Älvsborgs Fästning, 30 October, 2014

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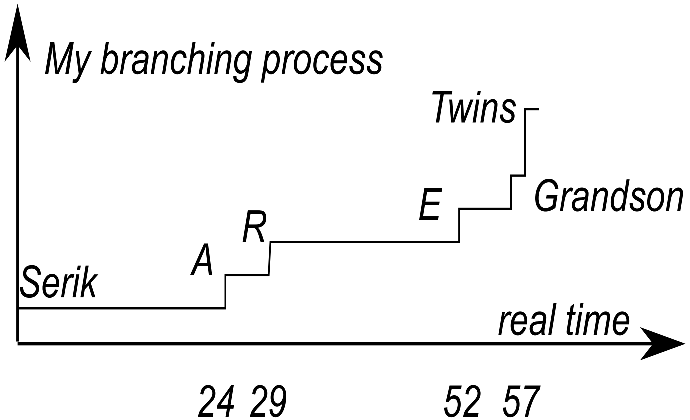
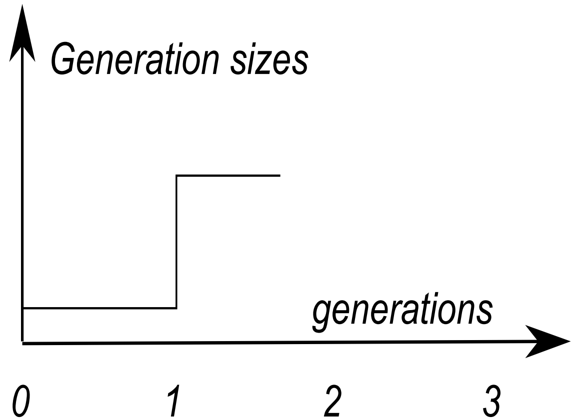
Chalmers and GU

Tail generating functions for Markov branching processes





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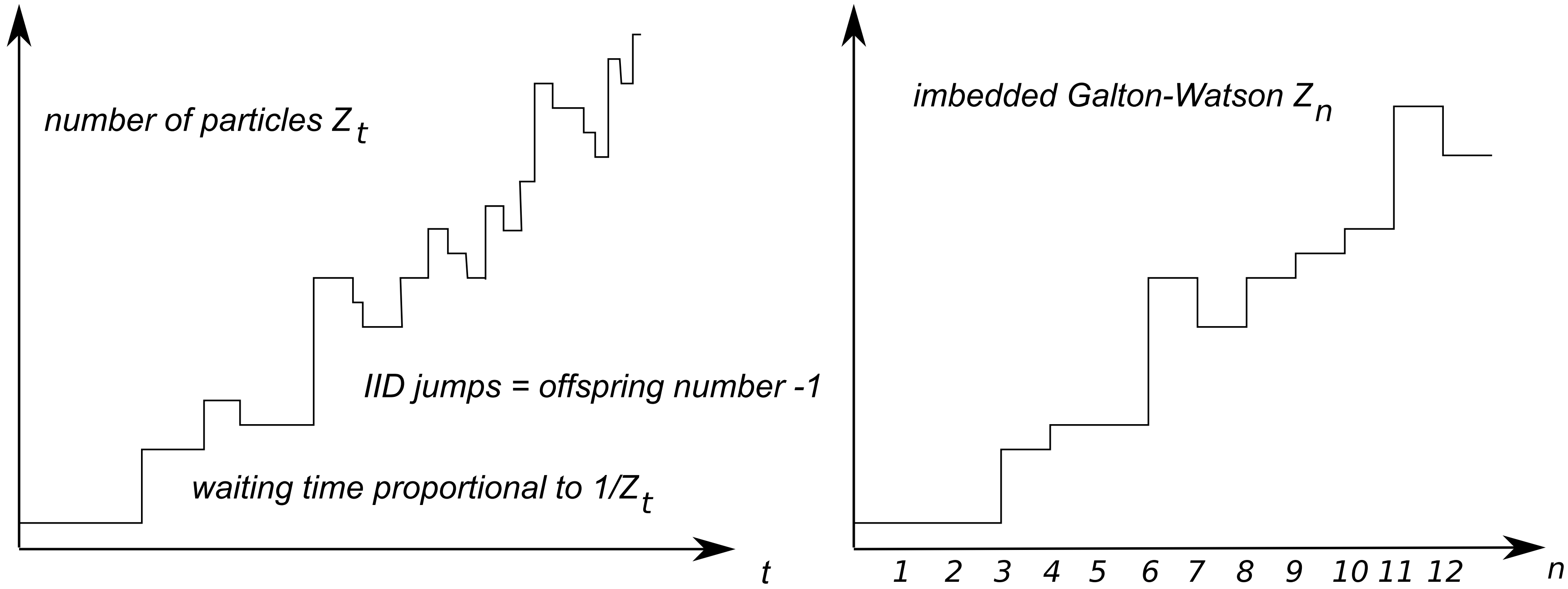
13 slides

