ERRATUM TO *THE SOUND OF SYMMETRY*

Z. LU & J. ROWLETT

1. Misprints

There is a typo at the bottom of p. 823, where it is written $\lambda_2(\Omega_1) - \lambda_2(\Omega_1) = \lambda_2(\Omega_k) - \lambda_1(\Omega_k) \leq \cdots$. Clearly the left side of the equality should be $\lambda_2(\Omega_1) - \lambda_1(\Omega_1)$. Lemma 11 should be corrected to state that the length of the shortest closed geodesic that is not contained entirely in the boundary is twice the height; see [2, Proposition 14]. There is a typo in [3, equation(3.9)] due to a missing set of parentheses. Since

$$\left[(\alpha(\pi-\alpha))^{-1}\right]' = -\frac{\pi - 2\alpha}{\alpha^2(\pi-\alpha)^2},$$

[3, equation(3.9)] should read

$$g'(\alpha) = -\frac{\pi-2\alpha}{\alpha^2(\pi-\alpha)^2}\left(f(\alpha)-f(\beta)\right), \quad f(x) = \frac{x^2(\pi-x)^2\cos(x)}{(\pi-2x)\sin^2 x}.$$

Below that the statement should be corrected to read that an equivalent expression for $f(\alpha) = -\frac{\csc(x)\cot(x)}{((\alpha(\pi-\alpha))^{-1})'}$. To prove [3, Lemma 12], in [3, (3.10)] we defined

$$u(\alpha) := \frac{f'(\alpha)}{f(\alpha)} = \log(f(\alpha))' = \frac{1}{\pi/2-\alpha} + \frac{2}{\alpha} - 2\cot\alpha - \frac{2}{\pi-\alpha} - \tan\alpha.$$

We claimed that $u(\alpha) < 0$ for $\alpha \in (0, \pi/2)$, however there are two misprints in the proof of the claim. The -2 in the last equation on p. 831 should be -4, and the -4 in the first equation on p. 832 should be -8. Although with these misprints the proof in [3] of the claim no longer holds, here we present two proofs that may be of independent interest.

2. Proof of [3, Claim on p.830]

Proposition 1. *The function*

$$u(\alpha) := \frac{1}{\pi/2-\alpha} + \frac{2}{\alpha} - 2\cot\alpha - \frac{2}{\pi-\alpha} - \tan\alpha < 0, \quad \alpha \in (0, \pi/2).$$

Proof. Changing variables to $z = \frac{\alpha}{\pi}$ the proposition is equivalent to proving that

$$-2\cot(\pi z) - \tan(\pi z) + \frac{2}{\pi z} - \frac{2}{\pi - \pi z} + \frac{1}{\pi/2 - \pi z} < 0$$

$$\iff 2\pi\cot(\pi z) + \pi\tan(\pi z) - \frac{2}{z} + \frac{2}{1-z} - \frac{2}{1-2z} > 0, \quad z \in (0, 1/2).$$

We further make the change of variables $w = \frac{1}{2} - z$ and note that $\cot(\pi(1/2 - w)) = \tan(\pi w)$, so this is equivalent to

$$2\pi\tan(\pi w) + \pi\cot(\pi w) - \frac{1}{w} - \frac{4}{1-2w} + \frac{4}{1+2w} > 0, \quad w \in (0, 1/2). \tag{1}$$

By [1, 1.421.3]

$$\pi\cot(\pi w) = \sum_{k\in\mathbb{Z}} \frac{1}{k+w} = \frac{1}{w} - \sum_{k=1}^{\infty} \frac{2w}{k^2-w^2}. \tag{2}$$

By [1, 1.421.1]

$$2\pi\tan(\pi w) = 16w\sum_{k\geq 0} \frac{1}{(2k+1)^2-4w^2}. \tag{3}$$

Since

$$\frac{4}{1+2w} - \frac{4}{1-2w} = -\frac{16w}{1-4w^2}, \quad \frac{16w}{(2k+1)^2-4w^2} = \frac{4w}{\left(k+\frac{1}{2}\right)^2-w^2}$$

by (2) and (3), observing that $w>0$, (1) is equivalent to

$$\wp(w^2) := \sum_{k\geq 1} \frac{2}{\left(k+\frac{1}{2}\right)^2-w^2} - \sum_{k=1}^{\infty} \frac{1}{k^2-w^2} > 0, \quad w\in(0,1/2). \tag{4}$$

