x \in E\},$$

and

$$\Omega_\varepsilon := \{(x, y) \mid x \in E, \quad \varepsilon h_\varepsilon^-(x) \leq y \leq \varepsilon h_\varepsilon^+(x)\}.$$

THEOREM 3. *Let E, $h_\varepsilon^\pm$, σ_ε, and Ω_ε be as above, and let*

$$U_\varepsilon(\delta) := \{x \in E \mid h_\varepsilon(x) \geq \sigma_\varepsilon - \delta\}.$$

If

$$\operatorname{diam} U_\varepsilon(\delta) \to 0, \ \textit{as } \delta \to 0, \ \textit{uniformly in } \varepsilon,$$

then

$$\lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) \to \infty, \qquad \textit{as } \varepsilon \to 0.$$

A more refined theorem was proven by Borisov and Freitas in a slightly different setting [**4**]. They determined the asymptotics of both the eigenvalues and eigenfunctions for domains in $\mathbb{R}^d$ with smooth boundary which collapse to a domain in $\mathbb{R}^{d-1}$. Since simplices and polygonal domains do not satisfy the hypotheses of [ibid] due to the presence of corners, those results cannot be used to prove Theorem 3. We note however that in two dimensions Theorem 2 can be deduced from Friedlander and Solomyak's results [**13**]. Both [**4**] and [**13**] require sophisticated functional analysis, whereas our proofs are rather elementary.

2. Collapsing triangular domains

2.1. Preliminaries. The variational principle, also known as the mini-max principle, states that the eigenvalues are given by the infima of the Rayleigh-Ritz quotient,

$$\lambda_{k+1} = \inf \left\{ \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2} \,\middle|\, 0 = \int_\Omega f f_j, j = 0, \dots, k \right\} \tag{2.1}$$

where $f_0 \equiv 0$, and f_j is an eigenfunction for λ_j for $j \geq 1$. We refer to [**6**] for the proof. The test functions above and in the equivalent "maxi-min" principle (2.2) are $\mathcal{C}^2$ functions which vanish on the boundary of Ω and are not identically zero,

$$\lambda_k = \inf_{\dim(L)=k} \left\{ \sup_{f \in L} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2} \right\}. \tag{2.2}$$

The maxi-min principle can be used to prove domain monotonicity (see [**7**])

$$\Omega \subset \Omega' \implies \lambda_k(\Omega) \geq \lambda_k(\Omega'), \quad \forall k \in \mathbb{N}.$$

We shall use the standard "big-O" and "little-o" from asymptotic analysis. We shall also use the following notations for functions $f, g : \mathbb{R} \to \mathbb{R}$,

$$f \lesssim g \text{ as } t \to t_0 \iff \exists C, \varepsilon > 0 \text{ such that } f(t) \leq Cg(t) \quad \forall t \in [t_0 - \varepsilon, t_0 + \varepsilon],$$
$$f \gtrsim g \quad \text{as } t \to t_0 \iff -f \lesssim -g \text{ as } t \to t_0,$$
$$f \sim g \quad \text{as } t \to L \iff f \lesssim g \text{ and } g \lesssim f \text{ as } t \to t_0,$$
$$f \approx g \iff \lim_{t \to t_0} \frac{f}{g} = 1.$$

In some arguments we will use the following constants[1]

$$c_1' = -a_1' 2^{2/3} \approx 1.62, \quad c_2' = -a_2' 2^{2/3} \approx 5.16. \tag{2.3}$$

Above a_i' is the i^{th} negative zero of the derivative of the Airy function of the first kind.

[1]This constant arises from the asymptotic formula for the first two zeros of the derivative of the Bessel function which is related to the first two Dirichlet eigenvalue of an obtuse isosceles triangle; see [**11**].

2.2. The moduli space of triangles. Since the gap function is invariant under scaling, we restrict to triangles with diameter one. Such a triangle has angles

$$0 < \alpha\pi \leq \beta\pi \leq \pi - \alpha\pi - \beta\pi.$$

The moduli space of triangles, which we shall call P, can thus be identified with a triangle in the $\alpha \times \beta$ plane; see Figure 1. We are interested in the behavior of ξ approaching the boundary of P which is the dashed vertical segment in Figure 1 and corresponds to triangles which degenerate to a segment.

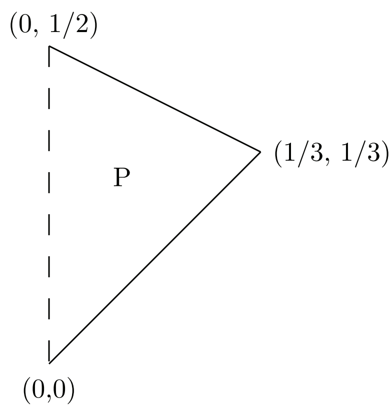

FIGURE 1. Moduli space of triangles.

Consider the triangle T with angles $0 < \alpha\pi \leq \beta\pi \leq \pi - \alpha\pi - \beta\pi$, and assume for some fixed $\varepsilon > 0$, $\beta \geq \varepsilon$. Let the side opposite $\alpha\pi$ have length A, the side opposite $\beta\pi$ have length B, and the third side have length one. The Law of Sines states that

$$\frac{\sin(\alpha\pi)}{A} = \frac{\sin(\beta\pi)}{B} = \frac{\sin(\pi - \alpha\pi - \beta\pi)}{1}.$$

Then

$$B = \frac{\sin(\beta\pi)}{\sin(\alpha\pi + \beta\pi)} \implies |1 - B| = O(\alpha), \text{ as } \alpha \to 0. \tag{2.4}$$

We approximate $\xi(T)$ using domain monotonicity with two sectors both of opening angle $\alpha\pi$. The larger sector S has radius 1 and contains T, whereas the smaller sector σ has radius B and is contained in T. By domain monotonicity,

$$\lambda_2(T) \geq \lambda_2(S), \quad \lambda_1(T) \leq \lambda_1(\sigma).$$

Then,

$$\lambda_2(S) - \lambda_1(\sigma) \leq \xi(T) \leq \lambda_2(\sigma) - \lambda_1(S).$$

Since

$$\lambda_k(\sigma) = B^{-2}\lambda_k(S), \quad \forall k \in \mathbb{N},$$

we have the estimate

$$\lambda_2(S) - B^{-2}\lambda_1(S) \leq \xi(T) \leq B^{-2}\lambda_2(S) - \lambda_1(S). \tag{2.5}$$

By (2.4) and (1.4), there are constants c, $c' > 0$ such that

$$\xi(S) - \frac{c}{\alpha} = \frac{c_2 - c_1}{\alpha^{4/3}} - \frac{c}{\alpha} \leq \xi(T) \leq \xi(S) + \frac{c'}{\alpha} = \frac{c_2 - c_1}{\alpha^{4/3}} + \frac{c'}{\alpha}.$$

Consequently,

$$\lim_{\alpha\to 0}\frac{\xi(T)}{\xi(S)}=1. \tag{2.6}$$

More generally, we have the following.

