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Linear-fractional branching processes with countably many types

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Notation for vectors of possibly infinite dimension

$\mathbf{s} = (s_1, s_2, \dots) \quad \mathbf{h} = (h_1, h_2, \dots) \quad \mathbf{g} = (g_1, g_2, \dots) \quad \mathbf{0} = (0, 0, \dots)$

Definition 1

Random vector $\mathbf{Z}$ *has a linear-fractional distribution* $LF(\mathbf{h}, \mathbf{g}, m)$

> $(h_0, h_1, h_2, \dots)$ *is a probability distribution on* $\{0, 1, 2, \dots\}$
> $(g_1, g_2, \dots)$ *is a probability distribution on* $\{1, 2, \dots\}$
> m *is a positive constant*

if

$$\mathbb{E}(\mathbf{s}^{\mathbf{Z}}) = h_0 + \frac{\sum_{i=1}^{\infty} h_i s_i}{1 + m - m \sum_{i=1}^{\infty} g_i s_i}, \tag{1}$$

so that in particular

$$\mathbb{P}(\mathbf{Z} = \mathbf{0}) = h_0.$$

General linear-fractional generating functions

$$\mathbb{E}(\mathbf{s}^{\mathbf{Z}}) = h_0 + \frac{h_1 s_1 + h_2 s_2 + \ldots}{1 + m - m(g_1 s_1 + g_2 s_2 + \ldots)}$$

Such a multivariate linear-fractional distribution can be viewed as a multivariate geometric distribution modified at zero:

$$\mathbf{Z} = \mathbf{X} + (\mathbf{Y}_1 + \ldots + \mathbf{Y}_N) \cdot 1_{\{\mathbf{X} \neq \mathbf{0}\}} \tag{2}$$

in terms of mutually independent random entities $(\mathbf{X}, N, \mathbf{Y}_1, \mathbf{Y}_2, \ldots)$.

Here vectors $\mathbf{X}$ and $\mathbf{Y}_j$ have multivariate Bernoulli distributions

$$\mathbb{P}(\mathbf{X} = \mathbf{e}_i) = h_i, \ i \geq 1,$$
$$\mathbb{P}(\mathbf{X} = \mathbf{0}) = h_0,$$
$$\mathbb{P}(\mathbf{Y}_j = \mathbf{e}_i) = g_i, \ i \geq 1, \ j \geq 1,$$

and N is a geometric random variable with mean m.

LF branching processes with countably many types

Definition 2

LF branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$

sub-stochastic matrix $\mathbf{H}$ *with non-negative rows* $\mathbf{h}_1, \mathbf{h}_2, \ldots$
probability distribution $\mathbf{g}$ *on* $\{1, 2, \ldots\}$
positive constant m

if particles of type i
reproduce according to a $LF(\mathbf{h}_i, \mathbf{g}, m)$ *distribution,* $i = 1, 2, \ldots$.

Remark

Strong limitation on the reproduction law: parameters $(\mathbf{g}, m)$ are ignorant on mother's type. This is needed for iterations of the corresponding generating functions to be also linear-fractional.

The matrix of mean offspring numbers for a LF branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$ is computed as

$$\mathbf{M} = \mathbf{H} + m\mathbf{H}\mathbf{1}^{\mathrm{t}}\mathbf{g}, \tag{3}$$

where $\mathbf{1}^{\mathrm{t}}$ is the column version of $\mathbf{1} = (1, 1, \ldots)$.

It follows,

$$\mathbf{H} = \mathbf{M} - \frac{m}{1+m}\mathbf{M}\mathbf{1}^{\mathrm{t}}\mathbf{g}. \tag{4}$$

Proposition 1 *Consider a linear-fractional branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$ starting from a type i particle. The vector of its n-th generation sizes has a LF$(\mathbf{h}_i^{(n)}, \mathbf{g}^{(n)}, m^{(n)})$ distribution with*

$$m^{(n)}\mathbf{g}^{(n)} = m\mathbf{g}(\mathbf{I} + \mathbf{M} + \cdots + \mathbf{M}^{n-1}), \tag{5}$$

$$m^{(n)} = m\sum_{j=0}^{n-1} \mathbf{g}\mathbf{M}^j\mathbf{1}^{\mathrm{t}}, \tag{6}$$

$$\mathbf{H}^{(n)} = \mathbf{M}^n - \frac{m^{(n)}}{1+m^{(n)}}\mathbf{M}^n\mathbf{1}^{\mathrm{t}}\mathbf{g}^{(n)}, \tag{7}$$

where $\mathbf{H}^{(n)}$ *is the matrix with rows* $\mathbf{h}_1^{(n)}, \mathbf{h}_2^{(n)}, \ldots$.

