

EIGENVALUES OF COLLAPSING DOMAINS AND DRIFT LAPLACIANS

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ABSTRACT. By introducing a weight function to the Laplace operator, Bakry and Émery defined the “drift Laplacian” to study diffusion processes. Our first main result is that, given a Bakry–Émery manifold, there is a naturally associated family of graphs whose eigenvalues converge to the eigenvalues of the drift Laplacian as the graphs collapse to the manifold. Applications of this result include a new relationship between Dirichlet eigenvalues of domains in \mathbb{R}^n and Neumann eigenvalues of domains in \mathbb{R}^{n+1} and a new maximum principle. Using our main result and maximum principle, we are able to generalize *all the results in Riemannian geometry based on gradient estimates to Bakry–Émery manifolds*.

1. Introduction

Bakry–Émery geometry was introduced in [3] to study diffusion processes. For a Riemannian manifold (M, g) and $\phi \in \mathcal{C}^2(M)$, the Bakry–Émery manifold is a triple (M, g, ϕ) , where the measure on M is the weighted measure $e^{-\phi} dV_g$. If Ric and Δ are, respectively, the Ricci curvature and Laplacian with respect to the Riemannian metric g , then the Bakry–Émery Ricci curvature is defined to be¹

$$\text{Ric}_\infty = \text{Ric} + \text{Hess}(\phi)$$

and the Bakry–Émery Laplacian is

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla.$$

The operator can be extended as a self-adjoint operator with respect to the weighted measure $e^{-\phi} dV_g$; it is also known as a “drifting” or “drift” Laplacian.

Theorem 1. *Let (M, g, ϕ) be a compact Bakry–Émery manifold. Let*

$$M_\varepsilon := \{(x, y) \mid x \in M, \quad 0 \leq y \leq \varepsilon e^{-\phi(x)}\} \subset M \times \mathbb{R}^+,$$

with $\phi \in \mathcal{C}^2(M)$ and $e^{-\phi} \in \mathcal{C}(M \cup \partial M)$. Let $\{\mu_k\}_{k=0}^\infty$ be the eigenvalues of the Bakry–Émery Laplacian on M . If $\partial M \neq \emptyset$, assume the Neumann boundary condition. Let $\mu_k(\varepsilon)$ be the Neumann eigenvalues of M_ε for $\tilde{\Delta} := \Delta + \partial_y^2$. Then $\mu_k(\varepsilon) = \mu_k + O(\varepsilon^2)$ for $k \geq 0$.

A corollary of Theorem 1 gives a relationship between the Dirichlet eigenvalues in \mathbb{R}^n and Neumann eigenvalues in \mathbb{R}^{n+1} .

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¹In the notation of [18], this is the ∞ Bakry–Émery Ricci curvature; see also [24].

Corollary 1. *Let M be a bounded domain in \mathbb{R}^n with smooth boundary, and let ϕ_1 be the first Dirichlet eigenfunction of the Euclidean Laplacian on M . Define*

$$M_\varepsilon := \{(x, y) \in \mathbb{R}^{n+1} \mid x \in M, 0 \leq y \leq \varepsilon \phi_1(x)^2\}.$$

Let $\{\lambda_k\}_{k=1}^\infty$ be the Dirichlet eigenvalues of M , and let $\{\mu_k(\varepsilon)\}_{k=0}^\infty$ be the Neumann eigenvalues of M_ε . Then $\lim_{\varepsilon \rightarrow 0} \mu_{k-1}(\varepsilon) = \lambda_k - \lambda_1$, for all $k \in \mathbb{N}$.

In the second part of the paper, we establish a new maximum principle which, together with Theorem 1, imply the following.

Principle. *There is a one-one correspondence between the gradient estimate on a Riemannian manifold and on a Bakry–Émery manifold. More precisely, the eigenvalue estimate on the Bakry–Émery manifold (M, g, ϕ) is equivalent to that on the Riemannian manifold $(M_\varepsilon, g + dy^2)$ for ε small enough.*

The method of gradient estimates in eigenvalue problems was first used by Li–Yau [15]. The papers [22, 25, 26, 14, 4, 28, 2, 27] are the most influential to this work. Gradient estimates on Riemannian manifolds are often quite complicated. The point of the above **Principle** is that one may apply all the proofs of gradient estimates directly to Bakry–Émery geometry without repeating the calculations.

We end this section by discussing an alternative but related setting.² In [18], Lott considered the product $M \times S^q$ for $q > 0$, with warped product metric

$$g_i^{M \times S^q} = g^M + \frac{1}{i^2} \phi^{\frac{2}{q}} g^{S^q}.$$

A curvature bound of the type

$$\text{Ric}_\phi^q = \text{Ric}^M + \text{Hess}(\phi) - \frac{1}{q} d\phi \otimes d\phi \geq r g^M$$

for some constant r implies a lower bound of the curvature on $M \times S^q$:

$$\text{Ric}^{M \times S^q} \geq r_1 g^{M \times S^q}$$

for a possibly smaller r_1 , as $i \rightarrow \infty$. In this setting, the manifolds (M_i, g_i) converge with respect to Gromov–Hausdorff norm to the manifold (M, g^M) . On the one hand, $q \in \mathbb{N}$ allows higher dimensional collapse, whereas we consider only one collapsing direction. On the other hand, the fibers in [18] are *closed*, which eliminates the need for estimates regarding the boundary of M_ε . In order to prove convergence of the eigenvalues in the setting of [18], one could apply Theorem 7.3 from Cheeger and Colding [6], but one must first prove that the gradient of the k th eigenfunction is bounded above *uniformly in i* (cf. equation (7.4) of [6]). Moreover, to prove the *uniform* eigenvalue convergence (for each k) as done here, one would need to demonstrate technical results analogous to those in Section 5. In the case of one collapsing direction, the L^2 bounds on the eigenfunctions are sufficient to prove uniform convergence of the eigenvalues. When more than one dimension collapses, some of the arguments in Section 3 will not apply. However, in the closed case, further techniques are available and boundary estimates are obsolete, so it may be possible to generalize our arguments. This will be carried out elsewhere.

The eigenvalues of the Laplacian on manifolds *with boundary* which collapse or are “very thin in one direction” has been studied by several authors. An estimate

²We are grateful to O. Munteanu for bringing this reference to our attention.

and convergence result for the Dirichlet Laplacian on planar domains were obtained by [11] and later refined in [8]. In all dimensions, there is motivation from physics to understand the spectrum of the Laplacian for “thin tubes”, see [10]. Our work can be seen as the higher dimensional Neumann analogue of [11] in the more general setting of Bakry–Émery geometry. It would be interesting to determine further terms in the expansion of $\mu_k(\varepsilon)$ as $\varepsilon \rightarrow 0$ in the spirit of Friedlander and Solomyak’s work in two dimensions [8]. This investigation will be the subject of forthcoming work.

The organization of the paper is as follows. In Section 2, we present the variational principles for the drift Laplacian, which we use heavily in our proof. The proof of Theorem 1 and Corollary 1 comprise Section 3. We prove the new maximum principle and discuss its applications in Section 4; finally, Section 5 contains technical results on Schauder estimates which are of independent interest.

2. Variational principles

On a Riemannian manifold (M, g) with boundary ∂M , the Laplace operator can be written as

$$\Delta = \frac{1}{\sqrt{\det(g)}} \sum_{i,j} \partial_i g^{ij} \sqrt{\det(g)} \partial_j,$$

and in particular on \mathbb{R}^n with the Euclidean metric,

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

The Dirichlet (respectively, Neumann) eigenvalues of the Laplace operator are the real numbers λ for which there exists an eigenfunction

$$u \in \mathcal{C}^\infty(M) \text{ such that } -\Delta u = \lambda u \text{ and } u|_{\partial M} = 0, \left(\text{respectively, } \frac{\partial u}{\partial n} \Big|_{\partial M} = 0 \right).$$

The eigenvalues of the drift Laplace operator are defined analogously.

We shall use λ to denote Dirichlet eigenvalues, μ to denote Neumann eigenvalues, and index the Dirichlet eigenvalues by \mathbb{N} and the Neumann eigenvalues by $0 \cup \mathbb{N}$. The Dirichlet and Neumann³ eigenvalues, respectively, satisfy the following variational principles [5]:

$$\lambda_k = \inf_{\varphi \in \mathcal{C}^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2}{\int_M \varphi^2} \mid \varphi|_{\partial M} = 0, \varphi \not\equiv 0 = \int_M \varphi \phi_l, 0 \leq l < k \right\},$$

$$\mu_j = \inf_{\varphi \in \mathcal{C}^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2}{\int_M \varphi^2} \mid \varphi \not\equiv 0 = \int_M \varphi \phi_l, -1 \leq l < j \right\},$$

for $k \geq 1$ and $j \geq 0$, where ϕ_j and φ_l are, respectively, eigenfunctions for λ_j and μ_l (assuming that $\phi_0 \equiv 0$ and $\varphi_{-1} \equiv 0$).