PROPOSITION 1. *Let $\{T_n\}$ be a sequence of triangles with diameter one and angles $\alpha_n\pi$, $\beta_n\pi$ and $\pi(1-\alpha_n-\beta_n)$. Let S_n be the sector with opening angle $\alpha_n\pi$ and radius* 1. *Assume that $\alpha_n=o(\beta_n^3)$ and $\beta_n\to 0$ as $n\to\infty$. Then*

$$\xi(T_n)\approx\xi(S_n)=\frac{c_2-c_1}{\alpha_n^{4/3}}\quad \text{as } \beta_n\to 0.$$

PROOF. For simplicity, we shall abuse notation and drop the subscript n. Estimating with two sectors S and σ as above with opening angle $\alpha\pi$ and radii 1 and B, respectively, we again have (2.5). In this case since both α and β tend to 0, by (2.4)

$$|1-B|=O\left(\frac{\alpha}{\beta}\right).$$

Therefore (2.5) becomes

$$\xi(S)-\frac{c\alpha}{\beta}\lambda_1(S)\le\xi(T)\le\xi(S)+\frac{c'\alpha}{\beta}\lambda_2(S),$$

for some positive constants c and c'. By (1.4) for some positive constants C and C' we have

$$\xi(S)-\frac{C}{\alpha\beta}=\frac{c_2-c_1}{\alpha^{4/3}}-\frac{C}{\alpha\beta}\le\xi(T)\le\xi(S)+\frac{C'}{\alpha\beta}=\frac{c_2-c_1}{\alpha^{4/3}}+\frac{C'}{\alpha\beta}.$$

Since $\alpha=o(\beta^3)$, (2.6) follows. ∎

We next consider obtuse isosceles triangles with diameter one. By [**11**],

$$\lambda_i(T)=\frac{4}{\alpha^2}+\frac{4c_i'}{\alpha^{4/3}}+O(\alpha^{-2/3}),\quad i=1,2.$$

Then,

$$\xi(T)\approx\frac{4(c_2'-c_1')}{\alpha^{4/3}}+O(\alpha^{-2/3})\qquad\text{as } \alpha\to 0.$$

More generally, we have the following.

PROPOSITION 2. *Let $\{T_n\}$ be a sequence of triangles with diameter one and angles $\alpha_n\pi$, $\beta_n\pi$ and $\pi(1-\alpha_n-\beta_n)$. Assume that $\alpha_n=\beta_n+o(\beta_n^3)$ and $\beta_n\to 0$ as $n\to\infty$. Then*

$$\xi(T_n)\approx\frac{4(c_2'-c_1')}{\alpha_n^{4/3}}\qquad \text{as } n\to\infty.$$

PROOF. By abuse of notation we shall again drop the subscript n. We estimate the gap function on the triangle PRT in Figure 2 using a smaller isosceles triangle QRT and a larger isosceles triangle PST.

By domain monotonicity

$$\lambda_k(PST)\le\lambda_k(PRT)\le\lambda_k(QRT).$$

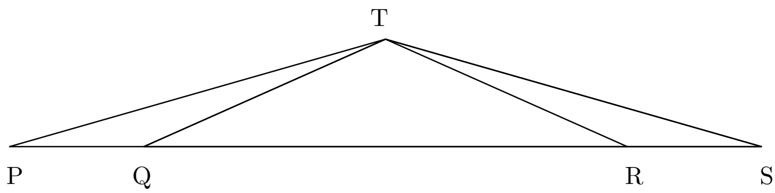

FIGURE 2. Triangle with two angles collapsing and approximating isosceles triangles.

The triangle QRT has angles $\beta\pi$, $\beta\pi$, and $\pi - 2\beta\pi$, whereas the triangle PST has angles $\alpha\pi$, $\alpha\pi$, and $\pi - 2\alpha\pi$. Let B denote the length of QT, A the length of PT, Γ the length of QR, and γ the length of PQ. By [**11**],

$$\lambda_k(QRT) = \frac{4}{\Gamma^2}\left(\frac{1}{\beta^2} + \frac{c'_k}{\beta^{4/3}}\right) + O(\beta^{-2/3}), \quad k = 1, 2, \tag{2.7}$$

and

$$\lambda_k(PST) = \frac{4}{(\Gamma + 2\gamma)^2}\left(\frac{1}{\alpha^2} + \frac{c'_k}{\alpha^{4/3}}\right) + O(\alpha^{-2/3}), \quad k = 1, 2. \tag{2.8}$$

By the Law of Cosines

$$\Gamma = 2B\cos(\beta\pi),$$

and by the Law of Sines

$$B = \frac{A\sin(\alpha\pi)}{\sin(\beta\pi)} \implies |A - B| = o(\beta^2).$$

By the Law of Sines

$$\gamma = \frac{B\sin(\beta\pi - \alpha\pi)}{\sin(\alpha\pi)} = o(\beta^2).$$

Since PRT has unit diameter,

$$\Gamma + \gamma = 1, \quad |\Gamma - 1| = o(\beta^2).$$

The estimate for $\xi(PRT)$ is

$$\lambda_2(PST) - \lambda_1(QRT) \le \xi(PRT) \le \lambda_2(QRT) - \lambda_1(PST). \tag{2.9}$$

By (2.5) and (1.4)

$$\xi(PRT) \approx 4(c'_2 - c'_1)\beta^{-4/3}, \qquad \text{as } \beta \to 0.$$

□

The above propositions imply the following.

COROLLARY 1. *The gap function is not polyhomogeneous on P.*

PROOF. Polyhomogeneity would imply that the gap function $\lambda_2 - \lambda_1$ on triangles with unit diameter admits an expansion of the form

$$\xi(\alpha, \beta) \sim \sum_{k=1}^{\infty} \alpha^{z_k}\log(\alpha)^{p_k} b_k(\beta - \alpha), \quad \alpha \to 0. \tag{2.10}$$

Above, the coefficient functions b_k are smooth in $\beta - \alpha$, the real parts of the (possibly complex) powers z_k tend toward infinity as $k \to \infty$, and the powers p_k

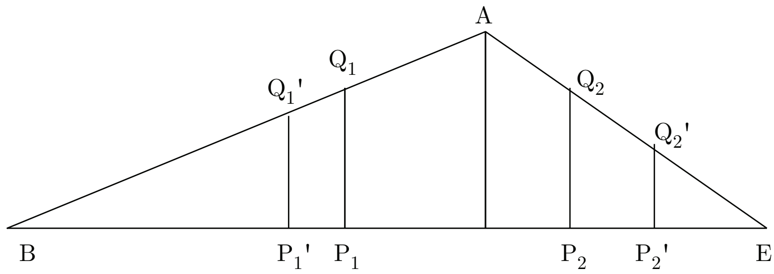

FIGURE 3. Arbitrary collapsing triangle.

are non-negative integers. For all points away from $(\alpha, \beta) = (0, 0)$, ξ is asymptotic to

$$\xi(\alpha, \beta) \approx \frac{c_2 - c_1}{\alpha^{4/3}}, \qquad \text{as } \alpha \to 0.$$

This corresponds to an expansion (2.10) with $z_1 = -4/3$, $p_1 = 0$, and $b_1(x)$ a smooth function with $b_1(x) > 0$ for $x > 0$, and $b_1(0) = c_2 - c_1$. However, along trajectories approaching $(\alpha, \beta) = (0, 0)$ with $\beta - \alpha = o(\beta^3)$, by Proposition 2 the gap is asymptotic to $4(c_2' - c_1')\alpha^{-4/3}$. This would require $b_1(0) = 4(c_2' - c_1')$. Since

$$4(c_2' - c_1') \approx 14, \quad c_2 - c_1 \approx 3,$$

there can be no smooth function b_1 which satisfies these conditions. □

REMARK 1. In forthcoming work by Grieser and Melrose the eigenvalues are shown to be polyhomogeneous on the moduli space of triangles with an appropriate blow-up at the point $(\alpha, \beta) = (0, 0)$ [**19**].