In particular,

$$\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}) = (1 + m^{(n)})^{-1}\mathbf{M}^n\mathbf{1}^{\mathrm{t}}, \tag{8}$$

where $\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0})$ *is a vector with elements* $\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0} | \mathbf{Z}^{(0)} = \mathbf{e}_i)$.

Given that $\mathbf{M}$ is irreducible and aperiodic and all powers $\mathbf{M}^n = (m_{ij}^{(n)})_{i,j=1}^{\infty}$ are element-wise finite, which is always true in the linear-fractional case, the asymptotical behavior of these powers is described by the Perron-Frobenius theory for countable matrices.

Due to Theorem 6.1 in [Seneta] all elements of the matrix power series

$$\mathbf{M}(s) = \sum_{n=0}^{\infty} s^n \mathbf{M}^n \tag{9}$$

have a common convergence radius $0 \leq R < \infty$.

Furthermore, one of the two alternatives holds:

R-transient case: $\sum_{n=0}^{\infty} m_{ii}^{(n)} R^n < \infty$ for all i
R-recurrent case: $\sum_{n=0}^{\infty} m_{ii}^{(n)} R^n = \infty$ for all i

Classification of BP with countably many types

In the R-recurrent case there exist unique up to constant multipliers *positive* vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$R\mathbf{M}\mathbf{u}^{\mathrm{t}} = \mathbf{u}^{\mathrm{t}}, \;\; R\mathbf{v}\mathbf{M} = \mathbf{v}.$$

Renormalization $Rv_j m_{ji}/v_i$ transforms $\mathbf{M}$ into a *stochastic* matrix

The R-recurrent case is further divided in two sub-cases:

R-null case, when $\mathbf{v}\mathbf{u}^{\mathrm{t}} = \infty$

R-positive case, when $\mathbf{v}\mathbf{u}^{\mathrm{t}} < \infty$

In the R-null case (and clearly also in the R-transient case)

$$R^n m_{ij}^{(n)} \to 0, \;\; i, j = 1, 2, \ldots$$

Classification of BP with countably many types

In the R-positive case one can scale eigenvectors so that $\mathbf{v}\mathbf{u}^{\mathrm{t}} = 1$ and

$$R^n m_{ij}^{(n)} \to u_i v_j, \;\; i, j = 1, 2, \ldots.$$

These suggest a double classification of BP with a mean matrix $\mathbf{M}$

- usual classification based on the value of the growth rate $\rho = 1/R$: subcritical $\rho < 1$, critical $\rho = 1$, or supercritical $\rho > 1$
- an additional classification due to recurrence property of the branching process with respect to the infinite type space.

Definition 3 *A branching process will be called R-transient, R-recurrent, R-null recurrent, or R-positively recurrent depending on the respective property of its matrix of mean offspring numbers* $\mathbf{M}$.

Consider a LF branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$.

The monotone function

$$f(s) = \sum_{n=1}^{\infty} s^n \mathbf{g}\mathbf{H}^n \mathbf{1}^{\mathrm{t}} \tag{10}$$

grows from zero to infinity as s goes from zero to infinity. Therefore equation

$$mf(R) = 1 \tag{11}$$

always has a unique positive solution R.

Observe that $mf(Rs)$ is the generating function for a probability distribution with mean

$$\beta = m \sum_{n=1}^{\infty} nR^n \mathbf{g}\mathbf{H}^n \mathbf{1}^{\mathrm{t}}. \tag{12}$$

Theorem 1

Consider a LF branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$ *assuming that its mean matrix* $\mathbf{M}$ *is irreducible and aperiodic.*

The convergence parameter R *of* $\mathbf{M}$ *is defined by* (11) *and*

$$\boxed{\mathbf{M} \textit{ is always } R\textit{-recurrent}}$$

If $\beta = \infty$, *then* $\mathbf{M}$ *is* R*-null recurrent.*

If $\beta < \infty$, *then* $\mathbf{M}$ *is* R*-positively recurrent and*

$$R^n \mathbf{M}^n \to \mathbf{u}^{\mathrm{t}} \mathbf{v}, \ \ n \to \infty. \tag{13}$$

Here positive vectors

$$\mathbf{u}^{\mathrm{t}} = (1+m)\beta^{-1}\sum_{k=1}^{\infty} R^k \mathbf{H}^k \mathbf{1}^{\mathrm{t}}, \tag{14}$$

$$\mathbf{v} = \frac{m}{1+m}\sum_{k=0}^{\infty} R^k \mathbf{g}\mathbf{H}^k \tag{15}$$

are such that $\mathbf{v}\mathbf{u}^{\mathrm{t}} = \mathbf{v}\mathbf{1}^{\mathrm{t}} = 1$ *and* $\mathbf{g}\mathbf{u}^{\mathrm{t}} = \frac{1+m}{m\beta}$.

Remark

Formulae (14) and (15) are new even in the case of finitely many types.

