Measure-branching renewal processes

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Abstract

Consider a generalized renewal process where elements are replaced by a random number of new elements. The corresponding generalization of the residual lifetime at $t$ is a random measure $\mu'(du)$ on $[0, \infty)$. The measure-valued process $\{\mu'(du), t \geq 0\}$ is a homogeneous Markov process. We obtain a measure-branching approximation for $\{n^{-1} \mu^T(T' du), t \geq 0\}$ as $n \to \infty$ and $T = r(n) \to \infty$.

Keywords: General branching process; Immigration; Residual lifetime; Measure-branching process

1. Introduction

Consider a population of individuals with a common reproduction law. At its death each individual gives birth to a random number of daughters. The reproduction law of such an individual is the joint distribution of the number of its daughters and their lifelengths. Call this a branching renewal model if the branching property holds: all reproduction acts have independent outcomes.

The word renewal emphasizes our intention to treat this reproduction model as a non-linear renewal process when elements are replaced by a random number of new elements. Write $\mathbb{R}_+ = [0, \infty)$.

- $\mu'(\mathbb{R}_+)$ = the population size at time $t$;
- $\mu'([0,u])$ = the number of individuals at time $t$ who will die by time $t + u$.

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The measure $\mu'(du)$ is a counterpart of the residual lifetime concept of renewal theory. For a branching renewal population the measure-valued process $\{\mu^t, t \geq 0\}$ is a homogeneous Markov process. Call it a measure-branching renewal process.

The aim of this paper is to establish a weak convergence of the form

$$\{n^{-1}\mu^T(tdu), t \geq 0\} \rightarrow \{\pi'(du), t \geq 0\}, \quad n \rightarrow \infty, \quad (1.1)$$

for a suitable time scale $T = T(n)$. A necessary condition for (1.1) is

$$n^{-1}\mu^0(tdu) \rightarrow \pi^0(du), \quad n \rightarrow \infty. \quad (1.2)$$

Throughout we confine ourselves to finite random measures defined on the Borel subsets of $\mathbb{R}$ (Kallenberg (1975)). This facilitates technicalities but causes a nuisance. The initial state $\mu^0$ has to depend on the series number $n$ to ensure

$$P\{|\pi^0(\mathbb{R}+) > 0\} > 0, \quad P\{|\pi^0(\mathbb{R}+) < \infty\} = 1.$$ 

Our limit theorem is based on the following (Dynkin-Lamperti) renewal theorem (cf. Bingham et al. (1987)). Consider a renewal process with a lifetime distribution function $A(t)$ such that

$$\int_0^\infty u \, dA(u) = t^{1-\beta} L(t), \quad \beta \in (0, 1], \quad (1.3)$$

where $L(t)$ varies slowly as $t \rightarrow \infty$. Denote by $m^t(du)$ the residual lifetime distribution at time $t$. Then, there is weak convergence of the probability measures

$$m^T(tdu) \rightarrow M^t(du), \quad T \rightarrow \infty, \quad (1.4)$$

$M^t(du)$ being the Dirac measure $\delta_0(du)$ concentrated at zero if $\beta = 1$, and

$$M^t([0,u]) = 1 - \frac{\sin \pi \beta}{\pi \beta} \int_0^u (t + v)^{-\beta} \, dv, \quad u \geq 0,$$

if $\beta \in (0, 1)$.

A comparison with other branching models shows that the branching renewal model is, in a sense, equivalent to the general (Crump-Mode-Jagers) branching model (cf. Jagers (1975)). It suffices to observe that the dead individuals in a branching renewal population form a general branching process with immigration.

The corresponding immigration process is defined by $\mu^0(du)$, so that condition (1.2) is a condition on immigration (cf. e.g. Badalbaev and Zubkov (1983)). The definition of $\mu^t(du)$ in terms of the general branching model reveals a new Markov structure within general branching framework (cf. Jagers (1989)).

A remarkable fact is that a martingale, introduced for the general branching model by Nerman (1981) (cf. also Jagers and Nerman (1984)), in terms of the measure $\mu^t$ looks particularly simple:

$$e^{-at} \int_0^\infty e^{-xu} \mu^t(du), \quad t \geq 0.$$
where $\alpha$ is the Malthusian parameter. We will use the martingale property of the process $\{\mu'(\mathbb{R}_+), t \geq 0\}$.

**Remark.** Consider a system of particles that move on $\mathbb{R}_+$ towards zero with unit speed. Each particle coming at zero pulls the trigger of a device that casts on $(0, \infty)$ a group of new particles. The measure $\mu'(du)$ could be interpreted as the distribution of the particles on $\mathbb{R}_+$ at time $t$.