2.3. A general collapsing triangle. For the general case consider the triangle $T = ABE$ in Figure 3. We make the following assumptions: the smallest angle of the triangle is at the vertex B and measures $\alpha\pi$, $|DE| \le |BD|$, and $|BE| = 1$. Fix

$$0 < \varepsilon < \frac{2}{9}.$$

Let P_1 be between B and D so that $|P_1 D| = \alpha^\varepsilon$, and P_1' be between B and P_1 so that $|P_1' D| = 2\alpha^\varepsilon$. If $|DE| > 2\alpha^\varepsilon$, let P_2 be between D and E so that $|P_2 D| = \alpha^\varepsilon$, and P_2' be between P_2 and E so that $|P_2' D| = 2\alpha^\varepsilon$. If $|DE| \le 2\alpha^\varepsilon$, we do not define or use the points P_2, P_2'. If $|DE| > 2\alpha^\varepsilon$ let U be the trapezoid $AQ_1P_1P_2Q_2$ and similarly let $U' = AQ_1'P_1'P_2'Q_2'$. If $|DE| \le 2\alpha^\varepsilon$, we let $U = AQ_1P_1E$ and $U' = AQ_1'P_1'E$. Let $V = ABE \setminus U$ and $V' = ABE \setminus U'$. In the estimates to follow, we show that we may estimate $\lambda_2(ABE) - \lambda_1(ABE)$ using $\lambda_2(U') - \lambda_1(U')$.

Let f_i be the eigenfunction for $\lambda_i = \lambda_i(ABE)$, for $i = 1, 2$. Assume that

$$\int_{ABE} f_i^2 dxdy = 1, \quad i = 1, 2,$$

and define

$$\eta_i := \int_V f_i^2 dxdy, \quad i = 1, 2. \tag{2.11}$$

In the following arguments the notation "$dxdy$" shall be suppressed whenever it is clear from context. We identify the base of the triangle BE with the segment from $(0,0) \in \mathbb{R}^2$ to $(1,0) \in \mathbb{R}^2$. Let $h(x)$ denote the height of the triangle over a point $(x,0)$ in the base. Since the eigenfunctions f_i vanish on the boundary of the vertical segment from $(x,0)$ to $(x,h(x))$, the variational principle for the first eigenvalue of this vertical segment implies

$$\frac{\int_0^{h(x)} |\nabla f_i|^2}{\int_0^{h(x)} f_i^2} dy \geq \frac{\pi^2}{h(x)^2} \implies \int_0^{h(x)} |\nabla f_i|^2 dy \geq \frac{\pi^2}{h(x)^2} \int_0^{h(x)} f_i^2 dy. \tag{2.12}$$

This is known as the one-dimensional Poincaré inequality. The heights of V, U, and U' satisfy

$$h(V) \lesssim (1-\alpha^\varepsilon)\alpha, \quad h(U) \lesssim \alpha, \quad h(U') \lesssim \alpha.$$

Integrating the inequality (2.12) with respect to x over the points $(x,0)$ in the base of each of V, U, and U', we have the estimates

$$\frac{\int_V |\nabla f_i|^2}{\int_V f_i^2} \gtrsim \frac{1}{(1-\alpha^\varepsilon)^2\alpha^2}, \quad \frac{\int_U |\nabla f_i|^2}{\int_U f_i^2} \gtrsim \frac{1}{\alpha^2}, \text{ and } \frac{\int_{U'} |\nabla f_i|^2}{\int_{U'} f_i^2} \gtrsim \frac{1}{\alpha^2}.$$

By the variational principle (2.1),

$$\frac{\eta_i}{(1-\alpha^\varepsilon)^2\alpha^2} + \frac{1-\eta_i}{\alpha^2} \lesssim \int_V |\nabla f_i|^2 + \int_U |\nabla f_i|^2 = \lambda_i.$$

Since the triangle ABD is a right triangle, by [**11**] and domain monotonicity

$$\lambda_i \leq \lambda_2(ABD) \lesssim \frac{1}{\alpha^2} + \frac{c_2}{\alpha^{4/3}} + O(\alpha^{-1}).$$

Therefore,

$$\eta_i \lesssim \frac{\alpha^{2/3} c_2 (1-\alpha^\varepsilon)^2}{\alpha^\varepsilon(2-\alpha^\varepsilon)} \lesssim \alpha^{2/3-\varepsilon}, \quad i = 1,2. \tag{2.13}$$

2.3.1. *Estimate for* λ_1. To estimate $\lambda_2 - \lambda_1 = \xi(T)$ from below, by domain monotonicity

$$\lambda_1 \leq \lambda_1(U') \implies \xi(T) \geq \lambda_2 - \lambda_1(U').$$

In the following arguments, we will show that we can estimate $\lambda_2(U')$ from above in terms of λ_1 and λ_2.

2.3.2. *Estimate for* λ_2. Let ρ be a smooth compactly supported function so that

$$\rho|_U \equiv 1, \quad \rho|_{V'} \equiv 0. \tag{2.14}$$

We may choose ρ so that

$$|\nabla \rho| \leq \frac{1}{\alpha^\varepsilon}, \quad \text{and} \quad |\Delta\rho|, |\Delta(\rho^2)| \leq \frac{1}{\alpha^{2\varepsilon}}. \tag{2.15}$$

Note that

$$-(\rho f_i)\Delta(\rho f_i) = \lambda_i \rho^2 f_i^2 - f_i^2 \rho\Delta\rho - 2 f_i \rho (\nabla\rho)(\nabla f_i). \tag{2.16}$$

We would like to use ρf_2 in the variational principle (2.1) for $\lambda_2(U')$, but since ρf_2 is not à priori orthogonal to the first eigenfunction for U', we must modify it a bit. Since ρf_1 is not orthogonal to the first eigenfunction for U' because both are positive, there is some $a \in \mathbb{R}$ such that $\rho f_2 + a\rho f_1$ is orthogonal to the first eigenfunction for U'. We may then use $\rho f_2 + a\rho f_1$ as a test function for the variational principle (2.1) for $\lambda_2(U')$,

$$\lambda_2(U') \leq \frac{\int_{U'} |\nabla(\rho f_2 + a\rho f_1)|^2}{\int_{U'} (\rho f_2 + a\rho f_1)^2}. \tag{2.17}$$

Since

$$\lambda_2 = \lambda_2 \frac{\int_{U'}(\rho f_2 + a\rho f_1)^2}{\int_{U'}(\rho f_2 + a\rho f_1)^2} = \frac{\lambda_2 \int_{U'} \rho^2 f_2^2 + a^2 \lambda_2 \int_{U'} \rho^2 f_1^2 + \lambda_2 \int_{U'} 2a\rho^2 f_1 f_2}{\int_{U'}(\rho f_2 + a\rho f_1)^2},$$

expanding the numerator and denominator in (2.17) we have

$$\lambda_2(U') \leq \lambda_2 + \frac{I + II + III}{\int_{U'}(\rho f_2 + a\rho f_1)^2},$$

where

$$I := \int_{U'} |\nabla(\rho f_2)|^2 - \lambda_2 \int_{U'} \rho^2 f_2^2, \quad II := a^2 \left(\int_{U'} |\nabla(\rho f_1)|^2 - \lambda_2 \int_{U'} \rho^2 f_1^2 \right),$$

and

$$III := \int_{U'} 2a \nabla(\rho f_1) \cdot \nabla(\rho f_2) - \lambda_2 \int_{U'} 2a\rho^2 f_1 f_2.$$

In the arguments below we estimate I, II, and III from above. Using integration by parts

$$\int_{U'} |\nabla \rho f_i|^2 = - \int_{U'} \rho f_i \Delta(\rho f_i) = \lambda_i \int_{U'} \rho^2 f_i^2 - \frac{1}{2} \int_{U'} \nabla \rho^2 \nabla f_i^2 - \int_{U'} f_i^2 \rho \Delta \rho. \tag{2.18}$$

We estimate using integration by parts, the fact that both $\Delta\rho$ and $\nabla\rho$ vanish identically on U, and (2.11), (2.13),

$$\left| \int_{U'} |\nabla(\rho f_i)|^2 - \lambda_i \int_{U'} \rho^2 f_i^2 \right| \leq \frac{1}{2} \left| \int_{U'} f_i^2 \Delta \rho^2 \right| + \int_{U'} |\rho \Delta \rho| f_i^2,$$

$$\lesssim \alpha^{-2\varepsilon} \int_{U' \setminus U} f_i^2 \lesssim \alpha^{-2\varepsilon} \int_V f_i^2 \lesssim \alpha^{2/3 - 3\varepsilon}. \tag{2.19}$$