Continuous vs discrete time

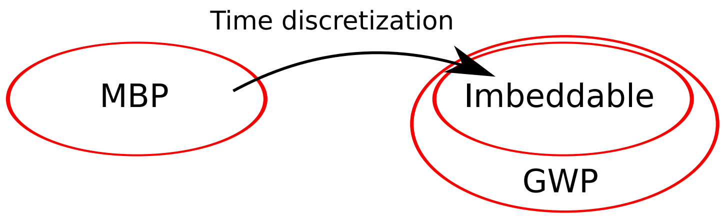


MBP: Markov branching process $\{Z_t\}_{t \in [0,\infty)}$

GWP: Galton-Watson process $\{Z_n\}_{n \in \{0,1,\ldots\}}$

The traditional approach:

study first GWP and then MBP using time discretization



Probability generating functions

$F_t(s) = E(s^{Z_t})$

Semigroup property

$$F_{t+u}(s) = F_t(F_u(s))$$

For a δ-imbedded GWP

$$\hat{F}_n(s) = F_{n\delta}(s)$$

are n-fold iterations of $\hat{F}_1(s) = F_\delta(s)$ which is the generating function for the offspring distribution over time δ

Backward Kolmogorov equation for MBP

$$\frac{\partial F_t(s)}{\partial t} = \lambda\Big[f(F_t(s)) - F_t(s)\Big], \qquad F_0(s) = s,$$

$f(s) = Es^{\nu}$ is the offspring distribution generating function

λ = parameter of the exponential lifelength distribution

Classification of branching processes

Repr. regime	$m = f'(1)$	$q = P(Z_\infty = 0)$	$E(Z_t)$
subcritical	$(0, 1)$	1	$e^{-\lambda(1-m)t}$
critical	1	1	1
supercritical	$(1, \infty)$	$[0, 1)$	$e^{\lambda(m-1)t}$

GWP-formula $E(Z_n) = m^n$

An alternative approach (for example, Zolotarev, 1957)

study directly MBP without referring to the GWP theory

Start from the integral version of backward Kolmogorov

$$\lambda t = \int_s^{F_t(s)} \frac{dx}{f(x) - x}$$

and focus on the non-negative roots of $f(x) = x$:

$$f(q) = q, \qquad f(r) = r, \qquad q < r$$

October 2014 preprint suggests such a direct analysis of MBP using a new tool: tail generating functions

Tail generating functions for Markov branching processes
http://arxiv.org/abs/1410.1497

Tail generating functions

For a generating function (not necessarily pgf)