³Note that the Neumann boundary condition is automatically satisfied if no boundary condition is imposed in the variational principle.

Proposition 1. *Let (M, g, ϕ) be a Bakry–Émery manifold with boundary. Then, the Dirichlet and Neumann eigenvalues of the associated drift Laplacian satisfy the following variational principles:*

$$\lambda_k = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 e^{-\phi}}{\int_M \varphi^2 e^{-\phi}} \mid \varphi|_{\partial M} = 0, \varphi \not\equiv 0 = \int_M \varphi \phi_j e^{-\phi}, 0 \leq j < k, \right\},$$

$$\mu_j = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 e^{-\phi}}{\int_M \varphi^2 e^{-\phi}} \mid \varphi \not\equiv 0 = \int_M \varphi \phi_l e^{-\phi}, -1 \leq l < j \right\}$$

for $k \geq 1$ and $j \geq 0$.

Remark 1. Let $\{\lambda_k\}_{k=1}^\infty$ be the Dirichlet eigenvalues and let the associated orthonormal basis of eigenfunctions be $\{\phi_k\}_{k=1}^\infty$. Setting the weight function $\phi = -2 \log \phi_1$, the variational principle for (M, g, ϕ) is that for all $k \geq 1$,

$$\lambda_k - \lambda_1 = \inf_{\varphi \in C^1(M)} \left\{ \frac{\int_M |\nabla \varphi|^2 \phi_1^2}{\int_M \varphi^2 \phi_1^2} \mid \varphi \not\equiv 0 = \int_M \varphi \phi_j \phi_1, 0 \leq j < k \right\}.$$

When $k = 2$, and the domain $M \subset \mathbb{R}^n$, the following variational principle is Corollary 1.3 of [13] and is based on results of [7]. The following proposition is a useful tool.

Proposition 2. *For $k \geq 1$, let ξ_0, \dots, ξ_{k-1} be a nontrivial orthogonal set with respect to the weighted \mathcal{L}^2 measure; that is*

$$\int_M \xi_i \xi_j e^{-\phi} = 0$$

for $i \neq j$ and $\xi_i \not\equiv 0$. Then, we have for the Neumann eigenvalues

$$\sum_{j=0}^k \mu_j \leq \sum_{j=0}^k \frac{\int_M |\nabla \xi_j|^2 e^{-\phi}}{\int_M |\xi_j|^2 \phi_1^2}.$$

The proof is well known and is omitted. □

We demonstrate that the difference between the k th and the first Dirichlet eigenvalues is the Neumann eigenvalue of a certain drift Laplacian. This result was known to Singer–Wong–Yau–Yau [22].

Proposition 3. *For a bounded domain $M \subset \mathbb{R}^n$, let $\{\lambda_k\}_{k=1}^\infty$ be the Dirichlet eigenvalues of the Euclidean Laplacian with orthonormal basis of eigenfunctions $\{\phi_k\}_{k=1}^\infty$, and let $\{\mu_k\}_{k=0}^\infty$ be the Neumann eigenvalues of the drift Laplacian on M with respect to the weight function $-2 \log \phi_1$. Then, $\lambda_k - \lambda_1 = \mu_{k-1}$ for all $k \in \mathbb{N}$.*

Proof. This follows from the following formula (cf. [19, 20, 22, 29]):

$$\Delta \left(\frac{\phi_k}{\phi_1} \right) + 2 \nabla \log \phi_1 \nabla \left(\frac{\phi_k}{\phi_1} \right) = -(\lambda_k - \lambda_1) \left(\frac{\phi_k}{\phi_1} \right). \quad \square$$

Finally, throughout this paper we will use the following notations: for a function $f(t)$ and fixed $k \geq 0$,

$f(t) = O(t^k)$ as $t \rightarrow 0$ if there exists $C, \delta > 0$ such that $|f(t)| \leq Ct^k$ for all $|t| \leq \delta$;

$f(t) = o(t^k)$ as $t \rightarrow 0$ if $\lim_{t \rightarrow 0} \frac{f(t)}{t^k} = 0$.

Also, throughout this paper, a constant C is independent of ε , but may differ from line to line.

3. Eigenvalue convergence: a coarse estimate

In this section, we prove a coarse version of Theorem 1. Let (M, g, ϕ) be the compact Bakry–Émery manifold, with or without boundary, and let

$$(3.1) \quad M_\varepsilon = \{(x, y) \mid x \in M, \quad 0 \leq y \leq \varepsilon f(x)\} \subset M \times \mathbb{R}^+, \quad f(x) := e^{-\phi(x)}.$$

Let $\{\mu_k\}_{k=0}^\infty$ and $\{\psi_k\}_{k=0}^\infty$ be respectively the eigenvalues and eigenfunctions for the drift Laplacian Δ on M (if $\partial M \neq \emptyset$, we endow it the Neumann boundary condition), and let $\{\mu_k(\varepsilon)\}_{k=0}^\infty$ be the eigenvalues for $\tilde{\Delta} = \Delta + \partial_y^2$ on M_ε with corresponding orthogonal eigenfunctions $\{\varphi_{j,\varepsilon}\}_{j=0}^\infty$. We assume the eigenfunctions are normalized so that

$$\int_{M_\varepsilon} \varphi_{j,\varepsilon} \varphi_{k,\varepsilon} = \varepsilon \delta_j^k.$$

In particular, the volume of M_ε is ε . This normalization depends only on f and M .

We use ∇ and Δ as the gradient and Laplace operators, respectively, of M , and $\tilde{\nabla} = (\nabla, \frac{\partial}{\partial t})$ and $\tilde{\Delta}$ as the gradient and Laplace operators, respectively, of $M_\varepsilon \subset M \times \mathbb{R}^+$.

We prove the theorem by induction. For $k = 0$, the statement of Theorem 1 is trivial. We shall prove the theorem for $k \geq 1$, assuming that for $1, \dots, k-1$, the theorem has been proven. By a theorem of Uhlenbeck [23], for generic manifold μ_1, \dots, μ_k are simple; that is, all eigenspaces with respect to the eigenvalues μ_1, \dots, μ_k are of multiplicity one. Since the eigenvalues are continuous with respect to continuous deformations of the domain, it is sufficient to prove the theorem under this additional assumption.

Lemma 1. *Using the above notation, $\mu_k(\varepsilon) \leq \mu_k + O(\varepsilon^2)$.*

Proof. Considering ψ_k as functions on M_ε , they are orthogonal with respect to the measure $dV_g dy$. By Proposition 2, we have

$$\sum_{j=0}^k \mu_j(\varepsilon) \leq \sum_{j=0}^k \mu_j.$$

By the inductive assumption, we have

$$\mu_j \leq \mu_j(\varepsilon) + O(\varepsilon^2)$$

for all $j < k$. The lemma follows from the above two inequalities. \square

For any $0 \leq r \leq \varepsilon$, and for $0 \leq i \leq k$, let

$$b_i(x, r) := \varphi_{i, \varepsilon}(x, rf(x)) \text{ and } A_k = \sum_{j=0}^k \int_{M_\varepsilon} \left(\frac{\partial \varphi_{j, \varepsilon}}{\partial y} \right)^2(x, y).$$

Intuitively, since the domain M_ε is very thin, the eigenfunctions $\varphi_{i, \varepsilon}$ should be almost constant along the y -direction. The following lemma confirms this intuition by quantifying the heuristic argument.

Lemma 2. *Using the above notations, we have*

$$(3.2) \quad \left| \int_M b_i(x, r) b_j(x, r) f(x) - \delta_i^j \right| \leq C \sqrt{\varepsilon A_k} \quad \forall 0 \leq i, j \leq k$$

for all $0 \leq r \leq \varepsilon$.