### 2. The semigroup $\{V', t \geq 0\}$

Write

$$C_\beta^+ = \{\text{continuous functions } f: \mathbb{R}_+ \to \mathbb{R}_+ \text{ with } \|f\| < \infty\},$$

where $\|\cdot\|$ is the supremum norm. If $\beta \in (0, 1]$ and $f \in C_\beta^+$, then the non-linear integral equation

$$X(t) = \langle f, M' \rangle - \int_0^t X^2(t - u) \, du$$

with

$$\langle f, M' \rangle = \int_0^\infty f(u) M'(du)$$

has no more than one solution $X \in C_\beta^+$. Define the non-linear operator $V': C_\beta^+ \to C_\beta^+$ by

$$V'f(u) = \begin{cases} X(t - u), & 0 \leq u < t, \\ f(u - t), & u \geq t. \end{cases}$$

In this section we verify the correctness of this definition and show that the family $\{V', t \geq 0\}$ forms a semigroup.

For $\beta = 1$ Eq. (2.1) yields

$$V'f(u) = \begin{cases} (t - u + f^{-1}(0))^{-1}, & 0 \leq u < t, \\ f(u - t), & u \geq t. \end{cases}$$

and the semigroup property holds evidently.

Fix $\beta \in (0, 1)$ and $f \in C_\beta^+$. Put $\lambda = (4\|f\|)^{-1}$.

**Lemma 2.1.** There exists a function $X \in C_\beta^+$ complying with Eq. (2.1) for every $t \in [0, \lambda]$.

**Proof.** Denote by $C_f$ the set of all continuous functions $g:[0, \lambda] \to \mathbb{R}_+$ satisfying $g(t) \leq \langle f, M' \rangle, \ t \in [0, \lambda]$.  

Put $\rho(g_1, g_2) = \|g_1 - g_2\|$. It suffices to prove that the operator $K$:

$$Kg(t) = \langle f, M^t \rangle - \int_0^t g^2(t - u) \, du$$

is a contraction of the complete metric space $(C_f, \rho)$ into itself. Clearly,

$$Kg(t) \geq 0, \quad t \in [0, \bar{t}], \quad g \in C_f$$

is the only property of the operator $K$ which needs a proof.

Since the measure $M'(du)$ has a density function, the representation

$$\langle f, M^t \rangle = \int_0^t Bf(t - u) \, du$$

is valid with

$$Bf(t) = \gamma \int_0^\infty f(u)(t + u)^{-\beta} \, du;\quad \gamma = \pi^{-1} \sin \pi \beta$$

This representation implies (2.4):

$$\int_0^t g^2(t - u) \, du \leq \|f\| (Bf) \ast G_\beta \ast G_\beta(t) \leq 2 \|f\| t^\beta (Bf) \ast G_\beta(t) \leq \langle f, M^t \rangle,$$

where $G_\beta(t) = t^\beta$. \hfill \Box

The correctness of the definition (2.2) follows from Lemma 2.1 and the next one.

**Lemma 2.2.** If $X \in C_b^+$ satisfies Eq. (2.1) for all $t \in [0, t_0]$, then so does $X(t)$ for all $t \in [0, 2t_0]$.

**Proof.** Using the well-defined family $\{V', t \in [0, t_0]\}$ denote for $s \in [0, t_0]$

$$Y_s(t) = \begin{cases} X(t), & 0 \leq t \leq s, \\ V_{t-s} \ast f(0), & s < t \leq s + t_0. \end{cases}$$

It suffices to verify that $Y_s(\cdot)$ complies with (2.1) for $t \in (s, s + t_0]$:

$$V_{t-s} \ast f(0) = \langle f, M^t \rangle - \int_0^{t-s} (V_{t-s-u} \ast f(0))^2 \, du - \int_{t-s}^t X^2(t - u) \, du.$$  \hspace{1cm} (2.6)

According to the definition of $\{V', t \in [0, t_0]\}$ we have

$$V_{t-s} \ast f(0) = \langle V_{s}, M^{t-s} \rangle - \int_0^{t-s} (V_{t-s-u} \ast f(0))^2 \, du.$$
Hence the relation (2.6) could be transformed into

\[ \langle V^s f, M^{t-s} \rangle = \langle f, M^t \rangle - \int_{t-s}^t X^2(t-u) \, du. \]  

(2.7)

Due to (2.5) the LHS of (2.7) equals

\[ \int_0^{t-s} BV^sf(t-s-u) \, du \]

and

\[ BV^sf(t-s-u) = \gamma \int_0^s X(s-v)(t-u+v-s)^{-\beta-1} \, dv \]

\[ + \gamma \int_s^{\infty} f(v-s)(t-u+v-s)^{-\beta-1} \, dv \]

\[ = \gamma \int_{t-s}^t X(t-v)(v-u)^{-\beta-1} \, dv + Bf(t-u). \]