We compute using integration by parts: $\int_{U'} \nabla(\rho f_1) \nabla(\rho f_2) =$

$$- \int_{U'} \rho f_1 \Delta(\rho f_2) = \lambda_2 \int_{U'} \rho^2 f_1 f_2 - 2 \int_{U'} \rho f_1 \nabla \rho \cdot \nabla f_2 - \int_{U'} \rho f_1 f_2 \Delta \rho, \tag{2.20}$$

and $\int_{U'} \nabla(\rho f_1) \nabla(\rho f_2) =$

$$- \int_{U'} \rho f_2 \Delta(\rho f_1) = \lambda_1 \int_{U'} \rho^2 f_1 f_2 - 2 \int_{U'} \rho f_2 \nabla \rho \cdot \nabla f_1 - \int_{U'} \rho f_2 f_1 \Delta \rho. \tag{2.21}$$

This gives the inequality

$$\begin{aligned} &\left| 2 \int_{U'} \nabla(\rho f_1) \nabla(\rho f_2) - (\lambda_1 + \lambda_2) \int_{U'} \rho^2 f_1 f_2 \right| \\ &\lesssim \left| \int_{U' \setminus U} \frac{f_1 f_2}{\alpha^{2\varepsilon}} \right| + 2 \left| \int_{U' \setminus U} \rho \nabla \rho \nabla (f_1 f_2) \right| \\ &\lesssim \alpha^{-2\varepsilon} \left| \int_V f_1 f_2 \right| \lesssim \alpha^{2/3 - 3\varepsilon}, \end{aligned} \tag{2.22}$$

which follows from integration by parts, the Schwarz inequality, and (2.11), (2.13). By (2.19),

$$|I| \lesssim \alpha^{2/3-3\varepsilon}. \tag{2.23}$$

To estimate II, we note that $\lambda_1 \le \lambda_2$ so that

$$II \le a^2 \left(\int_{U'} |\nabla \rho f_1|^2 - \lambda_1 \int_{U'} \rho^2 f_1^2 \right) \lesssim a^2 \alpha^{2/3-3\varepsilon}, \tag{2.24}$$

which follows from (2.19).

Note that by the orthogonality of f_1 and f_2, the Schwarz inequality and (2.13),

$$\left| \int_{U'} \rho^2 f_1 f_2 \right| = \left| \int_{ABE} (1-\rho^2) f_1 f_2 \right| \le \left| \int_V f_1 f_2 \right| \lesssim \alpha^{2/3-\varepsilon}. \tag{2.25}$$

By (2.22), (2.25), and adding and subtracting $\lambda_1 \int_{U'} a\rho^2 f_1 f_2$, we estimate III,

$$\begin{aligned} III &\le |a| \left| 2\int_{U'} \nabla(\rho f_1)\cdot\nabla(\rho f_2) - (\lambda_1+\lambda_2)\int_{U'} \rho^2 f_1 f_2 \right| + |a|(\lambda_2-\lambda_1)\left|\int_{U'} \rho^2 f_1 f_2\right| \\ &\lesssim |a|(\alpha^{2/3-3\varepsilon} + (\lambda_2-\lambda_1)\alpha^{2/3-\varepsilon}). \end{aligned} \tag{2.26}$$

Next we estimate the denominator in (2.17). By (2.25), and (2.13) which implies $\int_{U'} f_i^2 \gtrsim 1 - \alpha^{2/3-\varepsilon}$,

$$\int_{U'} (\rho f_2 + a\rho f_1)^2 \gtrsim (1+a^2)(1-\alpha^{2/3-\varepsilon}) - 2|a|\alpha^{2/3-\varepsilon}. \tag{2.27}$$

This inequality together with our estimates (2.23), (2.24), (2.26), and (2.27) in the variational principle for $\lambda_2(U')$ (2.17) show that

$$\lambda_2(U') \lesssim \lambda_2 + \frac{\alpha^{2/3-3\varepsilon} + a^2\alpha^{2/3-3\varepsilon} + |a|\alpha^{2/3-3\varepsilon} + |a|(\lambda_2-\lambda_1)\alpha^{2/3-\varepsilon}}{(1+a^2)(1-\alpha^{2/3-\varepsilon}) - 2|a|\alpha^{2/3-\varepsilon}}.$$

This simplifies a bit to

$$\lambda_2(U') \lesssim \lambda_2 + \alpha^{2/3-3\varepsilon}\left(\frac{1+a^2+|a|+|a|(\lambda_2-\lambda_1)\alpha^{2\varepsilon}}{1+a^2-(1+a^2)\alpha^{2/3-\varepsilon} - 2|a|\alpha^{2/3-\varepsilon}} \right).$$

We therefore have the bound

$$\lambda_2(U') \lesssim \lambda_2 + \alpha^{2/3-3\varepsilon} + (\lambda_2-\lambda_1)\alpha^{2/3-\varepsilon}.$$

2.3.3. *Gap estimate.* Using our estimates for λ_1 and λ_2, and the choice of $\varepsilon \in (0, 2/9)$ which implies $\alpha^{2/3-3\varepsilon} \to 0$ as $\alpha \to 0$, we have

$$\lambda_2 - \lambda_1 \gtrsim \lambda_2(U') - \lambda_1(U') - (\lambda_2-\lambda_1)\alpha^{2/3-\varepsilon}.$$

We then have

$$(\lambda_2-\lambda_1) \gtrsim \frac{\lambda_2(U')-\lambda_1(U')}{1+\alpha^{2/3-\varepsilon}}.$$

By the main theorem of [**31**], since the diameter of U' is at most $4\alpha^\varepsilon$,

$$\lambda_2(U') - \lambda_1(U') \ge \frac{\pi^2}{64\alpha^{2\varepsilon}},$$

which shows that

$$\lambda_2 - \lambda_1 \gtrsim C\alpha^{-2\varepsilon}$$

and is therefore unbounded as $\alpha \to 0$. We have shown that for any triangle with one or two small angles, $\xi(T)$ becomes unbounded as the triangle collapses to a segment. $\square$

3. Collapsing polygonal domains

The geometric feature which seems to determine whether the gap remains bounded or diverges as a polygonal domain collapses to a segment is the rate at which boundary points approach the segment. In the familiar example of rectangles this rate is perfectly uniform, and the gap remains bounded. In the case of triangles, points near B collapse "more quickly" than points near A (see Figure 3), and the gap diverges. The theorem below generalizes this observation to convex domains.

To state the theorem we shall require the following notions. Let $\{Q_n\}_{n\in\mathbb{N}}$ be a family of convex domains in $\mathbb{R}^2$. The arguments in both this section and the following can be generalized to convex domains with diameter 1 which collapse to the diameter, but for the sake of simplicity we shall assume Q_n is the convex hull of the graph of a Lipschitz continuous function over the unit interval and identify the base of Q_n with $S_n = \{(x, 0) : 0 \le x \le 1\}$. Without loss of generality we assume that the domains are contained in the upper half plane.

Definition 3.1. *The* height h_n *of* Q_n *is defined by*

$$h_n := \max\{y \mid (x, y) \in Q_n\}. \tag{3.2}$$

We say that the domains Q_n collapse as $n \to \infty$ *if*

$$\lim_{n\to\infty} h_n = 0.$$

The following definition describes collapse in which all boundary points are the same distance from the base up to $o(h^2)$, where h denotes the height.