Proof. ⁴ For any $0 \leq r \leq \varepsilon$, $0 \leq y \leq \varepsilon f(x)$, and $0 \leq i, j \leq k$,

$$(3.3) \quad \begin{aligned} |b_i(x, r) b_j(x, r) - \varphi_i(x, y) \varphi_j(x, y)| &\leq \int_0^{\varepsilon f(x)} |\partial_y (\varphi_i(x, y) \varphi_j(x, y))| dy \\ &\leq \int_0^{\varepsilon f(x)} \left(\left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) (x, y) dy. \end{aligned}$$

Note that for any $0 \leq r \leq \varepsilon$,

$$\varepsilon \int_M b_i(x, r) b_j(x, r) f(x) = \int_0^{\varepsilon f(x)} \int_M b_i(x, r) b_j(x, r) = \int_{M_\varepsilon} b_i(x, r) b_j(x, r),$$

and

$$\int_{M_\varepsilon} \varphi_i(x, y) \varphi_j(x, y) = \varepsilon \delta_i^j.$$

Then

$$\left| \varepsilon \int_M b_i(x, r) b_j(x, r) f(x) - \varepsilon \delta_i^j \right| = \left| \int_{M_\varepsilon} (b_i(x, r) b_j(x, r) - \varphi_i(x, y) \varphi_j(x, y)) \right|,$$

which by (3.3),

$$(3.4) \quad \begin{aligned} &\leq \int_{M_\varepsilon} \int_0^{\varepsilon f(x)} \left(\left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) (x, t), \\ &\leq \varepsilon \|f\|_\infty \int_{M_\varepsilon} \left(\left| \frac{\partial \varphi_i}{\partial y} \right| \cdot |\varphi_j| + \left| \frac{\partial \varphi_j}{\partial y} \right| \cdot |\varphi_i| \right) \\ &\leq \varepsilon \|f\|_\infty \left(\sqrt{A_k} \cdot \|\varphi_j\|_{L^2(M_\varepsilon)} + \sqrt{A_k} \cdot \|\varphi_i\|_{L^2(M_\varepsilon)} \right). \end{aligned}$$

Since $\|\varphi_i\|_{L^2(M_\varepsilon)} = \sqrt{\varepsilon}$,

$$\left| \varepsilon \int_M b_i(x, r) b_j(x, r) f(x) - \varepsilon \delta_i^j \right| \leq C \varepsilon^{3/2} \sqrt{A_k}.$$

□

⁴For simplicity of notation, we drop the subscript ε from φ .

Corollary 2. *Using the same notations as above, we have*

$$\left| \int_M b_i(x, r) b_j(x, r) f(x) - \delta_i^j \right| \leq C\varepsilon \quad \forall 0 \leq i, \quad j \leq k, \quad 0 \leq r \leq \varepsilon.$$

Proof. This follows from the fact that $A_k \leq \sum_{j=1}^k \|\tilde{\nabla} \varphi_j\|_{L^2(M_\varepsilon)}^2 = \sum_{j=1}^k \mu_j(\varepsilon) \varepsilon \leq C\varepsilon$, where the constant C depends only on k (not on ε). \square

The following result is a coarse version of Theorem 1.

Lemma 3. *Under the same condition as in Theorem 1, and assuming that Theorem 1 is true for $j < k$, we have*

$$\mu_k \leq \mu_k(\varepsilon) + C(\varepsilon^2 + \sqrt{\varepsilon A_k}).$$

In particular, this estimate and Lemma 1 imply that $\mu_k(\varepsilon) = \mu_k + O(\varepsilon)$ for all $k \geq 0$.

Proof. Define inductively that $\tilde{b}_0(x, r) = b_0(x, r)$,

$$\tilde{b}_k(x, r) = b_k(x, r) + \sum_{j=0}^{k-1} c_{kj}(r) b_j(x, r),$$

where for any $k \geq 0$, $c_{kj}(r)$ are functions of r such that

$$\tilde{b}_k \perp b_1, \dots, b_{k-1}$$

with respect to the measure $f dV_g$. The intuition behind defining the functions \tilde{b}_k is that the functions b_k are *almost orthogonal*, as proven in Lemma 2, but to obtain the estimate $\mu_j(\varepsilon) + O(\varepsilon^2) \leq \mu_j$ we need *orthogonal* test functions to apply Proposition 2. In the arguments below, we first apply Proposition 2 and then show that the modified functions \tilde{b}_k are very close to the original functions b_k , thereby obtaining the required estimate.

By Proposition 2, we have

$$(3.5) \quad \sum_{j=0}^k \mu_j \leq \sum_{j=0}^k \frac{\int_M |\nabla \tilde{b}_j|^2 f}{\int_M \tilde{b}_j^2 f}.$$

By Lemma 2, $c_{kj}(r) \sim O(\sqrt{\varepsilon A_k})$ uniformly for $0 < r \leq \varepsilon$. Thus by the definition of \tilde{b}_k , using Lemma 2 again, we have

$$\sum_{j=0}^k \mu_j \leq \sum_{j=0}^k \frac{\int_M |\nabla \tilde{b}_j|^2 f}{\int_M \tilde{b}_j^2 f} \leq (1 + C\sqrt{\varepsilon A_k}) \sum_{j=0}^k \int_M |\nabla b_j|^2 f.$$

Since the above inequality holds for all r , integrating from 0 to ε , we have

$$(3.6) \quad \varepsilon \sum_{j=0}^k \mu_j \leq (1 + C\sqrt{\varepsilon A_k}) \sum_{j=0}^k \int_0^\varepsilon \int_M |\nabla b_j|^2 f.$$

We compute

$$\nabla b_j(x, r) = (\nabla \varphi_j)(x, r f(x)) + r \frac{\partial \varphi_j}{\partial y}(x, r f(x)) \nabla f(x).$$

Using the Cauchy inequality, we get

$$(3.7) \quad |\nabla b_j(x, r)|^2 \leq (1 + \varepsilon^2 |\nabla f|^2) |\tilde{\nabla} \varphi_{j, \varepsilon}|^2.$$

Therefore,

$$\int_0^\varepsilon \int_M |\nabla b_j(x, r)|^2 f(x) \leq (1 + C\varepsilon^2) \int_{M_\varepsilon} |\tilde{\nabla} \varphi_{j, \varepsilon}|^2 = (1 + C\varepsilon^2) \mu_j(\varepsilon) \varepsilon$$

for $j \leq k$.

The above estimate together with (3.7) show that

$$\varepsilon \sum_{j=0}^k \mu_j \leq \varepsilon (1 + C\sqrt{\varepsilon A_k}) (1 + C\varepsilon^2) \sum_{j=0}^k \mu_j(\varepsilon).$$

Dividing by ε and letting $\varepsilon \rightarrow 0$, this estimate together with an induction argument completes the proof of the lemma. The precise estimate $\mu_k(\varepsilon) = \mu_k + O(\varepsilon^2)$ to complete the proof of Theorem 1 will be demonstrated in the final section. \square

Proof of Corollary 1. The corollary follows immediately from Lemma 3 and Proposition 3. \square

4. A maximum principle

The Neumann eigenvalues are continuous functions with respect to the manifold M . Therefore, to estimate the eigenvalues, we may use an exhaustion of M ,

$$M^\delta = \{x \in M \mid \text{dist}(x, \partial M) \geq \delta\}, \quad \delta > 0.$$

On M^δ , f has a positive lower bound. Thus using the variational principle, we may, without loss of generality assume that f is not only positive but is a constant in a neighborhood of ∂M^δ . For the rest of the paper, we make such an assumption.

The usual maximum principle for the gradient estimate is the following. Let

$$H = \frac{1}{2} |\nabla \varphi|^2 + F(\varphi),$$

where F is a smooth function of one variable, and let x_0 be an interior point of M at which H reaches its maximum. Then

$$0 \geq |\nabla^2 \varphi|^2 + \nabla \varphi \nabla (\Delta \varphi) + \text{Ric}(\nabla \varphi, \nabla \varphi) + F'(\varphi) \Delta \varphi + F''(\varphi) |\nabla \varphi|^2.$$

The above inequality is very useful for obtaining lower bounds on the first eigenvalue of a Laplace or Schrödinger operator; for more details, we refer to the book [21].

However, it is not appropriate to apply the above maximum principle directly to the manifold M_ε for the following reasons:

- (1) M_ε need not be convex, even if M is. As we know, if M is convex, the maximum of H must be reached in the interior of M . In general, we do not have such a property for M_ε .
- (2) The natural Ricci curvature attached to the problem is Ric_∞ , not the Ricci curvature of M_ε , which is essentially Ric .

Remark 2. The choice of F is highly technical. In the Li–Yau’s case [15], which is the simplest case, $F(x) = \frac{1}{2}x^2$. In Zhong–Yang’s case [28], F is (up to a constant)

$$F(x) = 1 - x^2 + a \left(\frac{4}{\pi} \left(\arcsin x + x\sqrt{1-x^2} - 2x \right) \right),$$

where a is a positive constant. More sophisticated choices of F can be found in [16] and [17].