Therefore relation 2.7 could be further transformed in

\[ \gamma \int_{t-s}^t X(t-v) \int_0^{t-s} (v-u)^{-\beta-1} \, du \, dv = \int_{t-s}^t \varphi(t-u) \, du, \]

(2.8)

where

\[ \varphi(t) = Bf(t) - X^2(t). \]

Finally, relation (2.8) follows from (cf. (2.1))

\[ X(t) = \int_0^t \varphi(t-u) \, du, \quad t \in [0, t_0], \]

and the equality

\[ \int_0^{z-t+s} \int_0^{t-s} (z-u-y)^{-\beta-1} \, du \, dy = \frac{\pi \beta}{\sin \pi \beta} z^{\beta-1}, \quad z \in [t-s, t]. \]

Now it follows that

\[ V^{t+s} f(0) = X(t+s) = Y_s(t+s) = V^t V^s f(0), \quad s \geq 0, \ t \geq 0. \]

This yields the semigroup property:

\[ V^t V^s f(u) = V^{t-s} V^s f(0) I \{ u < t \} + V^s f(u-t) I \{ u \geq t \} \]

\[ = V^{t+s} u f(0) I \{ u < t \} + f(u-t-s) I \{ u \geq t+s \} \]

\[ = V^{t+s} f(u). \]
3. A limit theorem

Take a group of siblings from a branching renewal population. Denote by $N$ the group size and by $0 < \tau^1 \leq \cdots \leq \tau^N < \infty$ the lifelengths of these siblings. Put

$$N(t) = \max \{j: \tau^j \leq t\}, \quad A(t) = EN(t).$$

Let the branching be critical:

$$EN = 1, \quad \sigma^2 = \text{Var} N \in (0, \infty). \quad (3.1)$$

In the critical case the function $A(\cdot)$ possesses all the properties of a distribution function of a positive random value. In a sense (cf. Nerman (1984)), the function $A(\cdot)$ is the lifelength distribution function for a "typical mother" in the critical branching renewal population.

**Theorem.** Let conditions (3.1) and (1.3) hold, and $T = T(n)$ comply with

$$T^{-\beta} L(T) \sim \frac{\sigma^2 \sin \pi \beta}{2n \pi (1 - \beta)} n \to \infty.$$

Weak convergence of the random measures (1.2) implies weak convergence of the measure-valued homogeneous Markov processes (1.1). The limit $\{\pi'(du), t \geq 0\}$ is a measure-branching process governed by the semigroup $V'$ via

$$E e^{-\langle f, \pi' \rangle} = E e^{-\langle V' f, \pi' \rangle}, \quad t, s \in \mathbb{R}_+, f \in C^+_b.$$

**Corollary.** Put $Z(t) = \mu'(\mathbb{R}_+)$. Under the hypotheses of the theorem the weak convergence

$$\{n^{-1} Z(Tn), t \geq 0\} \to \{\xi(t), t \geq 0\}, \quad n \to \infty,$$

not necessarily Markov processes, takes place.

When $\beta = 1$ and $\pi^0 = \delta_0$ the relation

$$E e^{-\langle f, \pi' \rangle} = e^{-\nu f(0)}$$

and formula (2.3) yields that all the measures $\pi'$ are concentrated at zero:

$$\pi'(du) = \xi(t)\delta_0(du),$$

and the process $\xi(\cdot)$ coincides with a well-known diffusion approximation for branching processes (Athreya and Ney (1972) p. 260).

When $\beta = 1$ and $\pi^0$ is not concentrated at zero, we have

$$\pi'((0, \infty)) = \pi^0((0, \infty)), \quad t \geq 0.$$

If, furthermore, $\pi^0(du)$ has stationary, independent increments, then the process $\{\pi'\{0\}, t \geq 0\}$ is a CBI process of Kawazu and Watanabe (1971).
In the case $\beta \in (0, 1)$ generation overlappings totally distort the usual limit picture. In particular, the measure $\pi^t$, with $\pi^0 = \delta_0$, has no mass at zero at all:

$$E\pi^t(\{0\}) = M^t(\{0\}) = 0, \quad t > 0.$$ 

**Example.** Let $\tau^1, \ldots, \tau^N$ be the numbers of successful trials in a Bernoulli array. If the probability of success at the $i$th trial equals

$$i^{-\rho} \left( \frac{1}{\sum_{k=1}^{\infty} k^{-\rho}} \right)^{-1}, \quad \rho > 1, \quad i = 1, 2, \ldots,$$

then conditions (3.1) and (1.3) hold with $\beta = \min\{1, \rho - 1\}$.