We therefore calculate

$$\wp'(x) = \sum_{k=1}^{\infty} \frac{2}{((k+\frac{1}{2})^2-x)^2} - \sum_{k=1}^{\infty} \frac{1}{(k^2-x)^2}.$$

Since we consider $x=w^2\in(0,1/2)$, we have for $x\in(0,1/4)$,

$$\sum_{k=1}^{\infty} \frac{2}{((k+\frac{1}{2})^2-x)^2} \leq \sum_{k=1}^{\infty} \frac{2}{((k+\frac{1}{2})^2-1/4)^2} = \sum_{k=1}^{\infty} \frac{2}{k^2(k+1)^2}$$
$$= 2\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)^2 = 2\sum_{k=1}^{\infty}\frac{1}{k^2} + 2\sum_{k=1}^{\infty}\frac{1}{(k+1)^2} - 4\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$$
$$= \frac{\pi^2}{3} + 2\left(\frac{\pi^2}{6}-1\right) - 4 < 0.6,$$

having used

$$\sum_{k\geq 1}\frac{1}{k(k+1)} = \sum_{k\geq 1}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1, \quad \sum_{k\geq 1}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

Since $x\in(0,1/4)$,

$$\sum_{k=1}^{\infty}\frac{1}{(k^2-x)^2} \geq \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90} > 1 \implies \wp'(x) < 0.6-1 < 0.$$

Thus $\wp$ is a strictly decreasing function, and for $w=x^2\in(0,1/4)$

$$\wp(w) > \wp(1/4) = \sum_{k=1}^{\infty}\frac{2}{k(1+k)} - \sum_{k=1}^{\infty}\frac{1}{k^2-1/4}$$
$$= 2\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) - \sum_{k=1}^{\infty}\left(\frac{1}{k-1/2}-\frac{1}{k+1/2}\right) = 0.$$

$\square$

3. Laurent series method

An alternative proof that may be of independent interest is obtained using the Laurent and Taylor series expansions of the function $u(\alpha)$ for $\alpha \in (0, \pi/4]$ and $\alpha \in (\pi/4, \pi/2)$.

Proposition 2. *The function*

$$u(\alpha) := \frac{1}{\pi/2 - \alpha} + \frac{2}{\alpha} - 2\cot\alpha - \frac{2}{\pi - \alpha} - \tan\alpha$$

is strictly negative for $\alpha \in (0, \pi/4]$.

Proof. Recall the Laurent expansion [1, 1.411.7]

$$\cot(z) = \sum_{n\geq 0} \frac{(-1)^n 4^n B_{2n} z^{2n-1}}{(2n)!},$$

with B_{2n} the $2n^{th}$ Bernoulli number. Consequently

$$\frac{1}{z} - \cot(z) = -\sum_{n\geq 1} \frac{(-1)^n 4^n B_{2n} z^{2n-1}}{(2n)!}.$$

One also has the expansion [1, 1.411.5]

$$\tan z = \sum_{n\geq 1} \frac{(-1)^{n-1} 4^n (2^{2n}-1) B_{2n} z^{2n-1}}{(2n)!}. \tag{5}$$

We calculate the geometric series

$$\frac{1}{\pi/2 - \alpha} = \frac{2}{\pi(1 - 2\alpha/\pi)} = \frac{2}{\pi}\sum_{n\geq 0}\left(\frac{2\alpha}{\pi}\right)^n, \quad 0 < \alpha < \frac{\pi}{2},$$

and

$$-\frac{2}{\pi - \alpha} = -\frac{2}{\pi}\frac{1}{1 - \alpha/\pi} = -\frac{2}{\pi}\sum_{n\geq 0}\left(\frac{\alpha}{\pi}\right)^n.$$

So we have

$$u(\alpha) = \frac{2}{\pi}\sum_{n\geq 0}\left[-\left(\frac{\alpha}{\pi}\right)^n + \left(\frac{2\alpha}{\pi}\right)^n\right] - \tan\alpha - 2\sum_{n\geq 1}\frac{(-1)^n 4^n B_{2n}\alpha^{2n-1}}{(2n)!}.$$

Using the series expansion of the tangent (5)we therefore combine and simplify

$$u(\alpha) = \frac{2}{\pi}\sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n (2^n - 1) + \sum_{n\geq 1}\frac{4^n B_{2n}\alpha^{2n-1}}{(2n)!}(-1)^n (4^n - 3).$$