Definition 3.3. *We say that* Q_n *are* asymptotically rectangular as $n \to \infty$ *if for each* n*, there exist rectangles* $R_n = [C_n, C_n + A_n] \times [0, h_n] \supset Q \supset r_n = [c_n, c_n + a_n] \times [0, b_n]$ *and a constant* $\gamma > 0$ *such that* $a_n \ge \gamma$ *for all* $n \in \mathbb{N}$ *and*

$$\frac{h_n}{b_n} = 1 + o(h_n^2) \qquad \text{as } n \to \infty. \tag{3.4}$$

Triangular domains do not collapse asymptotically rectangularly. The following notion generalizes the "non-uniform" collapse of triangular domains.

Definition 3.5. *We say that* Q_n *collapse* non-uniformly *as* $n \to \infty$ *if there exist convex inscribed polygons* $U_n \subset Q_n$ *such that the following hold.*

(1) *The diameter of* $U_n \to 0$ *as* $n \to \infty$.
(2) *The longest side* Σ_n *of* U_n *is contained in* S_n.
(3) *The height of* $U_n = h_n$.
(4) *The height of* $V_n := Q_n \setminus U_n$*, satisfies* $h(V_n) \sim (1 - h_n^\varepsilon)h_n$ *for some* $\varepsilon \in (0, 2/9)$.

These two types of collapse determine whether the gap remains bounded or diverges as the domains collapse.

Theorem 4. *Let* $\{Q_n\}_{n\in\mathbb{N}}$ *be a family of convex domains as above, and assume that* Q_n *collapse as* $n \to \infty$*. If* Q_n *are asymptotically rectangular as* $n \to \infty$*, then* *the gap function $\xi(Q_n)$ is bounded as $n \to \infty$. If Q_n collapse non-uniformly as $n \to \infty$, then the gap function $\xi(Q_n) \to \infty$ as $n \to \infty$.*

PROOF. Assume first that the collapsing domains are asymptotically rectangular. By domain monotonicity

$$\lambda_2(Q_n) - \lambda_1(Q_n) \le \lambda_2(r_n) - \lambda_1(R_n).$$

For simplicity we shall drop the subscript and denote $\lambda_i = \lambda_i(Q)$ for $i = 1, 2$. Since $b \le h \to 0$, we may assume that $a \ge b$. Then

$$\lambda_2(r) - \lambda_1(R) = \pi^2 \left(\frac{h^2 - b^2}{h^2 b^2} + \frac{4}{a^2} - \frac{1}{A^2} \right).$$

Since $Q \subset R$, $A \ge 1$, and by assumption a is also bounded below as $n \to \infty$. Thus

$$\xi(Q) \lesssim \pi^2 \left(\frac{(h/b)^2 - 1}{h^2} + \frac{4}{a^2} - \frac{1}{A^2} \right).$$

By the definition of asymptotically rectangular (3.4), it follows that $\xi(Q)$ is bounded as $n \to \infty$.

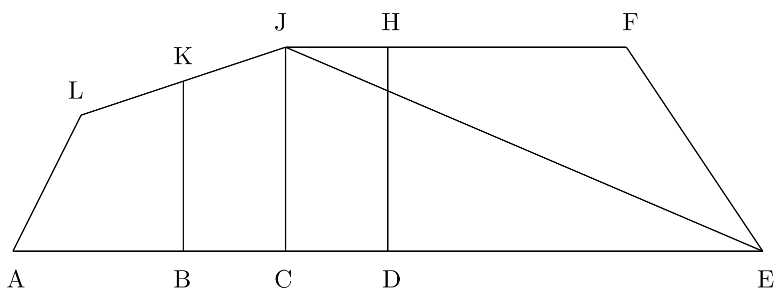

FIGURE 4. Collapsing polygon with unbounded gap.

For the case of non-uniform collapse we shall generalize the estimates used in the preceding section for triangles. Let $U' \supset U$ be a convex inscribed polygon so that one side Γ of U' satisfies $S \supset \Gamma \supset \Sigma$ and $|\Gamma - \Sigma| \sim h^{\varepsilon}$, where $\varepsilon \in (0, 2/9)$ is defined by Definition (3.4). We can define such an inscribed polygon since the diameter of $U \to 0$, and S has length 1. With this definition the diameter of $U' \to 0$ as $h \to 0$.

Let f_i be the eigenfunction for Q with eigenvalue λ_i, for $i = 1, 2$. By convexity, Q contains an inscribed right triangle T of height $h = h(U)$ and base at least $1/2$. For example in Figure 4, $Q = AEFJL$, $T = JCE$, $h = |JC|$, and $U = JKBDH$. Assume f_i are normalized so that

$$\int_Q f_i^2 = 1, \text{ and let } \eta_i := \int_V f_i^2, \quad i = 1, 2.$$

By the one dimensional Poincaré inequality,

$$\frac{\int_V |\nabla f_i|^2}{\int_V f_i^2} \gtrsim \frac{\pi^2}{(h(V))^2},$$

and

$$\frac{\int_U |\nabla f_i|^2}{\int_U f_i^2} \gtrsim \frac{\pi^2}{(h(U))^2}, \quad \frac{\int_{U'} |\nabla f_i|^2}{\int_{U'} f_i^2} \gtrsim \frac{\pi^2}{(h(U))^2},$$

since U and U' have the same height. For $h \approx 0$, the measure of the smallest angle of $T \sim h = h(U)$. By domain monotonicity

$$\lambda_i(Q) \leq \lambda_2(T) \leq \lambda_2(\sigma) \lesssim \frac{\pi^2}{(h(U))^2} + \frac{c_2}{(h(U))^{4/3}} + O(h(U)^{-1}),$$

where σ is a sector of the same opening angle as T which is contained in T and has radius bounded below by a fixed constant.

Estimating η_i as we did for triangles,

$$\frac{\eta_i \pi^2}{(h(V))^2} + \frac{\pi^2(1-\eta_i)}{(h(U))^2} \lesssim \int_V |\nabla f_i|^2 + \int_U |\nabla f_i|^2 = \lambda_i \leq \lambda_2(T)$$
$$\lesssim \frac{\pi^2}{(h(U))^2} + \frac{c_1}{(h(U))^{4/3}} + O(h(U)^{-1}).$$

This gives the following estimate for η_i,

$$\eta_i \lesssim \frac{h(U)^{2/3} h(V)^2}{h(U)^2 - h(V)^2}, \quad i = 1, 2. \tag{3.6}$$

By assumption $h = h(U)$ and $h(V) \sim h(1 + h^{\varepsilon})$ for some $\varepsilon \in (0, 2/9)$. It follows that

$$\eta := \max\{\eta_1, \eta_2\} \lesssim h^{2/3-\varepsilon}. \tag{3.7}$$

Let

$$\delta := \frac{2}{3} - \varepsilon. \tag{3.8}$$

3.0.4. *Estimates for λ_1.* By domain monotonicity

$$\lambda_1 \leq \lambda_1(U') \implies \lambda_2 - \lambda_1 \geq \lambda_2 - \lambda_1(U').$$

3.0.5. *Estimates for λ_2.* We shall again define a cut-off function ρ as in (2.14) so that,

$$|\nabla \rho| \leq h^{-\varepsilon}, \quad |\Delta \rho| \leq h^{-2\varepsilon}.$$

Since the function ρf_2 is not à priori an admissible test function for the variational principle for $\lambda_2(U')$ we again modify it to make it orthogonal to the first eigenfunction on U'. Since the first eigenfunction of U' and the first eigenfunction of Q are both positive on the interior, we shall consider the test function

$$\rho f_2 + a\rho f_1,$$

where a is chosen so that $\rho f_2 + a\rho f_1$ is $\mathcal{L}^2$ orthogonal to the first eigenfunction of U'.