As in the previous sections, we assume M is a compact manifold with or without boundary. Let U be an open set of M and let (x_1, \dots, x_n) be a local coordinates system on U . Let $\varphi_{k,\varepsilon}$ be the Neumann eigenfunctions of the eigenvalues $\mu_k(\varepsilon)$ with the L^2 norm normalized to be $\sqrt{\varepsilon}$. We let

$$\psi(x) = \varphi_{k,\varepsilon}(x, 0), \quad x \in M.$$

The technical heart of this paper is Theorem 5, which implies the following key results of this section.

Lemma 4. *With the above notation, as $\varepsilon \rightarrow 0$,*

$$(4.1) \quad \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} - \nabla \log f(x) \nabla \psi = o(1),$$

$$(4.2) \quad \left(\nabla \psi, \nabla \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right) - (\nabla^2 \log f)(\nabla \psi, \nabla \psi) - \nabla^2 \psi (\nabla \psi, \nabla \log f) = o(1).$$

Proof. Since $\varphi_{k,\varepsilon}$ satisfies the Neumann condition, we have

$$(4.3) \quad \begin{aligned} \frac{\partial \varphi_{k,\varepsilon}}{\partial y}(x, 0) &= 0; \\ \frac{\partial \varphi_{k,\varepsilon}}{\partial y}(x, \varepsilon f(x)) - \varepsilon \nabla f(x) \nabla \varphi_{k,\varepsilon}(x, \varepsilon f(x)) &= 0, \end{aligned}$$

for any $x \in M$. Applying the mean value theorem to the above equations, we have

$$\varepsilon f(x) \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, \xi(x)) - \nabla \log f(x) \nabla \varphi_{k,\varepsilon}(x, \varepsilon f(x)) \right) = 0,$$

where $\xi(x) \in (0, \varepsilon f(x))$. Theorem 5 then implies (4.1).

Assume now that at x , the local coordinate system is normal. For the second statement, taking partial derivatives with respect to x_j in the second equation of (4.3) gives

$$\begin{aligned} & \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, \varepsilon f(x)) + \varepsilon \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, \varepsilon f(x)) \frac{\partial f}{\partial x_j} \\ & - \varepsilon \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, \varepsilon f(x)) - \varepsilon \frac{\partial f(x)}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j}(x, \varepsilon f(x)) \\ & - \varepsilon^2 \frac{\partial f}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y}(x, \varepsilon f(x)) \frac{\partial f}{\partial x_j} = 0. \end{aligned}$$

Since $\partial \varphi_{k,\varepsilon} / \partial y = 0$ on $\{y = 0\}$, we have

$$\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, 0) = 0, \quad \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial x_i \partial y}(x, 0) = 0.$$

The mean value theorem implies

$$\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y}(x, \varepsilon f(x)) = \varepsilon f(x) \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, \xi(x))$$

for some $\xi(x) \in (0, \varepsilon f(x))$. Using Theorem 5 again, we have

$$\begin{aligned} & \varepsilon f(x) \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, 0) + \varepsilon \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, 0) \frac{\partial f}{\partial x_j} - \varepsilon \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, 0) \\ & - \varepsilon \frac{\partial f(x)}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j}(x, 0) - \varepsilon^2 \frac{\partial f}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y}(x, 0) \frac{\partial f}{\partial x_j} = o(\varepsilon f(x)) \end{aligned}$$

as $\varepsilon \rightarrow 0$. Thus, we have

$$\begin{aligned} & \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, 0) + \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}(x, 0) \frac{\partial \log f}{\partial x_j} - \frac{1}{f} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i}(x, 0) \\ & - \frac{\partial \log f(x)}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j}(x, 0) = o(1). \end{aligned}$$

Using (4.1), we get

$$\begin{aligned} & \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2}(x, 0) + \nabla \log f(x) \nabla \psi(x) \frac{\partial \log f(x)}{\partial x_j} - \frac{1}{f(x)} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \frac{\partial \psi(x)}{\partial x_i} \\ & - \frac{\partial \log f(x)}{\partial x_i} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) = o(1), \end{aligned}$$

which implies (4.2). □

For our new maximum principle, we consider

$$H(x, y) = \frac{1}{2} |\tilde{\nabla} \varphi_{k,\varepsilon}|^2 + F(\varphi_{k,\varepsilon}),$$

where F is a smooth function of one variable. Assume that $(x_0, 0)$ is the point at which H reaches the maximum on $\{y = 0\}$, where x_0 is in the interior of M .

At $(x_0, 0)$, we have

$$\nabla H(x_0, 0) = 0 \text{ and } \Delta H(x_0, 0) \leq 0.$$

The difficulty is that H satisfies an elliptic equation with respect to $\tilde{\Delta}$, rather than Δ . To obtain the new maximum principle, we need to estimate the second derivative of H in the y -direction.

Lemma 5. *At $(x_0, 0)$,*

$$\frac{\partial^2 H}{\partial y^2} = \nabla^2 \log f(\nabla \psi, \nabla \psi) + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

Proof. Using the normal coordinates at x_0 , we have

$$\frac{\partial H}{\partial y} = \sum_{i=1}^n \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y} + \sum_{i=1}^n \frac{\partial \varphi_{k,\varepsilon}}{\partial y} \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} + F'(\varphi_{k,\varepsilon}) \frac{\partial \varphi_{k,\varepsilon}}{\partial y},$$

and

$$\begin{aligned} \frac{\partial^2 H}{\partial y^2} &= \sum_{i=1}^n \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y} \right)^2 + \sum_{i=1}^n \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i} \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_i \partial y^2} \\ &\quad + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + \frac{\partial \varphi_{k,\varepsilon}}{\partial y} \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial y^3} + F''(\varphi_{k,\varepsilon}) \left(\frac{\partial \varphi_{k,\varepsilon}}{\partial y} \right)^2 + F'(\varphi_{k,\varepsilon}) \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}. \end{aligned}$$

Since $\varphi_{k,\varepsilon}$ satisfies the Neumann boundary condition, $\frac{\partial \varphi_{k,\varepsilon}}{\partial y}$ and $\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_i \partial y}$ vanish on $\{y = 0\}$. Thus we have

$$\frac{\partial^2 H}{\partial y^2} = \sum_{i=1}^n \frac{\partial \varphi_{k,\varepsilon}}{\partial x_i} \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_i \partial y^2} + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + F'(\varphi_{k,\varepsilon}) \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}.$$

Using Lemma 4, we have

$$\frac{\partial^2 H}{\partial y^2} = \nabla^2 \log f(\nabla \psi, \nabla \psi) + \nabla^2 \psi(\nabla \psi, \nabla \log f) + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + F'(\psi) \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} + o(1).$$

Since at x_0 , $\nabla H = 0$, we have

$$\sum_{j=1}^n \frac{\partial^2 \psi}{\partial x_i \partial x_j} \frac{\partial \psi}{\partial x_j} + F'(\psi) \frac{\partial \psi}{\partial x_i} = 0, \quad \text{for each } 1 \leq i \leq n.$$

Using the above equality and Lemma 4, the second and fourth terms on the right side of the expression for $\frac{\partial^2 H}{\partial y^2}$ cancel. \square

Theorem 2 (Maximum Principle). *With the above notations, we have at $(x_0, 0)$*

$$o(1) \geq |\nabla^2 \psi|^2 + \nabla \psi \nabla(\tilde{\Delta} \varphi_{k,\varepsilon}) + \text{Ric}_\infty(\nabla \psi, \nabla \psi) + F'(\psi) \tilde{\Delta} \varphi_{k,\varepsilon} + F''(\psi) |\nabla \psi|^2.$$

Proof. By the Bochner formula, we have

$$\begin{aligned} \tilde{\Delta} H &= |\tilde{\nabla}^2 \varphi_{k,\varepsilon}|^2 + \text{Ric}(\tilde{\nabla} \varphi_{k,\varepsilon}, \tilde{\nabla} \varphi_{k,\varepsilon}) + \tilde{\nabla} \varphi_{k,\varepsilon} \tilde{\nabla}(\tilde{\Delta} \varphi_{k,\varepsilon}) \\ &\quad + F'(\varphi_{k,\varepsilon}) \tilde{\Delta} \varphi_{k,\varepsilon} + F''(\varphi_{k,\varepsilon}) |\tilde{\nabla} \varphi_{k,\varepsilon}|^2. \end{aligned}$$

On $\{y = 0\}$, we have

$$\begin{aligned} |\tilde{\nabla}^2 \varphi_{k,\varepsilon}|^2 &= |\nabla^2 \psi|^2 + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2, \\ \text{Ric}(\tilde{\nabla} \varphi_{k,\varepsilon}, \tilde{\nabla} \varphi_{k,\varepsilon}) &= \text{Ric}(\nabla \psi, \nabla \psi), \\ \tilde{\Delta} \varphi_{k,\varepsilon} &= \Delta \psi + \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2}, \end{aligned}$$

Thus, we have

$$\begin{aligned} \tilde{\Delta} H &= |\nabla^2 \psi|^2 + \left(\frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right)^2 + \text{Ric}(\nabla \psi, \nabla \psi) \\ &\quad + \nabla \psi \nabla(\tilde{\Delta} \varphi_{k,\varepsilon}) + F'(\psi) \tilde{\Delta} \varphi_{k,\varepsilon} + F''(\psi) |\nabla \psi|^2. \end{aligned}$$

Using Lemma 5, noting that at $(x_0, 0)$

$$\tilde{\Delta}H = \Delta H + \frac{\partial^2 H}{\partial y^2} \leq \frac{\partial^2 H}{\partial y^2},$$

completes the proof. \square

4.1. Applications. Our work not only has applications to Bakry–Émery geometry but also to Ricci solitons. We recall the main result of Futaki and Sano [9].