The proof is based on

$$\mathbf{M}(s) = \mathbf{H}(s) + \frac{m}{1 - mf(s)}\mathbf{H}(s)\mathbf{1}^{\mathrm{t}}\mathbf{g}(\mathbf{H}(s) + \mathbf{I}), \tag{16}$$

where $\mathbf{M}(s) = \sum_{n=0}^{\infty} s^n \mathbf{M}^n$ and $\mathbf{H}(s) = \sum_{n=0}^{\infty} s^n \mathbf{H}^n$.

Apply the renewal theorem taken from Chapter XIII.4 in [Feller-2]:

Proposition 2 *Let $A(s) = \sum_{n=0}^{\infty} a_n s^n$ be a probability generating function and $B(s) = \sum_{n=0}^{\infty} b_n s^n$ is a generating function for a non-negative sequence so that $A(1) = 1$ while $B(1) \in (0, \infty)$.*

Then the non-negative sequence defined by

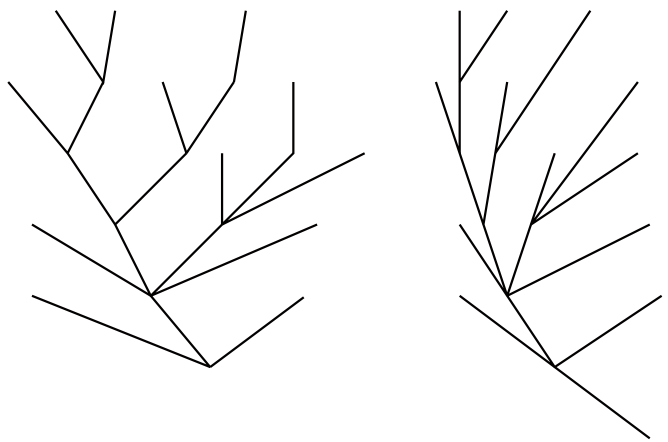
$$\sum_{n=0}^{\infty} t_n s^n = \frac{B(s)}{1 - A(s)}$$

is such that $t_n \to \frac{B(1)}{A'(1)}$ as $n \to \infty$.

If the initial particle of the LF branching process with parameters $(\mathbf{H}, \mathbf{g}, m)$ has distribution $\mathbf{g}$

$$\mathbb{P}(\mathbf{Z}^{(0)} = \mathbf{e}_i) = g_i,\ i = 1, 2, \dots, \tag{17}$$

then we get a *branching renewal property* implying an alternative description of this reproduction system in terms of *individuals* defined as sequences of first-born *particles*.



Individual's life length L has a discrete phase-type distribution governed by the pair $(\mathbf{H}, \mathbf{g})$. Its tail probabilities are computed as

$$\mathbb{P}(L > n) = \mathbf{g}\mathbf{H}^n\mathbf{1}^{\mathrm{t}}, \ n \geq 0, \tag{18}$$

allowing to express the generating function (10) in the form

$$f(s) = \sum_{n=1}^{\infty} s^n \mathbb{P}(L > n). \tag{19}$$

Clearly,

$$f(s) = \frac{1 - \mathbb{E}(s^L)}{1 - s} - 1$$

and R is found from

$$\mathbb{E}(R^L) = R \cdot \frac{1 + m}{m} - \frac{1}{m}.$$

The life length has always a positive mean (finite or infinite)

$$\lambda := \mathbb{E}(L) = 1 + f(1). \tag{20}$$

Every year during its life (except the last year) the individual produces a geometric number of offspring with mean m.

Thus the mean total offspring number per individual is

$$\mu = m(\lambda - 1) = mf(1). \tag{21}$$

Depending on $\mu < 1$, $\mu = 1$, or $\mu > 1$, the branching process should be called sub-, critical, or supercritical resulting exactly in the same classification defined in terms of the convergence parameter R.

Total population sizes $Z^{(n)} = \mathbf{Z}^{(n)}\mathbf{1}^{\mathrm{t}}$ form a discrete time CMJ process. There are many triplets $(\mathbf{H}, \mathbf{g}, m)$ resulting in the same CMJ process with the same values of R and β.

Mean age at childbearing $\beta = \sum_{k=1}^{\infty} kmR^k \mathbb{P}(L > k)$

The difference between such related branching processes is in the labeling rules for particles and will be seen in their Perron-Frobenius eigenvectors.

Remark

Making births very rare events we can approximate the linear-fractional CMJ processes by the continuous time CMJ processes with Poisson reproduction, recently studied by Lambert.

There is a particular labeling system allowing for more explicit formulae for the Perron-Frobenius eigenvectors.

We relabel individuals by looking into the future and assigning label i to individuals whose remaining life length equals i for $i = 1, 2, \ldots$. For example, an individual with label 1 must die in the next year.