$$f(s) = \sum_{k=0}^{\infty} s^k p_k, \quad p_k \geq 0, \quad s \in [0, R]$$

and a given $a \in [0, R]$, define a tail generating function by

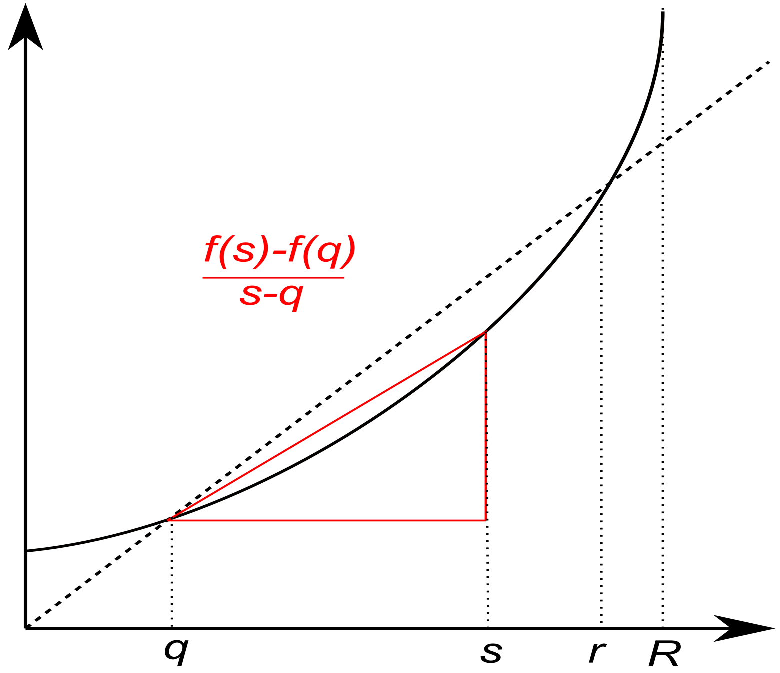
$$\nabla_a f(s) = \frac{f(s) - f(a)}{s - a}, \quad \nabla_a f(a) = f'(a)$$

For $n \geq 1$, define recursively

$$\nabla_a^n f(s) = \nabla_a(\nabla_a^{n-1} f)(s), \quad \nabla_a^0 f(s) = f(s).$$

If $a = 1$, and $f(s) = Es^\nu$ is a probability generating function, then

$$\nabla_1 f(s) = \sum_{k=0}^{\infty} s^k P(\nu > k)$$



R = radius of convergence

q = smallest non-negative root of $f(x) = x$

r = larger non-negative root of $f(x) = x$, if any

$$\nabla_a \nabla_b = \nabla_b \nabla_a, \qquad \nabla_a^n f(a) = \frac{f^{(n)}(a)}{n!}$$

$$f(s) = \sum_{i=0}^{n-1} \nabla_a^i f(a)(s-a)^i + \nabla_a^n f(s)(s-a)^n$$

$$\nabla_a^n f(s) = \sum_{k=0}^{\infty} p_{k+n} \sum_{i=0}^{k} s^i \binom{k-i+n-1}{n-1} a^{k-i}$$

PROPOSITION. For given $a \in (0, R]$ and $n \geq 1$, the moment condition

$$\sum_{k=2}^{\infty} p_k a^k k^{n-1} \ln k < \infty$$

is equivalent to

$$\int_0^a \nabla_a^n f(x) dx < \infty.$$

Theorem 1 *Let* $t \geq 0$ *and* $s \in [0,1)$. *If* $m = 1$, *then*

$$\int_s^{F_t(s)} \frac{dx}{(1-x)^2 \nabla_1^2 f(x)} = \lambda t$$

If $m \neq 1$, *then*

$$\nabla_q F_t(s) = \gamma^t \exp\left\{ -\int_s^{F_t(s)} \frac{\nabla_q^2 f(x) dx}{1 - \nabla_q f(x)} \right\}$$

$$\gamma := e^{\lambda(f'(q)-1)} \in (0,1)$$

If $1 < m < \infty$, *then*

$$\nabla_q F_t(s) = \gamma^t \left[\nabla_1 F_t(s)\right]^\beta \exp\left\{ \int_s^{F_t(s)} \frac{\nabla_1 \nabla_q^2 f(x) - \beta \nabla_1^2 \nabla_q f(x)}{\nabla_1 \nabla_q f(x)} dx \right\}$$

$$\beta := \tfrac{1-f'(q)}{m-1} \in (0,1)$$

We illustrate using the linear-fractional reproduction law

$$f(s) = p_0 + (1-p_0)\frac{ps}{1-(1-p)s}, \quad p_0 \in [0,1], \quad p \in (0,1].$$

Tail generating functions are also linear-fractional

$$\nabla_{a_1} \ldots \nabla_{a_n} f(s) = \frac{p(1-p_0)}{1-p} \prod_{i=1}^{n} \frac{1-p}{1-(1-p)a_i} \cdot \frac{1}{1-(1-p)s}.$$

Mean offspring number $m = \frac{1-p_0}{p}$.