Theorem 3 (Futaki–Sano). *Let M^n be a compact smooth manifold of dimension at least 4. If g is a non-trivial gradient shrinking Ricci soliton on M (see definition 1.1 of [9]), then the diameter of M with respect to g is bounded below by $\frac{10\pi}{13\sqrt{\gamma}}$, where γ is a constant determined by g .*

This result is proven by using Ling’s gradient estimates [16] to demonstrate a lower bound for the first non-zero eigenvalue of a certain Bakry–Émery Laplacian. Our **Principle** shows that one may directly apply Ling’s estimates to the Bakry–Émery Laplacian to obtain the result. It is reasonable to expect that one may similarly express elliptic geometric equations, like the Ricci soliton equation, in terms of a Bakry–Émery Laplacian and exploit the eigenvalue estimates from Riemannian geometry together with our **Principle** to produce interesting results.

Another application arises from the so-called *fundamental gap*: the difference between the first two Dirichlet eigenvalues of a domain in \mathbb{R}^n . Andrews and Clutterbuck [2] recently demonstrated an optimal lower bound of $3\pi^2/d^2$ for the fundamental gap of any convex domain in \mathbb{R}^n with diameter d . By Proposition 3, the fundamental gap can be interpreted as the first Neumann eigenvalue on certain Bakry–Émery manifold, and in particular, techniques of [1, 2] together with our work imply the following.

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with piecewise smooth boundary and diameter d . Let $f \in C^2(\Omega)$. If f satisfies*

$$(\nabla \log f(y) - \nabla \log f(x)) \cdot \frac{y-x}{|y-x|} \geq \frac{4\pi}{d} \tan\left(\frac{\pi|y-x|}{d}\right) \quad \forall \quad x \neq y \text{ in } \Omega,$$

then the first non-trivial Neumann eigenvalue of the Bakry–Émery Laplacian with respect to the weight function $\phi = -\log(f^2)$ is bounded below by $3\pi^2/d^2$. Moreover, the first Neumann eigenfunction for the Euclidean Laplacian on

$$\Omega_\varepsilon := \{(x, y) \mid x \in \Omega, 0 \leq y \leq \varepsilon f^2(x)\} \subset \mathbb{R}^{n+1}$$

is bounded below by $3\pi^2/d^2 - C\varepsilon^2$, where C is a fixed constant that depends only on n and Ω .

5. The approximation of eigenfunctions

It is not hard to write down the eigenfunctions formally. Let φ be a Neumann eigenfunction of M_ε with eigenvalue λ . Write

$$\varphi = \sum_{k=0}^{\infty} y^k \varphi_k,$$

where φ_k are functions on M . Then we have (formally)

$$\Delta\varphi_k + \lambda\varphi_k + (k+1)(k+2)\varphi_{k+2} = 0$$

for all $k \geq 0$. Since $\partial\varphi/\partial y = 0$ on $\{y=0\}$, we have $\varphi_1 = 0$ and hence $\varphi_{2k+1} = 0$ for all k . Let

$$H\varphi = -\Delta\varphi - \lambda\varphi.$$

Then

$$\varphi_{2k+2} = \frac{H\varphi_{2k}}{(2k+1)(2k+2)} = \frac{H^{k+1}\varphi_0}{(2k+2)!}.$$

Formally, we have

$$\varphi = \sum_{k=0}^{\infty} \frac{y^{2k} H^k}{(2k)!} \varphi_0 = \cosh(y\sqrt{H}) \varphi_0.$$

The differential equation for φ_0 follows from the Neumann boundary condition

$$\sqrt{H} \sinh(\varepsilon f(x)\sqrt{H}) \varphi_0 - \varepsilon \nabla f \cdot \nabla \left(\cosh(y\sqrt{H}) \varphi_0 \Big|_{y=\varepsilon f(x)} \right) = 0.$$

We are not able to prove the full regularity of the above equation at this moment. But a partial solution, namely, a good approximation to the eigenfunctions, is enough for our application. Very roughly speaking, in this section, we prove

$$\varphi = \varphi_0 + y^2 \varphi_2 + O(\varepsilon^3).$$

To state our results precisely, we recall the global Schauder estimates [12]*Theorem 6.6, Theorem 6.30 and the interpolation inequalities.

We let

$$\begin{aligned} B_{\text{I}} &= \{(x, y) \in M_\varepsilon \mid y = 0\}; \\ B_{\text{II}} &= \{(x, y) \in M_\varepsilon \mid y = \varepsilon f(x)\}; \\ B_{\text{III}} &= \{(x, y) \in M_\varepsilon \mid x \in \partial M\}. \end{aligned}$$

Then $B_{\text{I}} \cup B_{\text{II}} \cup B_{\text{III}} = \partial M_\varepsilon$.

Let

$$u_1 = u|_{\partial M_\varepsilon} \text{ and } u_2 = \frac{\partial u}{\partial n} \Big|_{\partial M_\varepsilon} \text{ on the smooth part of } \partial M_\varepsilon.$$

Define the weighted Hölder norm by

$$\|u\|_{C_\varepsilon^{k,\alpha}} = \varepsilon^{k+\alpha} [u]_{C^{k,\alpha}} + \cdots + \varepsilon^\alpha [u]_{C^\alpha} + \|u\|_{C^0},$$

and $\|u\|_{C_\varepsilon^\alpha} = \|u\|_{C_\varepsilon^{0,\alpha}}$, where $[\]_{C^{k,\alpha}}$ are the standard notations defined in [12]. Using these weighted norms, the constants in the Schauder estimates on M_ε are independent of ε . Let $0 < \alpha < 1$. Let L be a second order uniform elliptic operator with C^α -bounded coefficients. Then we have the following version of global Schauder estimates on M_ε

$$(5.1) \quad \|u\|_{C_\varepsilon^{2,\alpha}} \leq C(\|u\|_{C^0} + \varepsilon^2 \|Lu\|_{C_\varepsilon^\alpha} + \varepsilon \|u_2\|_{C_\varepsilon^{1,\alpha}}),$$

and

$$(5.2) \quad \|u\|_{C_\varepsilon^{2,\alpha}} \leq C(\|u\|_{C^0} + \varepsilon^2 \|Lu\|_{C_\varepsilon^\alpha} + \varepsilon \|u_2\|_{C_\varepsilon^{1,\alpha}(B_{\text{III}})} + \|u_1\|_{C_\varepsilon^{2,\alpha}(B_{\text{I}} \cup B_{\text{II}})}).$$

The Sobolev inequality on M_ε is

$$(5.3) \quad \left(\int_{M_\varepsilon} |u|^{2\frac{n+1}{n-1}} \right)^{\frac{n-1}{n+1}} \leq C\varepsilon^{-\frac{2}{n+1}} \left(\int_{M_\varepsilon} |\nabla u|^2 + \int_{M_\varepsilon} |u|^2 \right).$$

Define the Hölder norm in the y -direction to be

$$[u]_{C_y^\alpha} = \max_{x \in M} \sup_{0 \leq y_1, y_2 \leq \varepsilon f(x)} \frac{|u(x, y_1) - u(x, y_2)|}{|y_1 - y_2|^\alpha}.$$

Then we have the following.