We calculate

$$\begin{aligned}\frac{2}{\pi}\sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n (2^n - 1) &= \sum_{k\geq 1}\frac{2}{\pi}(2^{2k-1} - 1)\frac{\alpha^{2k-1}}{\pi^{2k-1}} + \sum_{j\geq 1}\frac{2}{\pi}(2^{2j} - 1)\frac{\alpha^{2j-1}}{\pi^{2j}}\alpha \\ &= \sum_{k\geq 1}(2^{2k} - 2)\frac{\alpha^{2k-1}}{\pi^{2k}} + \sum_{j\geq 1}\frac{2\alpha}{\pi}(4^j - 1)\frac{\alpha^{2j-1}}{\pi^{2j}} \\ &= \sum_{n\geq 1}\alpha^{2n-1}\left[\frac{4^n - 2}{\pi^{2n}} + \frac{2\alpha}{\pi}\frac{(4^n - 1)}{\pi^{2n}}\right].\end{aligned}$$

We therefore have

$$u(\alpha) = \sum_{n\geq 1} \alpha^{2n-1} \left[\left[\frac{4^n - 2}{\pi^{2n}} + \frac{2\alpha}{\pi} \frac{(4^n - 1)}{\pi^{2n}} \right] + \frac{4^n (4^n - 3) (-1)^n B_{2n}}{(2n)!} \right]. \tag{6}$$

Note that the Bernoulli numbers satisfy

$$(-1)^n B_{2n} = -|B_{2n}| \forall n \geq 1.$$

Moreover, by [1, 9.616],

$$|B_{2n}| = \frac{(2n)!\zeta(2n)}{2^{2n-1}\pi^{2n}} \forall n \geq 1$$

with ζ the Riemann zeta function. Consequently the coefficients of α^{2n-1} in (6) are

$$\frac{1}{\pi^{2n}} \left(4^n - 2 + \frac{2\alpha}{\pi}(4^n - 1) - 2(4^n - 3)\zeta(2n) \right)$$

$$= \frac{1}{\pi^{2n}} \left((4^n - 1) \left(1 + \frac{2\alpha}{\pi} - 2\zeta(2n) \right) - 1 + 4\zeta(2n) \right).$$

If we assume that $\alpha \in (0, \pi/4]$, then using the very crude estimates that $1 < \zeta(2n) < 2$ for $n \geq 1$, we obtain the upper bound for the coefficients

$$\frac{1}{\pi^{2n}} \left(-\frac{1}{2}(4^n - 1) - 1 + 8 \right) < 0 \forall n \geq 2. \tag{7}$$

For $n = 1$ we explicitly evaluate the Riemann zeta function and obtain the exact value of the coefficient:

$$\frac{1}{\pi^2} \left(4 - 2 + \frac{6\alpha}{\pi} - 2\frac{\pi^2}{6} \right) = \frac{1}{\pi^2} \left(2 + \frac{6\alpha}{\pi} - \frac{\pi^2}{3} \right) < 0$$

$$\iff \alpha < \frac{\pi}{3} \left(\frac{\pi^2}{6} - 1 \right) \approx 0.675, \quad \frac{\pi}{4} \approx 0.785.$$

Consequently, to prove that $u(\alpha) < 0$ on $(0, \pi/4]$, we investigate precisely the first two terms using the wonderful exercise in Fourier analysis which shows that the Riemann zeta function satisfies

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}.$$

The sum of the first two terms in the series defining $u(\alpha)$ is

$$\frac{\alpha}{\pi^2} \left(2 + \frac{6\alpha}{\pi} - \frac{\pi^2}{3} \right) + \frac{\alpha^3}{\pi^4} \left(15 \left(1 + \frac{2\alpha}{\pi} - \frac{2\pi^4}{90} \right) - 1 + \frac{4\pi^4}{90} \right).$$

Since $\alpha, \pi > 0$, the sign of the above expression is equal to the sign of

$$2 + \frac{6\alpha}{\pi} - \frac{\pi^2}{3} + \frac{\alpha^2}{\pi^2} \left(14 + \frac{30\alpha}{\pi} - \frac{13\pi^4}{45} \right). \tag{8}$$

For α near zero, this expression is strictly negative because $2 < \frac{\pi^2}{3}$. The derivative of (8) with respect to α is

$$\frac{6}{\pi} + \frac{28\alpha}{\pi^2} + \frac{90\alpha^2}{\pi^3} - \frac{26\alpha\pi^2}{45}.$$

For α near zero, this is positive. This is a quadratic function, and the discriminant is

$$\left(\frac{28}{\pi^2} - \frac{26\pi^2}{45} \right)^2 - 4 \left(\frac{90}{\pi^3} \frac{6}{\pi} \right) < 0.$$