In the critical case, $p_0 = 1-p$

$$\nabla_1 F_t(s) = \frac{p}{1-p} \cdot \frac{\nabla_2 F_t(s)}{\lambda t + \ln \nabla_1 F_t(s)}.$$

Subcritical case, $p + p_0 > 1$, is extendable with $r = \frac{p_0}{1-p} > 1$. From

$$\nabla_1 F_t(s) = e^{-\lambda(1-m)t} \exp\Big\{ -\int_s^{F_t(s)} \frac{\nabla_1^2 f(x)dx}{1 - \nabla_1 f(x)} \Big\}$$

we get the following compact form

$$\nabla_1 F_t(s) = e^{-\lambda(1-m)t} \big[\nabla_r F_t(s)\big]^m.$$

In the supercritical case, $p + p_0 < 1$, $q = \frac{p_0}{1-p} < 1$, from

$$\nabla_q F_t(s) = \gamma^t \big[\nabla_1 F_t(s)\big]^\beta \exp\Big\{ \int_s^{F_t(s)} \frac{\nabla_1 \nabla_q^2 f(x) - \beta \nabla_1^2 \nabla_q f(x)}{\nabla_1 \nabla_q f(x)} dx \Big\}$$

we derive a dual equation to the subcritical one

$$\nabla_q F_t(s) = e^{-\lambda(1-1/m)t} \big[\nabla_1 F_t(s)\big]^{1/m}.$$

PROPOSITION 1.

If $m < 1$, then as $t \to \infty$

$$e^{t\lambda(1-m)} Q_t \to c \in [0, \infty), \qquad Q_t := P(Z_t > 0)$$

$$E(s^{Z_t} | Z_t > 0) \to \psi(s), \quad s \in [0, 1],$$

where the limit probability generating function $\psi(s)$ is determined by

$$\nabla_1 \psi(s) = \exp\left\{ \int_0^s \frac{\nabla_1^2 f(x) dx}{1 - \nabla_1 f(x)} \right\},$$

with $\{c > 0\} \Leftrightarrow \{\psi'(1) < \infty\} \Leftrightarrow \sum p_k k \ln k < \infty$.

When $c = 0$, there is a slowly varying monotone function $\mathcal{L}_1$ such that

$$Q_t \mathcal{L}_1(Q_t) = e^{-t\lambda(1-m)}.$$

Linear-fractional $f(s)$: $c = (\frac{r-1}{r})^m$ and $\psi(s) = 1 - (1 - s)(1 - s/r)^{-m}$

PROPOSITION 2.

If $1 < m < \infty$, then almost surely

$$Z_t e^{(1-m)t} \to W, \quad t \to \infty.$$

If $\sum p_k k \ln k < \infty$, then $\qquad Ee^{-\rho W} = q + (1-q)\phi(\rho),$

$$\phi(\rho) = q + (1-q)\Big(\frac{1-\phi(\rho)}{\rho}\Big)^{\beta} \exp\Big\{\int_{\phi(\rho)}^{1} \frac{\beta\nabla_1^2\nabla_q f(x) - \nabla_1\nabla_q^2 f(x)}{\nabla_1\nabla_q f(x)} dx\Big\}$$

Otherwise, $P(W = 0) = 1$.

Linear-fractional $f(s)$: $\qquad \phi(\rho) = q + (1-q)\Big(\frac{1-\phi(\rho)}{\rho}\Big)^{1/m}.$

Moreover, if $p_0 > 0$, then

$$E(s^{Z_t} | Z_t > 0, Z_\infty = 0) \to 1 - (1-s)(1-sq)^{-1/m}.$$