The variational principle for $\lambda_2(U')$ using $\rho f_2 + a\rho f_1$ as a test function

$$\lambda_2(U') \leq \frac{\int_{U'} |\nabla(\rho f_2 + a\rho f_1)|^2}{\int_{U'} (\rho f_2 + a\rho f_1)^2}. \tag{3.9}$$

Since

$$\lambda_2 = \lambda_2 \frac{\int_{U'} (\rho f_2 + a\rho f_1)^2}{\int_{U'} (\rho f_2 + a\rho f_1)^2},$$

expanding the numerator and denominator in (3.9) we have

$$\lambda_2(U') \leq \lambda_2 + \frac{I + II + III}{\int_{U'} (\rho f_2 + a\rho f_1)^2},$$

where

$$I := \int_{U'} |\nabla(\rho f_2)|^2 - \lambda_2 \int_{U'} \rho^2 f_2^2, \quad II := a^2 \left(\int_{U'} |\nabla(\rho f_1)|^2 - \lambda_2 \int_{U'} \rho^2 f_1^2 \right)$$

and

$$III := \int_{U'} 2a\nabla(\rho f_1) \cdot \nabla(\rho f_2) - \lambda_2 \int_{U'} 2a\rho^2 f_1 f_2.$$

Using (2.18), we estimate

$$\left| \int_{U'} |\nabla(\rho f_i)|^2 - \lambda_i \int_{U'} \rho^2 f_i^2 \right| \le \frac{1}{2} \left| \int_{U'} \Delta\rho^2 f_i^2 \right| + \int_{U'} |\rho\Delta\rho| f_i^2,$$

$$\lesssim h^{-2\varepsilon} \int_{U'-U} f_i^2 \lesssim h^{-2\varepsilon} \int_V f_i^2 \lesssim h^{\delta-2\varepsilon}, \tag{3.10}$$

since $\Delta\rho$ and $\nabla\rho$ vanish identically on U. By (2.20) and (2.21),

$$\begin{aligned}
&\left| 2\int_{U'} \nabla(\rho f_1)\nabla(\rho f_2) - (\lambda_1 + \lambda_2) \int_{U'} \rho^2 f_1 f_2 \right| \\
&\lesssim \int_{U'-U} \frac{|f_1 f_2|}{h^{2\varepsilon}} + 2 \left| \int_{U'-U} \rho\nabla\rho\nabla(f_1 f_2) \right|
\end{aligned}$$

$$\lesssim h^{-2\varepsilon} \left(\int_V |f_1 f_2| + 2 \int_V |f_1 f_2| \right) \lesssim h^{\delta-2\varepsilon}, \tag{3.11}$$

which follows from integration by parts and the Schwarz inequality.

By our calculations for triangles, our estimate for η, and the estimates (3.10) and (3.11),

$$I \lesssim h^{\delta-2\varepsilon}, \tag{3.12}$$

$$II \lesssim a^2 h^{\delta-2\varepsilon}, \text{ and} \tag{3.13}$$

$$III \lesssim |a|(h^{\delta-2\varepsilon} + (\lambda_2 - \lambda_1)h^\delta). \tag{3.14}$$

Moreover,

$$\int_{U'} (\rho f_2 + a\rho f_1)^2 \gtrsim (1 + a^2)(1 - h^\delta) - 2|a|h^\delta. \tag{3.15}$$

Using these estimates, we estimate the Rayleigh-Ritz quotient for $\rho f_2 + a\rho f_1$,

$$\lambda_2(U') \lesssim \lambda_2 + \frac{h^{\delta-2\varepsilon} + a^2 h^{\delta-2\varepsilon} + |a|h^{\delta-2\varepsilon} + |a|(\lambda_2 - \lambda_1)h^\delta}{(1 + a^2)(1 - h^\delta) - 2|a|h^\delta}. \tag{3.16}$$

This simplifies to

$$\lambda_2(U') \lesssim \lambda_2 + h^{\delta-2\varepsilon} \left(\frac{1 + a^2 + |a| + |a|(\lambda_2 - \lambda_1)h^{2\varepsilon}}{1 + a^2 - (1 + a^2)h^\delta - 2|a|h^\delta} \right).$$

We therefore have the bound as $h \to 0$,

$$\lambda_2(U') \lesssim \lambda_2 + h^{\delta-2\varepsilon} + h^\delta(\lambda_2 - \lambda_1).$$

3.0.6. *Gap estimate.* Using our estimates for $\lambda_i(U')$

$$\lambda_2 - \lambda_1 \gtrsim \lambda_2 - \lambda_1(U') \gtrsim \lambda_2(U') - \lambda_1(U') - h^{\delta - 2\varepsilon} - (\lambda_2 - \lambda_1) h^{\delta},$$

Since

$$\delta = 2/3 - \varepsilon, \quad \varepsilon < 2/9,$$

it follows that

$$\delta - 2\varepsilon = 2/3 - 3\varepsilon > 0.$$

Consequently,

$$\lambda_2 - \lambda_1 \gtrsim \frac{\lambda_2(U') - \lambda_1(U')}{1 + h^{\delta}}. \tag{3.17}$$

By hypothesis on U and definition of U', the diameter of U' vanishes as $h \to 0$, so (3.17) and the main theorem of [**31**] imply

$$\xi \sim \lambda_2 - \lambda_1 \to \infty, \qquad \text{as } h \to 0.$$

□

The theorem shows that the gap function is sensitive to the rate at which boundary points collapse to the base. We shall demonstrate in the following section that the same phenomenon is true for domains in $\mathbb{R}^n$ under one dimensional collapse.

4. The general case

To prove Theorem 3 we shall require the following proposition.

PROPOSITION 3. *Let $\Omega \subset \mathbb{R}^n$ be a convex bounded domain with Lipschitz boundary, and let ϕ be the first eigenfunction for the Euclidean Laplacian on Ω with the Dirichlet boundary condition. Let U be a convex subset of Ω. Define*

$$\mu := \inf \left\{ \frac{\int_U |\nabla \varphi|^2 \phi^2}{\int_U \varphi^2 \phi^2} \middle| \varphi \in \mathcal{C}^2(U), \quad \int_U \varphi \phi^2 = 0 \right\}.$$

Then

$$\mu \geq \frac{\pi^2}{4 d(U)^2}, \quad d(U) = \text{ diameter of } U.$$

PROOF. By the variational principle (see [**27**] §2), μ is the first positive eigenvalue for the Bakry-Émery Laplace operator

$$\mathcal{L} := \Delta + 2\nabla \log \phi \nabla,$$

with the Neumann boundary condition on ∂U. Analogous to the classical variational principle for the Laplacian, the infimum is achieved by a function φ which satisfies

$$\Delta \varphi + 2\nabla \log \phi \nabla \varphi = -\mu \varphi, \quad \left. \frac{\partial \varphi}{\partial n} \right|_{\partial U} = 0. \tag{4.1}$$

We define for $\varepsilon > 0$

$$g := \frac{1}{2} \left(|\nabla \varphi|^2 + (\mu + \varepsilon) \varphi^2 \right).$$

CLAIM 1. *Let x_0 be the point at which g achieves its maximum. Then $\nabla \varphi(x_0) = 0$.*

PROOF. To prove the claim we first note that if g attains its maximum at $x_0 \in U \setminus \partial U$, then $\nabla\varphi(x_0) = 0$, and $\Delta g(x_0) \leq 0$. On the other hand if g achieves its maximum at $x_0 \in \partial U$, then we have

$$0 \leq \nabla g \cdot \vec{n} = \left((\mu+\varepsilon)\varphi\nabla\varphi + \left(\sum_{i=1}^{n} \varphi_i \varphi_{ij} \right)_{j=1}^{n} \right) \cdot \vec{n},$$

where $\vec{n}$ is the outward normal vector at x_0. Since φ satisfies the Neumann boundary condition,

$$0 \leq \nabla g \cdot \vec{n} = \sum_{i.j=1}^{n} \varphi_i \varphi_{ij} \cdot n_j = -\sum_{i,j=1}^{n} h_{ij}\varphi_i\varphi_j,$$

where h_{ij} is the second fundamental form. By convexity of U, h_{ij} is non-negative definite so we have

$$0 \geq \nabla g \cdot \vec{n} = \sum_{i.j=1}^{n} \varphi_i \varphi_{ij} \cdot n_j = -\sum_{i,j=1}^{n} h_{ij}\varphi_i\varphi_j \leq 0.$$