Theorem 5. *For M_ε defined by (3.1) such that $f(x) = e^{-\phi(x)}$ is constant in a neighborhood of ∂M , the Neumann eigenfunctions $\varphi_{k,\varepsilon}$ of M_ε satisfy*

$$\begin{aligned} \nabla \varphi_{k,\varepsilon} &= O(1), \\ \left\| \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial y^2} \right\|_{C_y^\alpha} + \left\| \frac{\partial^2 \varphi_{k,\varepsilon}}{\partial x_j \partial y} \right\|_{C_y^\alpha} &= O(1), \quad 1 \leq j \leq n \\ \left\| \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_j \partial y^2} \right\|_{C_y^\alpha} + \left\| \frac{\partial^3 \varphi_{k,\varepsilon}}{\partial x_i \partial x_j \partial y} \right\|_{C_y^\alpha} &= O(1), \quad 1 \leq i, j \leq n \end{aligned}$$

for any $0 < \alpha < 1$, where (x_1, \dots, x_n) is any local coordinate system of M .

As defined in Section 3, let $\{\mu_k\}_{k=0}^\infty$ and $\{\psi_k\}_{k=0}^\infty$ be respectively the eigenvalues and eigenfunctions for the drift Laplacian. Moreover, we assume that $\{\psi_k\}_{k=0}^\infty$ is an orthonormal basis for the space of L^2 functions of M with respect to the weighted measure $e^{-\phi} dV_g$.

Define the functions

$$\eta_k := -\nabla \phi \nabla \psi_k,$$

on M , and let

$$U_k := \psi_k + \frac{1}{2} y^2 \eta_k$$

be functions on M_ε . By our definition of $f(x) = e^{-\phi(x)}$, η_k are smooth up to the boundary. Note that since ψ_k is a Bakry–Émery eigenfunction, it satisfies

$$(5.4) \quad \Delta \psi_k + \eta_k = -\mu_k \psi_k.$$

Since ψ_k and η_k are independent of y ,

$$(5.5) \quad \tilde{\Delta} U_k = -\mu_k \psi_k + \frac{1}{2} y^2 \Delta \eta_k = -\mu_k U_k + \frac{1}{2} y^2 (\Delta \eta_k + \mu_k \eta_k).$$

We compute directly,

$$(5.6) \quad \left. \frac{\partial U_k}{\partial n} \right|_{\partial M_\varepsilon} = \begin{cases} 0 & \text{on } B_I \cup B_{III}; \\ -\frac{\varepsilon^3 f^2 \nabla f \nabla \eta_k}{2(1 + \varepsilon^2 |\nabla f|^2)^{1/2}} & \text{on } B_{II}. \end{cases}$$

Define

$$w_{k,\varepsilon} = \alpha_{k,k} \varphi_{k,\varepsilon} + U_k + \sum_{j=0}^{k-1} \alpha_{k,j} \varphi_{j,\varepsilon},$$

where $\alpha_{k,j}$ are defined such that $w_{k,\varepsilon} \perp \varphi_{0,\varepsilon}, \dots, \varphi_{k,\varepsilon}$ in $\mathcal{L}^2(M_\varepsilon)$.

Heuristically, by (5.5) and (5.6), U_k should be very close to the eigenfunction $\varphi_{k,\varepsilon}$, and the function $w_{k,\varepsilon}$ should be approximately the difference between U_k and $\varphi_{k,\varepsilon}$. In what follows, we prove that $w_{k,\varepsilon}$ and its y derivative (denoted $r_{k,\varepsilon}$ in (5.12) below) are indeed small, thereby rigorously demonstrating our intuitive heuristic.

The following inequality will be used repeatedly in the rest of the paper: let φ be a function on M_ε . Then for any $p > 0$, we have

$$(5.7) \quad \int_{B_{II}} |\varphi|^p \leq C \left(\frac{1}{\varepsilon} \int_{M_\varepsilon} |\varphi|^p + p \sqrt{\int_{M_\varepsilon} |\varphi|^{p-1}} \cdot \sqrt{\int_{M_\varepsilon} |\varphi|^{p-1} |\nabla \varphi|^2} \right).$$

To prove (5.7), we observe that for any $0 \leq y \leq \varepsilon f(x)$, we have

$$|\varphi|^p(x, \varepsilon f(x)) \leq |\varphi|^p(x, y) + p \int_0^{\varepsilon f(x)} |\varphi|^{p-1} |\nabla \varphi| dy.$$

Integrating over M_ε to both sides of the above equation and using the Cauchy–Schwarz inequality implies (5.7).

Lemma 6. *For all $j < k$, $\alpha_{k,j} = O(\varepsilon^2)$, and $\alpha_{k,k} = O(1)$.*

Proof. Integrating by parts, (5.5) and (5.6) give

$$\begin{aligned} \mu_j(\varepsilon) \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k &= - \int_{M_\varepsilon} \tilde{\Delta} \varphi_{j,\varepsilon} U_k = \int_{\partial M_\varepsilon} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} - \int_{M_\varepsilon} \varphi_{j,\varepsilon} \tilde{\Delta} U_k \\ &= \int_{\partial M_\varepsilon} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} + \mu_k \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k - \frac{1}{2} \int_{M_\varepsilon} \varphi_{j,\varepsilon} y^2 (\Delta \eta_k + \mu_k \eta_k). \end{aligned}$$

Thus by (5.6) again, we have

$$(5.8) \quad (\mu_j(\varepsilon) - \mu_k) \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k = - \int_{B_{II}} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} - \frac{1}{2} \int_{M_\varepsilon} \varphi_{j,\varepsilon} y^2 (\Delta \eta_k + \mu_k \eta_k).$$

We clearly have

$$(5.9) \quad \left| \int_{M_\varepsilon} \varphi_{j,\varepsilon} y^2 (\Delta \eta_k + \mu_k \eta_k) \right| \leq C \sqrt{\int_{M_\varepsilon} y^4} \cdot \sqrt{\int_{M_\varepsilon} |\varphi_{j,\varepsilon}|^2} = O(\varepsilon^3).$$

Using (5.7) for $p = 1$, we have

$$\left| \int_{B_{II}} \varphi_{j,\varepsilon} \frac{\partial U_k}{\partial n} \right| \leq C \varepsilon^3.$$

By the generic assumption of the manifold M and Lemma 3, $\mu_j(\varepsilon) = \mu_j + O(\varepsilon) < \mu_k$ for all $j < k$ for ε sufficiently small. Thus, dividing by $(\mu_j(\varepsilon) - \mu_k)$ in (5.8) gives

$$(5.10) \quad \int_{M_\varepsilon} \varphi_{j,\varepsilon} U_k = O(\varepsilon^3) \implies \alpha_{k,j} = O(\varepsilon^2).$$

That $\alpha_{k,k}$ is bounded follows from its definition. \square

A straightforward computation gives

$$(5.11) \quad \begin{aligned} F_1 &:= \tilde{\Delta} w_{k,\varepsilon} + \mu_k(\varepsilon) w_{k,\varepsilon} \\ &= (\mu_k(\varepsilon) - \mu_k) U_k + \frac{1}{2} y^2 (\Delta \eta_k + \mu_k \eta_k) + \sum_{j=0}^{k-1} \alpha_{k,j} (\mu_k(\varepsilon) - \mu_j(\varepsilon)) \varphi_{j,\varepsilon} \end{aligned}$$

with the boundary conditions

$$(5.12) \quad \frac{\partial w_{k,\varepsilon}}{\partial n} = \begin{cases} 0 & \text{on } B_I \cup B_{III}, \\ \frac{\partial U_k}{\partial n} = O(\varepsilon^3) & \text{on } B_{II}. \end{cases}$$

Let

$$r_{k,\varepsilon} = \frac{\partial w_{k,\varepsilon}}{\partial y}.$$

Then

$$(5.13) \quad \begin{aligned} F_2 &:= \tilde{\Delta} r_{k,\varepsilon} + \mu_k(\varepsilon) r_{k,\varepsilon} \\ &= (\mu_k(\varepsilon) - \mu_k) y \eta_k + y (\Delta \eta_k + \mu_k \eta_k) + \sum_{j=0}^{k-1} \alpha_{k,j} (\mu_k(\varepsilon) - \mu_j(\varepsilon)) \frac{\partial \varphi_{j,\varepsilon}}{\partial y} \end{aligned}$$

with the boundary conditions

$$(5.14) \quad \begin{cases} r_{k,\varepsilon} = 0 & \text{on } B_I, \\ r_{k,\varepsilon} = \varepsilon \nabla f(x) \nabla w_{k,\varepsilon}(x, \varepsilon f(x)) + O(\varepsilon^3) & \text{on } B_{II}, \\ \frac{\partial r_{k,\varepsilon}}{\partial n} = 0 & \text{on } B_{III}. \end{cases}$$