Consequently, there are no real roots, and the derivative of (8) is positive, so (8) is an increasing function of α. Its maximum on $(0, \pi/4]$ occurs at $\alpha = \pi/4$. To compute the sign, we evaluate (8) at $\alpha = \pi/4$ obtaining

$$2 + \frac{6}{4} - \frac{\pi^2}{3} + \frac{1}{16}\left(14 + \frac{30}{4} - \frac{13\pi^4}{45}\right) = \frac{7}{2} - \frac{\pi^2}{3} + \frac{1}{16}\left(\frac{43}{2} - \frac{13\pi^4}{45}\right) \approx -0.2 < 0.$$

This shows that on $(0, \pi/4]$ the sum of the first two terms in the series defining $u(\alpha)$ is strictly negative. By (7) the rest of the sum is also negative, and therefore $u(\alpha) < 0$ on $(0, \pi/4]$. □

Proposition 3. *The function*

$$u(\alpha) := \frac{1}{\pi/2 - \alpha} + \frac{2}{\alpha} - 2\cot\alpha - \frac{2}{\pi - \alpha} - \tan\alpha$$

is strictly negative for $\alpha \in (\pi/4, \pi/2)$.

Proof. Since

$$\tan(\alpha) = \cot(\pi/2 - \alpha), \quad \cot(\alpha) = \tan(\pi/2 - \alpha),$$

$$u(\alpha) = \frac{1}{\pi/2 - \alpha} + \frac{2}{\pi/2 - \pi/2 + \alpha} - 2\tan(\pi/2 - \alpha) - \frac{2}{\pi/2 + \pi/2 - \alpha} - \cot(\pi/2 - \alpha).$$

It is convenient to make the substitution

$$y := \frac{\pi}{2} - \alpha.$$

Then $y \in (0, \pi/4)$ corresponds to $\alpha \in (\pi/4, \pi/2)$, and

$$\begin{aligned} u(\alpha) &= \frac{1}{y} - \cot y - 2\tan y + \frac{2}{\pi/2 - y} - \frac{2}{\pi/2 + y} \\ &= \frac{1}{y} - \cot y - 2\tan y + \frac{4}{\pi}\frac{1}{1 - \frac{2y}{\pi}} - \frac{4}{\pi}\frac{1}{1 - \left(-\frac{2y}{\pi}\right)}. \end{aligned}$$

We use the series expansions:

$$\frac{1}{y} - \cot y = -\sum_{n\geq 1} \frac{(-1)^n 4^n B_{2n} y^{2n-1}}{(2n)!}, \quad -2\tan y = -2\sum_{n\geq 1} \frac{(-1)^{n-1} 4^n (4^n - 1) B_{2n} y^{2n-1}}{(2n)!},$$

with B_{2n} the $2n^{th}$ Bernoulli number. So,

$$\frac{1}{y} - \cot y - 2\tan y = \sum_{n\geq 1} \frac{(-1)^n 4^n B_{2n} y^{2n-1}}{(2n)!} \left(2(4^n) - 3\right).$$

We also have

$$\frac{4}{\pi}\frac{1}{1 - \frac{2y}{\pi}} = \frac{4}{\pi}\sum_{n\geq 0}\left(\frac{2y}{\pi}\right)^n.$$

So,

$$u(\alpha) = \sum_{n\geq 1} \frac{(-1)^n 4^n B_{2n} y^{2n-1}}{(2n)!} \left(2(4^n) - 3\right) + \frac{4}{\pi}\sum_{n\geq 0}\left(\frac{2y}{\pi}\right)^n - \frac{2}{\pi/2 + y}.$$

Similarly we combine the series

$$\frac{4}{\pi}\frac{1}{1 - \frac{2y}{\pi}} - \frac{4}{\pi}\frac{1}{1 - \left(-\frac{2y}{\pi}\right)} = \frac{4}{\pi}\sum_{n\geq 0}\left(\frac{2y}{\pi}\right)^n - \left(-\frac{2y}{\pi}\right)^n$$

$$= \frac{8}{\pi} \sum_{n \geq 1} \left(\frac{2y}{\pi} \right)^{2n-1}.$$

So, in total we obtain

$$u(\alpha) = \sum_{n \geq 1} \frac{(-1)^n 4^n B_{2n} y^{2n-1}}{(2n)!} (2(4^n) - 3) + \frac{8}{\pi} \sum_{n \geq 1} \left(\frac{2y}{\pi} \right)^{2n-1}, \quad y = \frac{\pi}{2} - \alpha.$$