The relabeled process is again a linear-fractional branching process with a modified triplet of parameters $(\mathring{\mathbf{H}}, \mathring{\mathbf{g}}, m)$, where

$$\mathring{\mathbf{H}} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \ldots \end{pmatrix}, \quad \mathring{g}_n = \mathbb{P}(L = n).$$

Observe that the corresponding mean matrix $\mathring{\mathbf{M}}$ is not fully irreducible, since the type 1 is final type.

However, the Perron-Frobenius theorem can be extended to this case as well and the corresponding Perron-Frobenius eigenvectors could be computed using (14) and (15) as

$$\mathring{u}_i = \beta^{-1}(1 + \ldots + R^{i-1}), \; i = 1, 2, \ldots,$$
$$\mathring{v}_j = m(1+m)^{-1}\mathbb{E}(R^{L-j}; L \geq j), \; j = 1, 2, \ldots.$$

It follows

$$\mathring{m}_{ij}^{(n)} \sim \rho^{n+j}(\rho^{-1} + \ldots + \rho^{-i})m(1+m)^{-1}\beta^{-1}\mathbb{E}(\rho^{-L}; L \geq j),$$

thus in the critical case

$$\mathring{m}_{ij}^{(n)} \to i\lambda^{-1}\beta^{-1}\mathbb{P}(L \geq j).$$

Proposition 3 *In the subcritical case when $\rho < 1$, or equivalently $\mu < 1$ we have*

$$\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}) \sim \rho^n (1+m)^{-1}(1-\mu)\mathbf{u},$$

and a convergence in distribution

$$\left[\mathbf{Z}^{(n)} | \mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right] \stackrel{d}{\to} \mathbf{Y}.$$

Limit $\mathbf{Y}$ has a $LF(\tilde{\mathbf{h}}, \tilde{\mathbf{g}}, m)$ distribution with parameters

$$\tilde{\mathbf{h}} = (1+m)(1-\mu)^{-1}\mathbf{v} - (\mu+m)(1-\mu)^{-1}\tilde{\mathbf{g}}, \tag{22}$$

$$\tilde{\mathbf{g}} = \lambda^{-1}(1-\mu)\mathbf{g}(\mathbf{I}-\mathbf{M})^{-1} \tag{23}$$

being independent of the initial type. In particular, for $Y = \mathbf{Y}\mathbf{1}^{\mathrm{t}}$

$$\mathbb{P}(Y = k) = m^{k-1}(1+m)^{-k}, \; k = 1, 2, \ldots. \tag{24}$$

Proposition 4 *In the critical case when $\rho = 1$ we have*

$$\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}) \sim \; n^{-1}(1+m)^{-1}\beta\mathbf{u},$$

and

$$\left[n^{-1}\mathbf{Z}^{(n)} | \mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right] \stackrel{d}{\to} X\mathbf{v},$$

where X is exponentially distributed with mean $(1+m)\beta^{-1}$.

If furthermore a vector $\mathbf{w}$ is such that $\mathbf{v}\mathbf{w}^{\mathrm{t}} = 0$, then

$$\left[n^{-1/2}\mathbf{Z}^{(n)}\mathbf{w}^{\mathrm{t}} | \mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right] \stackrel{d}{\to} Y\sqrt{(1+m)\beta^{-1}\mathbf{w}^2\mathbf{v}^{\mathrm{t}}}, \qquad (25)$$

where $\mathbf{w}^2 = (w_1^2, w_2^2, \ldots)$ and Y has a Laplace distribution with parameter 1.

Proposition 5 *In the supercritical case when $\rho > 1$ we have*

$$\mathbb{P}(\mathbf{Z}^{(n)} \neq \mathbf{0}) \to \ (\rho - 1)(1 + m)^{-1}\beta\mathbf{u},$$

and

$$\left[\rho^{-n}\mathbf{Z}^{(n)} | \mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right] \xrightarrow{d} X\mathbf{v},$$

where X is exponentially distributed with mean $(1 + m)(\rho - 1)^{-1}\beta^{-1}$.
If furthermore a vector $\mathbf{w}$ is such that $\mathbf{v}\mathbf{w}^{\mathrm{t}} = 0$, then

$$\left[\rho^{-n/2}\mathbf{Z}^{(n)}\mathbf{w}^{\mathrm{t}} | \mathbf{Z}^{(n)} \neq \mathbf{0}, \mathbf{Z}^{(0)} = \mathbf{e}_i\right] \xrightarrow{d} Y\sqrt{(1 + m)(\rho - 1)^{-1}\beta^{-1}\mathbf{w}^2\mathbf{v}^{\mathrm{t}}},$$

where $\mathbf{w}^2 = (w_1^2, w_2^2, \ldots)$ and Y has a Laplace distribution with parameter 1.

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