This implies that $\nabla g(x_0) = 0$. To show that $\Delta g(x_0) \leq 0$, note that

$$\Delta g(x_0) = \Delta' g(x_0) + g_{nn}(x_0) + 2H\nabla g(x_0) \cdot \vec{n},$$

where H is the mean curvature, g_{nn} is the second derivative of g in the normal direction, and Δ' is the Laplace operator for ∂U. At the maximum point x_0,

$$\Delta' g(x_0) \leq 0.$$

We have shown above that $\nabla g(x_0) \cdot \vec{n} = 0$, and since x_0 is the maximum point, $g_{nn}(x_0) \leq 0$. It follows that

$$\Delta g(x_0) \leq 0.$$

Consequently, in any case we have $\nabla g(x_0) =$

$$(\mu+\varepsilon)\varphi\nabla\varphi + \left(\sum_{i=1}^{n} \varphi_i\varphi_{ij} \right)_{j=1}^{n} = 0 \implies (\mu+\varepsilon)\varphi\varphi_j = -\sum_{i=1}^{n} \varphi_i\varphi_{ij}, \tag{4.2}$$

for all $j = 1, \ldots, n$. Note that the above equalities and the calculations below are all for the maximum point x_0, so for the sake of simplicity we shall not write x_0. We also have at the maximum point

$$\Delta g = (\mu+\varepsilon)|\nabla\varphi|^2 + (\mu+\varepsilon)\varphi\Delta\varphi + \sum_{i,j=1}^{n} (\varphi_{ij}^2 + \varphi_i\varphi_{ijj}) \leq 0. \tag{4.3}$$

For each $j = 1, \ldots, n$ in (4.2), multiplying by φ_j and summing over j, we have

$$\left(\sum_{i,j=1}^{n} \varphi_i\varphi_{ij}\varphi_j \right)^2 = \left(\sum_{j=1}^{n} -(\mu+\varepsilon)\varphi\varphi_j^2 \right)^2 = (\mu+\varepsilon)^2\varphi^2|\nabla\varphi|^4. \tag{4.4}$$

To show that $\nabla\varphi(x_0) = 0$ we will assume for the sake of contradiction that $\nabla\varphi(x_0) \neq 0$. Then by the Cauchy Inequality in (4.4)

$$\sum_{i,j=1}^{n} (\varphi_{ij})^2 = \sum_{i,j=1}^{n} (\varphi_{ij})^2 \frac{|\nabla\varphi|^4}{|\nabla\varphi|^4} \geq \frac{\left(\sum_{i,j=1}^{n} \varphi_{ij}\varphi_i\varphi_j \right)^2}{|\nabla\varphi|^4} = (\mu+\varepsilon)^2\varphi^2.$$

Next we have by (4.1)

$$\sum_{i,j=1}^{n} \varphi_i \varphi_{ijj} = \sum_{i=1}^{n} \varphi_i (\Delta\varphi)_i = \sum_{i=1}^{n} \varphi_i (-\mu\varphi - 2\nabla \log\phi \cdot \nabla\varphi)_i.$$

By [**5**] the logarithm of ϕ is concave and therefore the Hessian of $\log\phi$ is negative definite, and we have

$$\sum_{i,j=1}^{n} -2\varphi_i\varphi_j \frac{\phi_{ij}}{\phi} \geq 0.$$

Consequently,

$$\sum_{i=1}^{n} \varphi_i (-\mu\varphi - 2\nabla \log\phi \cdot \nabla\varphi)_i \geq -\mu|\nabla\varphi|^2 - 2\sum_{i,j=1}^{n} \varphi_i \frac{\phi_j}{\phi}\varphi_{ij}.$$

By (4.2),

$$\sum_{i,j=1}^{n} \varphi_i \frac{\phi_j}{\phi}\varphi_{ij} = -\sum_{j=1}^{n} (\mu+\varepsilon)\varphi\varphi_j \frac{\phi_j}{\phi} = -(\mu+\varepsilon)\varphi\nabla\log\phi\nabla\varphi,$$

so we have

$$\sum_{i=1}^{n} \varphi_i (-\mu\varphi - 2\nabla \log\phi \cdot \nabla\varphi)_i \geq -\mu|\nabla\varphi|^2 + 2(\mu+\varepsilon)\varphi\nabla\log\phi\nabla\varphi.$$

Putting it all together in (4.3) we have

$$0 \geq (\mu+\varepsilon)|\nabla\varphi|^2 + (\mu+\varepsilon)\varphi\Delta\varphi + \sum_{i,j=1}^{n} (\varphi_{ij}^2 + \varphi_i\varphi_{ijj}) \geq$$

$$(\mu+\varepsilon)|\nabla\varphi|^2 + (\mu+\varepsilon)\varphi(-\mu\varphi - 2\nabla\log\phi\nabla\varphi) + (\mu+\varepsilon)^2\varphi^2 - \mu|\nabla\varphi|^2 + 2(\mu+\varepsilon)\varphi\nabla\log\phi\nabla\varphi$$
$$= \varepsilon^2\varphi^2 + +\varepsilon|\nabla\varphi|^2 + \mu\varepsilon\varphi^2.$$

Since $\varepsilon, \mu > 0$, we clearly must have $\nabla\varphi = 0$ which is a contradiction. $\square$

It follows that at the maximum point x_0 for g, $\nabla\varphi$ vanishes and hence

$$g(x) \leq g(x_0) = \frac{1}{2}(\mu+\varepsilon)\varphi^2(x_0) \leq \frac{1}{2}(\mu+\varepsilon)\max(\varphi^2).$$

Assume we have normalized φ such that the maximum of φ^2 is 1. Then,

$$\frac{1}{2}\left(|\nabla\varphi|^2 + (\mu+\varepsilon)\varphi^2\right) = g \leq \frac{1}{2}(\mu+\varepsilon)$$

and so

$$|\nabla\varphi|^2 \leq (\mu+\varepsilon)(1-\varphi^2).$$

Letting $\varepsilon \to 0$ we have

$$|\nabla\varphi|^2 \leq \mu(1-\varphi^2) \implies \frac{|\nabla\varphi|}{\sqrt{1-\varphi^2}} \leq \sqrt{\mu}. \tag{4.5}$$

Since

$$\int_U \varphi\phi^2 = 0,$$

and $\phi > 0$ on the interior of U, there exist points $y_0, y_1 \in U$ such that $\varphi(y_0) = 0$ and $\varphi(y_1)^2 = 1$. Without loss of generality we may assume that $\varphi(y_1) = 1$. Let $\gamma(t)$ be a unit-speed geodesic in U such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$. Then

$$\begin{aligned}\left|\int_\gamma \frac{\nabla\varphi}{\sqrt{1-\varphi^2}}d\sigma\right| &= \left|\int_0^1 \frac{\nabla\varphi(\gamma(t))\dot\gamma(t)}{\sqrt{1-\varphi(\gamma(t))^2}}dt\right| \\ &= |\arcsin(\varphi(\gamma(1))) - \arcsin(\varphi(\gamma(0)))| = \frac{\pi}{2}.\end{aligned}$$

By (4.5),

$$\left|\int_\gamma \frac{\nabla\varphi}{\sqrt{1-\varphi^2}}d\sigma\right| \le l(\gamma)\sqrt{\mu} \le d(U)\sqrt{\mu}.$$

It follows that

$$\frac{\pi}{2} \le d(U)\sqrt{\mu} \implies \frac{\pi^2}{4d(U)^2} \le \mu.$$

□

We now have the necessary ingredients to prove our main result.