Putting the series together we obtain

$$\sum_{n \geq 1} y^{2n-1} \left[\frac{(-1)^n 4^n (2^{2n+1} - 3) B_{2n}}{(2n)!} + \frac{2^{2n+2}}{\pi^{2n}} \right].$$

Note that

$$(-1)^n B_{2n} = -|B_{2n}| < 0 \forall n \geq 1.$$

By [1, 9.616],

$$|B_{2n}| = \frac{(2n)! \zeta(2n)}{2^{2n-1} \pi^{2n}} \forall n \geq 1$$

with ζ the Riemann zeta function. We therefore obtain

$$\frac{4^n (2^{2n+1} - 3) |B_{2n}|}{(2n)!} = \frac{2(2^{2n+1} - 3) \zeta(2n)}{\pi^{2n}}.$$

The coefficient of y^{2n-1} is therefore

$$\frac{1}{\pi^{2n}} \left(2^{2n+2} - 2(2^{2n+1} - 3) \zeta(2n) \right) = \frac{1}{\pi^{2n}} \left(2^{2n+2} (1 - \zeta(2n)) + 6 \zeta(2n) \right).$$

For $n = 1$ we compute the coefficient of y explicitly

$$\frac{1}{\pi^2} \left(16 - 2(5) \frac{\pi^2}{6} \right) = \frac{1}{\pi^2} \left(16 - \frac{5\pi^2}{3} \right) < -0.045. \tag{9}$$

We calculate that

$$2^{2n+2}(1 - \zeta(2n)) + 6\zeta(2n) = -2^{2n+2} \sum_{m \geq 2} m^{-2n} + 6 + 6 \sum_{m \geq 2} m^{-2n} = 2 + 6(4)^{-n} + \sum_{m \geq 3} (6 - 2^{2n+2}) m^{-2n}$$

$$= 2 + \frac{6}{4^n} + (6 - 2^{2n+2}) 3^{-2n} + \sum_{m \geq 4} (6 - 2^{2n+2}) m^{-2n}.$$

Note that for all $n \geq 1$ we have $(6 - 2^{2n+2}) < 0$, so the sum on the right above is negative. Moreover we also have

$$\frac{6}{4^n} + (6 - 2^{2n+2}) 3^{-2n} < 0 \forall n \geq 2.$$

Consequently, an upper bound for the coefficient of y^{2n-1} for $n \geq 2$ is $\frac{2}{\pi^{2n}}$. The series from $n \geq 2$ may therefore be estimated from above by

$$\sum_{n \geq 2} \frac{2}{\pi^{2n}} y^{2n-1} = \frac{2}{y} \sum_{n \geq 2} \left(\frac{y^2}{\pi^2} \right)^n = \frac{2}{y} \frac{y^4}{\pi^4} \sum_{n \geq 0} \left(\frac{y^2}{\pi^2} \right)^n$$

$$= \frac{2y^3}{\pi^4} \frac{1}{1 - \frac{y^2}{\pi^2}}.$$

So, in total we have the estimate that

$$u(\alpha) < \frac{1}{\pi^2}\left(16 - \frac{5\pi^2}{3}\right) y + \frac{2y^3}{\pi^4}\frac{1}{1-\frac{y^2}{\pi^2}}, \quad y = \pi/2 - \alpha \in (0, \pi/4).$$

It therefore suffices to prove that

$$\frac{1}{\pi^2}\left(16 - \frac{5\pi^2}{3}\right) + \frac{2y^2}{\pi^4}\frac{1}{1-\frac{y^2}{\pi^2}} < 0, \quad y \in (0, \pi/4).$$

This is an increasing function of $y \in (0, \pi/4)$, so its maximum occurs at $y = \pi/4$ with the value

$$\frac{1}{\pi^2}\left(16 - \frac{5\pi^2}{3}\right) + \frac{1}{8\pi^2}\frac{1}{1 - 1/16} < -0.03.$$

□

Acknowledgments

JR is supported by Swedish Research Council Grant 2018-03873. Both authors are grateful to Leonardo Valori for drawing our attention to the misprints.

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Eighth, Elsevier/Academic Press, Amsterdam, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition.

[2] H. Hezari, Z. Lu, and J. Rowlett, *The Neumann isospectral problem for trapezoids*, Annales de l'Institut Henri Poincaré **18** (2017), 3759–3752.

[3] Z. Lu and J. Rowlett, *The sound of symmetry*, American Math Monthly **122** (2015), no. 9, 815-835.