Inductively, we assume that the Theorem 5 is true for $j \leq k-1$. Then by Lemma 3 and Lemma 6, we have

$$(5.15) \quad \|F_1\|_{C_\varepsilon^\alpha} = O(B_k) \text{ and } \|F_2\|_{C_\varepsilon^\alpha} = O(\varepsilon),$$

where

$$B_k = \varepsilon^2 + \sqrt{\varepsilon A_k}.$$

Lemma 7. *With the above notations, we have*

$$\|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} = O(\varepsilon^{1/2} B_k) \text{ and } \|\tilde{\nabla} w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} = O(\varepsilon^{1/2} B_k).$$

Proof. Multiplying both sides of (5.11) by $w_{k,\varepsilon}$ and integrating by parts, using (5.15), we get

$$(5.16) \quad \begin{aligned} &\|\nabla w_{k,\varepsilon}\|_{L^2(M_\varepsilon)}^2 - \mu_k(\varepsilon) \|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)}^2 \\ &\leq C B_k \cdot \|w_{k,\varepsilon}\|_{L^1(M_\varepsilon)} + \left| \int_{B_{II}} w_{k,\varepsilon} \frac{\partial w_{k,\varepsilon}}{\partial n} \right|. \end{aligned}$$

By (5.7), we have

$$\left| \int_{B_{II}} w_{k,\varepsilon} \frac{\partial w_{k,\varepsilon}}{\partial n} \right| = \left| \int_{B_{II}} w_{k,\varepsilon} \frac{\partial U_k}{\partial n} \right| \leq C \varepsilon^{5/2} (\|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} + \|\nabla w_{k,\varepsilon}\|_{L^2(M_\varepsilon)}).$$

Thus, we have

$$\|\nabla w\|_{L^2(M_\varepsilon)}^2 - \mu_k(\varepsilon) \|w\|_{L^2(M_\varepsilon)}^2 \leq C \varepsilon^{5/2} (\|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} + \|\nabla w_{k,\varepsilon}\|_{L^2(M_\varepsilon)}).$$

By the Poincaré inequality, we have

$$\mu_{k+1}(\varepsilon) \int_{M_\varepsilon} |w_{k,\varepsilon}|^2 \leq \int_{M_\varepsilon} |\nabla w_{k,\varepsilon}|^2.$$

The lemma is proved since by our “generic” assumption, there is a gap between $\mu_{k+1}(\varepsilon)$ and $\mu_k(\varepsilon)$ that is independent of ε . \square

Using the above lemma we shall complete the proof of Theorem 1.

Proof of Theorem 1. Since $B_k = O(\varepsilon)$, by the above lemma, $\|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} = O(\varepsilon^{3/2})$. It follows that

$$A_k = \sum_{j=1}^k \int_{M_\varepsilon} \left| \frac{\partial \varphi_{k,\varepsilon}}{\partial y} \right|^2 = O(\varepsilon^3),$$

and thus Theorem 1 follows from Lemma 3. \square

Corollary 3. *With the above notations, we have*

$$\|w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} = O(\varepsilon^{5/2}) \quad \text{and} \quad \|\tilde{\nabla} w_{k,\varepsilon}\|_{L^2(M_\varepsilon)} = O(\varepsilon^{5/2}). \quad \square$$

Now we work towards the proof of Theorem 5.

Lemma 8. $w_{k,\varepsilon} = O(\varepsilon^2)$

Proof. We prove the lemma using Moser iteration. By (5.15) and Theorem 1, $F_1 = O(\varepsilon^2)$ is a bounded function. Using integration by parts, we get

$$p \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} |\nabla w_{k,\varepsilon}|^2 \leq \int_{M_\varepsilon} |F_1| |w_{k,\varepsilon}|^p + \mu_k(\varepsilon) \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} + \int_{B_{II}} |w_{k,\varepsilon}|^p \left| \frac{\partial w_{k,\varepsilon}}{\partial n} \right|.$$

By (5.6), we have

$$\int_{B_{II}} |w_{k,\varepsilon}|^p \left| \frac{\partial w_{k,\varepsilon}}{\partial n} \right| \leq C\varepsilon^3 \int_{B_{II}} |w_{k,\varepsilon}|^p.$$

By (5.7), we have

$$\int_{B_{II}} |w_{k,\varepsilon}|^p \leq \left(\frac{1}{\varepsilon} \int_{M_\varepsilon} |w_{k,\varepsilon}|^p + p \sqrt{\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1}} \cdot \sqrt{\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} |\nabla w_{k,\varepsilon}|^2} \right).$$

It follows that

$$\int_{B_{II}} |w_{k,\varepsilon}|^p \left| \frac{\partial w_{k,\varepsilon}}{\partial n} \right| \leq \frac{1}{4} p \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} |\nabla w_{k,\varepsilon}|^2 + \varepsilon^6 p \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} + \varepsilon^2 \int_{M_\varepsilon} |w_{k,\varepsilon}|^p.$$

By the Young’s inequality, we have

$$\varepsilon^2 |w_{k,\varepsilon}|^p + \varepsilon^6 |w_{k,\varepsilon}|^{p-1} \leq |w_{k,\varepsilon}|^{p+1} + \varepsilon^{2(p+1)}.$$

Thus, we have

$$\int_{B_{II}} |w_{k,\varepsilon}|^p \left| \frac{\partial w_{k,\varepsilon}}{\partial n} \right| \leq \frac{p}{4} \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} |\nabla w_{k,\varepsilon}|^2 + Cp \left(\left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} \right) + \varepsilon^{2(p+1)+1} \right),$$

from which we have

$$p \int_{M_\varepsilon} |w_{k,\varepsilon}|^{p-1} |\nabla w_{k,\varepsilon}|^2 \leq Cp \left(\left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} \right) + \varepsilon^{2(p+1)+1} \right).$$

Using the above inequality and the Sobolev inequality (5.3), we have

$$\left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{(p+1)\frac{n+1}{n-1}} \right)^{\frac{n-1}{n+1}} \leq Cp^2 \varepsilon^{-\frac{2}{n+1}} \left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} + \varepsilon^{2(p+1)+1} \right).$$

We thus have

$$\left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{(p+1)\frac{n+1}{n-1}} + \varepsilon^{2(p+1)\frac{n+1}{n-1}+1} \right)^{\frac{n-1}{n+1}} \leq Cp^2 \varepsilon^{-\frac{2}{n+1}} \left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{p+1} + \varepsilon^{2(p+1)+1} \right).$$

Let

$$a_k = 2 \left(\frac{n+1}{n-1} \right)^k$$

for $k \geq 0$. Let

$$b_k = \left(\int_{M_\varepsilon} |w_{k,\varepsilon}|^{a_k} + \varepsilon^{2(a_k)+1} \right)^{1/a_k}.$$

Then we have

$$b_{k+1} \leq (C(a_k)^2 \varepsilon^{-\frac{2}{n+1}})^{1/a_k} b_k.$$

The standard iteration process shows that

$$\|w_{k,\varepsilon}\|_{C^0} \leq C \varepsilon^{-\frac{1}{n+1} \sum 1/a_k} b_0 \leq C \varepsilon^2.$$

□

Lemma 9. $r_{k,\varepsilon} = O(\varepsilon^2)$.

Proof. We could run Moser iteration again to get the estimate. However, the following proof using the maximum principle seems to be simpler. Using (5.1), (5.15), and the boundary conditions (5.14), we have

$$\|w_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} \leq \varepsilon^2,$$

which implies that

$$|\tilde{\nabla}^2 w_{k,\varepsilon}| \leq C, \quad |\tilde{\nabla} w_{k,\varepsilon}| \leq C\varepsilon$$

by the interpolation inequalities. For $C > 0$ large, by (5.15), we have

$$\tilde{\Delta}(r_{k,\varepsilon} + Cy^2) > 0.$$

Since $r_{k,\varepsilon} = 0$ on B_I , $\frac{\partial r_{k,\varepsilon}}{\partial n} = 0$ on B_{III} . By the maximum principle, we only need to estimate the maximum value of $r_{k,\varepsilon}$ on B_{II} . But on B_{II} , $r_{k,\varepsilon} = \varepsilon \nabla f \nabla w_{k,\varepsilon} + O(\varepsilon^3) = O(\varepsilon^2)$. Thus, we have $r_{k,\varepsilon} \leq C\varepsilon^2$. Similarly estimating $r_{k,\varepsilon} - Cy^2$ gives the other side of the inequality. □

Lemma 10. $\|r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} = O(\varepsilon^2)$, and $|\tilde{\nabla}^2 w_{k,\varepsilon}| = O(\varepsilon)$.