Proof of Theorem 3. In what follows, the constant C is independent of δ and ε but may be different from line to line and within the same line. Let u_1 and u_2 be the eigenfunctions of Ω_ε for the eigenvalues $\lambda_1(\Omega_\varepsilon), \lambda_2(\Omega_\varepsilon)$, respectively, and assume

$$\int_{\Omega_\varepsilon} u_i^2 = 1, \quad i = 1, 2.$$

Define

$$\begin{aligned}E(\delta) &:= \{x \in E \mid h_\varepsilon(x) > \sigma_\varepsilon - \delta\};\\ U(\delta) &:= (E(\delta) \times \mathbb{R}) \cap \Omega_\varepsilon;\\ V(\delta) &:= \Omega_\varepsilon \setminus U(\delta)\end{aligned}$$

We shall again estimate the $\mathcal{L}^2$ norm of the eigenfunctions on $V(\delta)$ using the one dimensional Poincaré inequality which implies

$$\lambda_i(\Omega_\varepsilon) = \int_{\Omega_\varepsilon} |\nabla u_i|^2 \gtrsim \frac{\pi^2}{(\sigma_\varepsilon - \delta)^2\varepsilon^2}\int_{V(\delta)} u_i^2 + \frac{\pi^2}{\sigma_\varepsilon^2\varepsilon^2}\left(1 - \int_{V(\delta)} u_i^2\right), \tag{4.6}$$

for $i = 1, 2$. By the Lipschitz continuity of $h_\varepsilon^\pm$ and the uniform boundedness of the modulus of continuity, it follows that Ω_ε contains a generalized cylinder of height $\sim \varepsilon(\sigma_\varepsilon - \delta^2)$ over $E(\delta^2)$. The eigenvalues of such a cylinder are bounded above by $\lambda_i(E(\delta^2)) + \frac{\pi^2}{(\sigma_\varepsilon-\delta^2)^2\varepsilon^2}$. By domain monotonicity,

$$\lambda_i(\Omega_\varepsilon) \le \lambda_i(U(\delta^2)) \lesssim \lambda_i(E(\delta^2)) + \frac{\pi^2}{(\sigma_\varepsilon - \delta^2)^2\varepsilon^2}. \tag{4.7}$$

Let x_0 be the maximum point of $h_\varepsilon(x)$ so that $h_\varepsilon(x_0) = \sigma_\varepsilon$. Then by the Lipschitz continuity of $h_\varepsilon^\pm$ and the uniform boundedness of the modulus of continuity, there is a constant C, such that

$$E(\delta^2) \supset B_{x_0}(C\delta^2).$$

Therefore, we have by domain monotonicity

$$\lambda_i(E(\delta^2)) \lesssim \delta^{-4}.$$

Combining the above inequality with with (4.6) and (4.7), we have

$$\frac{\pi^2}{(\sigma_\varepsilon-\delta)^2\varepsilon^2}\int_{V(\delta)} u_i^2 + \frac{\pi^2}{\sigma_\varepsilon^2\varepsilon^2}\left(1-\int_{V(\delta)} u_i^2\right) \lesssim \frac{\pi^2}{\delta^4} + \frac{\pi^2}{(\sigma_\varepsilon-\delta^2)^2\varepsilon^2},$$

and hence we have

$$\int_{V(\delta)} u_i^2 \lesssim \delta + \frac{\varepsilon^2}{\delta^5}.$$

We choose $\delta = \varepsilon^{1/3}$. Then we have

$$\int_{V(\delta)} u_i^2 \lesssim \varepsilon^{1/3}. \tag{4.8}$$

Since u_1 is orthogonal to u_2, we have

$$\left|\int_{U(\delta)} u_1 u_2\right| = \left|\int_{V(\delta)} u_1 u_2\right| \lesssim \varepsilon^{1/3}, \tag{4.9}$$

where the last inequality follows from (4.8) and the Schwarz inequality.

Let

$$\psi := \frac{u_2}{u_1}$$

and let

$$\alpha := \frac{\int_{U(\delta)} \psi u_1^2}{\int_{U(\delta)} u_1^2}, \quad \tilde{\psi} := \psi - \alpha.$$

By the estimate (4.9) and definition of ψ,

$$\alpha = \frac{\int_{U(\delta)} u_1 u_2}{\int_{U(\delta)} u_1^2} \lesssim \frac{\varepsilon^{1/3}}{1-\varepsilon^{1/3}}.$$

Let $\mu = \mu(\delta)$ be defined as in Proposition 3, where in the statement of the proposition $\phi = u_1$, and $U = U(\delta)$. Note that by the definition of $h_\varepsilon^\pm$ as concave, non-positive and convex non-negative functions and the convexity of E it follows that U is a convex subset of $\mathbb{R}^n$. We compute that

$$\int_U \tilde{\psi} u_1^2 = 0.$$

Since μ is the infimum and $\nabla\tilde{\psi} = \nabla\psi$,

$$\mu \le \frac{\int_U |\nabla\psi|^2 u_1^2}{\int_U \tilde{\psi}^2\, u_1^2} \le \frac{\int_{\Omega_\varepsilon} |\nabla\psi|^2 u_1^2}{\int_U \tilde{\psi}^2\, u_1^2} = \frac{\lambda_2(\Omega_\varepsilon)-\lambda_1(\Omega_\varepsilon)}{\int_U \tilde{\psi}^2\, u_1^2}.$$

The final equality in the numerator follows from §2 of [**27**]. We compute

$$\begin{aligned}\int_U \tilde{\psi}^2 u_1^2 &= \int_U u_2^2 + \alpha^2\int_U u_1^2 - 2\int_U \alpha u_2 u_1 \\ &= \int_U u_2^2 - \alpha\int_U u_1 u_2 \gtrsim 1-\varepsilon^{1/3}.\end{aligned}$$

Consequently we have the estimate

$$\mu \lesssim \frac{(\lambda_2(\Omega_\varepsilon)-\lambda_1(\Omega_\varepsilon))}{1-\varepsilon^{1/3}}.$$

By Proposition 3

$$\mu \geq \frac{\pi^2}{4d(U)^2}.$$

By the assumption that $d(U) \to 0$ uniformly as $\varepsilon \to 0$, it follows that $\lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) \to \infty$ as $\varepsilon \to 0$. Finally we note that the diameter of Ω_ε is uniformly bounded from above and from below (by the diameter of E, for example), and so it follows that

$$\xi(\Omega_\varepsilon) \sim \lambda_2(\Omega_\varepsilon) - \lambda_1(\Omega_\varepsilon) \to \infty, \qquad \text{as } \varepsilon \to 0.$$

□

REMARK 2. The assumption that the diameter of $U_\varepsilon(\delta)$ tends to zero is necessary. For example, assume there exists a fixed constant $c > 0$ such that

$$\operatorname{diam}\{x \in E \mid h_\varepsilon(x) \equiv \sigma_\varepsilon\} \geq c, \quad \forall \varepsilon.$$

By domain monotonicity estimates using two generalized cylinders analogous to the two rectangles used in the proof of the bounded-gap case of Theorem 4, it is straightforward to show that in this case the gap remains bounded under one dimensional collapse. The details are left to the reader.

REMARK 3. In [**4**] and [**13**] much more precise results for the behavior of the eigenvalues and the eigenfunctions under one-dimensional collapse were obtained. We note that those results require that the collapsing domains are defined by

$$\Omega_\varepsilon = \{(x, y) \mid x \in E, \quad 0 \leq y \leq \varepsilon h(x)\} \to E \qquad \text{as } \varepsilon \to 0. \tag{4.10}$$

Since the collapse is described by scaling a *fixed* non-negative function h by the parameter ε, those results do not immediately apply to the cases considered here. It would be interesting to investigate whether the results of [**4**] and [**13**] may be extended to more general collapsing geometry.

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