Proof. For fixed y , let $w = w_{k,\varepsilon}$ and let $h = w(x, yf(x))$. Then h satisfies the equation

$$(5.17) \quad \begin{aligned} & \Delta h + \mu_k(\varepsilon)h \\ &= - (1 - y^2 |\nabla f|^2) \frac{\partial r_{k,\varepsilon}}{\partial y} + 2y \langle \nabla f, \nabla r_{k,\varepsilon} \rangle + yr_{k,\varepsilon} \Delta f + F_1(x, yf(x)). \end{aligned}$$

Let $\Omega \subset M$ such that on $M \setminus \Omega$, f is identically equal to a positive constant δ . By the Schauder interior estimate, we have

$$\|h\|_{C^{3,\alpha}(\Omega)} \leq C(\varepsilon^2 + \|r_{k,\varepsilon}\|_{C^{2,\alpha}}).$$

Note that the above $C^{2,\alpha}$ -norm is a function of y , and this norm is unscaled. By Lemma 9, (5.13) and (5.15) the global Schauder estimate (5.2) gives

$$\|r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} \leq C(\varepsilon^2 + \|\varepsilon \nabla f \nabla h\|_{C_\varepsilon^{2,\alpha}}) \leq C(\varepsilon^2 + \|h\|_{C_\varepsilon^{3,\alpha}(\Omega)}).$$

The relation between weighted and the usual Hölder norms is (up to a constant)

$$\varepsilon^{k+\alpha} \|u\|_{C^{k,\alpha}} \leq \|u\|_{C_\varepsilon^{k,\alpha}} \leq \varepsilon^{k+\alpha} \|u\|_{C^{k,\alpha}} + \|u\|_{C^0}.$$

Thus, we have

$$\begin{aligned} \|r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} &\leq C(\varepsilon^2 + \varepsilon^{3+\alpha} \|h\|_{C^{3,\alpha}(\Omega)} + \|h\|_{C^0(\Omega)}) \\ &\leq C(\varepsilon^2 + \varepsilon^{3+\alpha}(\varepsilon^2 + \|r_{k,\varepsilon}\|_{C^{2,\alpha}})) \leq C(\varepsilon^2 + \varepsilon \|r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}}). \end{aligned}$$

Therefore, $\|r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} = O(\varepsilon^2)$, which also implies that

$$\frac{\partial^2 w_{k,\varepsilon}}{\partial y^2}, \quad \frac{\partial^2 w_{k,\varepsilon}}{\partial x_j \partial y} = O(\varepsilon),$$

where (x_1, \dots, x_n) is any local coordinate system on M . Using the global Schauder estimate on (5.17) again, we get

$$\frac{\partial^2 w_{k,\varepsilon}}{\partial x_j \partial x_i} = O(\varepsilon). \quad \square$$

Lemma 11. $|\nabla r_{k,\varepsilon}| = O(\varepsilon^2)$.

Proof. We need to prove that for any first-order differential operator R on M ,

$$R(r_{k,\varepsilon}) = O(\varepsilon^2)$$

uniformly for any $0 \leq y \leq \varepsilon$. By Lemma 10, this is equivalent to

$$v = R(r_{k,\varepsilon}(x, yf(x))) = O(\varepsilon^2)$$

uniformly for any $0 \leq y \leq \varepsilon$.

We first assume that the vector field R is vertical to ∂M . Then by Lemma 10, $v = 0$ on B_I , and

$$(5.18) \quad v = R(\varepsilon \nabla f(x) \nabla w_{k,\varepsilon}(x, \varepsilon f(x))) = O(\varepsilon^2) \text{ on } B_{II}.$$

On B_{III} , since R is vertical to ∂M , $v = g \frac{\partial}{\partial n}$ on ∂M for some function g . Thus by Lemma 10 again,

$$(5.19) \quad v = g \frac{\partial}{\partial n} (r_{k,\varepsilon}(x, yf(x))) = O(\varepsilon^2).$$

By (5.13) and Lemma 10, we have

$$(5.20) \quad \tilde{\Delta}R(r_{k,\varepsilon}) + \mu_k(\varepsilon)R(r_{k,\varepsilon}) = R(F_2) + [\Delta, R]r_{k,\varepsilon} = O(1).$$

Thus for $C > 0$ large enough, we have

$$\tilde{\Delta}(R(r_{k,\varepsilon}) + Cy^2) > 0,$$

and by the maximum principle, $R(r_{k,\varepsilon}) \leq O(\varepsilon^2)$. Like in the proof of Lemma 9, the other side of the inequality can be obtained by estimating $R(r_{k,\varepsilon}) - Cy^2$.

Now we assume that R is tangential on ∂M . We have similar estimates on B_I and B_{II} as above. On B_{III} , we note that

$$(5.21) \quad \frac{\partial}{\partial n}(R(r_{k,\varepsilon}(x, yf(x)))) = \tilde{R}(r_{k,\varepsilon}(x, yf(x))) + O(\varepsilon^2),$$

where $\tilde{R} = [\frac{\partial}{\partial n}, R]$. Let ξ be a function on M such that on ∂M , $\xi = 0$; $\partial\xi/\partial n = 1$; and on M , $\varepsilon\Delta\xi$ bounded; $\xi = O(\varepsilon)$. We construct such a function ξ as follows: let $\rho(x)$ be a cut-off function on \mathbb{R} such that $\rho \geq 0$; $\rho'(0) = 0$ and $\rho(x) = 0$ for $x \geq \varepsilon$; $\varepsilon|\rho_1|$ and $\varepsilon^2|\rho_1''|$ are bounded. Let $\xi = -d\rho(d)$, where d is the distance function to the boundary ∂M . Let

$$\tilde{v} = R(r_{k,\varepsilon}(x, yf(x))) - C_1\xi \max|\nabla r_{k,\varepsilon}| + C_2y^2$$

for large C_1, C_2 . By choosing C_1 large enough, from (5.21), we have

$$\frac{\partial \tilde{v}}{\partial n} < 0$$

on B_{III} . Fixing C_1 , we choose C_2 large enough. Then by Lemma 9, we have $\tilde{\Delta}\tilde{v} > 0$. By the maximum principle, the maximum point of \tilde{v} must be reached on $B_I \cup B_{II}$. By the boundary conditions (5.14), we have

$$\tilde{v} \leq C(\varepsilon^2 + \varepsilon \max|\nabla r_{k,\varepsilon}|).$$

Thus we have

$$R(r_{k,\varepsilon}(x, yf(x))) \leq C(\varepsilon^2 + \varepsilon \max|\nabla r_{k,\varepsilon}|).$$

Since R is arbitrary, this yields

$$\max|\nabla r_{k,\varepsilon}| \leq C(\varepsilon^2 + C_1\delta \max|\nabla r_{k,\varepsilon}|). \quad \square$$

Proof of Theorem 5. The proof is similar to that of Lemma 10. On B_I , $R(r_{k,\varepsilon}) = 0$; on B_{II} , by (5.18), we have

$$(5.22) \quad \|R(r_{k,\varepsilon})\|_{C_\varepsilon^{2,\alpha}(B_{II})} \leq C\varepsilon^2 + C\varepsilon^{3+\alpha}\|h\|_{C^{4,\alpha}(\Omega)}.$$

On B_{III} , we have two cases. If R is vertical to ∂M , by (5.19),

$$\|R(r_{k,\varepsilon})\|_{C_\varepsilon^{2,\alpha}(B_{III})} \leq C(\varepsilon^2 + \varepsilon\|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}}).$$

If R is tangential to ∂M , by (5.21),

$$\left\| \frac{\partial}{\partial n} R(r_{k,\varepsilon}) \right\|_{C_\varepsilon^{1,\alpha}(B_{III})} \leq C\varepsilon^2 + \|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{1,\alpha}}.$$

By the Schauder estimate and (5.20),

$$(5.23) \quad \|R(r_{k,\varepsilon})\|_{C_\varepsilon^{2,\alpha}} \leq C(\varepsilon^2 + \varepsilon^{3+\alpha}\|h\|_{C^{4,\alpha}(\Omega)} + \|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{1,\alpha}}).$$

By the Schauder interior estimate,

$$(5.24) \quad \|h\|_{C^{4,\alpha}(\Omega)} \leq C(\varepsilon^2 + \|r_{k,\varepsilon}\|_{C^{3,\alpha}}) \leq C(\varepsilon^2 + \varepsilon^{-2-\alpha}\|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}}).$$

Combining (5.22), (5.23) and (5.24), we get

$$\|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} \leq C(\varepsilon^2 + \varepsilon\|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{2,\alpha}} + \|\nabla r_{k,\varepsilon}\|_{C_\varepsilon^{1,\alpha}}),$$

which implies the theorem by the interpolation inequalities.

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