**Remarks.** Measure-branching processes, introduced by Jirina (1962) are known mostly in connection with branching diffusions (cf. Ethier and Kurtz (1986)). The measure-branching process $x^t$ was initially obtained as a limit for the Bellman–Harris branching processes by Sagitov (1991). Bose and Kaj (1991) treated the general branching model in terms of the measure

$$X^t([u_1, u_2]) = \text{the number of individuals in the age interval } [u_1, u_2] \text{ at time } t.$$ 

We end this section by stating an important intermediate result, concerning the Laplace transform

$$Q(t, f) = 1 - E_0 e^{-\langle f, \mu^t \rangle},$$

where $E_0(\cdot)$ stands for $E(\cdot | \mu^0 = \delta_0)$.

The sign $\Rightarrow$ will indicate that convergence is uniform in $t \in [0, t_0]$ for any finite $t_0$.

**Proposition.** Let conditions (3.1) and (1.3) hold. If $g_n \in C_b^+$,

$$\|g_n\| \leq \lambda n^{-1}, \quad \lambda < \infty, \quad n = 1, 2, \ldots,$$

then

$$nQ(T_n, g_n) \Rightarrow V^t g(0), \quad n \to \infty.$$ 

4. The operator $\Psi$

Denote $B[0, 1] = \{\text{Borel functions } f: \mathbb{R}_+ \to [0, 1]\}$.

Define the non-linear operator $\Psi: B[0, 1] \to B[0, 1]$ by

$$\Psi[f](t) \equiv \Psi[f(\cdot)](t) = E\left( \prod_{j=1}^{N(\omega)} (1 - f(t - \tau^j)) - 1 + \sum_{j=1}^{N(\omega)} f(t - \tau^j) \right).$$
To verify that \( \Psi[f] \in B[0, 1] \) for any \( f \in B[0, 1] \) observe that
\[
0 \leq \prod_{j=1}^{n} (1 - b_j) - 1 + \sum_{j=1}^{n} b_j \leq \prod_{j=1}^{n} (1 - a_j) - 1 + \sum_{j=1}^{n} a_j
\]
for \( 0 \leq b_j \leq a_j \leq 1, j = 1, \ldots, n \). The estimate (4.1) shows also that the operator \( \Psi \) is monotone: if \( f, g \in B[0, 1] \) and \( f(t) \leq g(t) \) for all \( t \in \mathbb{R}_+ \), then
\[
\Psi[f](t) \leq \Psi[g](t), \quad t \in \mathbb{R}_+.
\]

**Lemma 4.1.** Let condition (3.1) hold and take \( \varepsilon \in (0, 1) \). If
\[
0 < f(u) \leq c \leq 1, \quad 0 \leq u \leq t,
\]
then
\[
\frac{\sigma^2}{2} \inf_{1 - \varepsilon < v < 1} f^2(tv) - c^2 \rho_1(e, t, c) \leq \Psi[f](t) \leq \frac{\sigma^2}{2} \sup_{1 - \varepsilon < v < 1} f^2(tv) + c^2 \rho_2(e, t)
\]
with
\[
\rho_1(e, t, c) \to 0, \quad t \to \infty, \quad c \to 0^+; \quad \rho_2(e, t) \to 0, \quad t \to \infty.
\]

**Proof.** Use the decomposition
\[
\Psi[f](t) = \Psi[f(t(1 - e) + \cdot)](te) + \Psi^\varepsilon[f](t) + \Psi_\varepsilon[f](t),
\]
where
\[
\Psi^\varepsilon[f](t) = E \left\{ 1 - \prod_{j=1}^{N(te)} (1 - f(t - \tau_j)) \right\} \left\{ 1 - \prod_{j=N(te)+1}^{N(t)} (1 - f(t - \tau_j)) \right\},
\]
\[
\Psi_\varepsilon[f](t) = E \left\{ \prod_{j=N(te)+1}^{N(t)} (1 - f(t - \tau_j)) - 1 + \prod_{j=N(te)+1}^{N(t)} f(t - \tau_j) \right\}.
\]
The monotonicity of \( \Psi \) implies
\[
\Psi[f(t(1 - e) + \cdot)](te) \leq \frac{\sigma^2}{2} \sup_{1 - \varepsilon < v < 1} f^2(tv).
\]
Condition (4.2) yields
\[
\Psi^\varepsilon[f](t) \leq c^2 E N(te)(N(t) - N(te)).
\]
According to (4.1) condition (4.2) yields as well
\[
\Psi_\varepsilon[f](t) \leq E \{ (1 - c)^{N(t) - N(te)} - 1 + c(N(t) - N(te)) \} \leq c^2 F(N(t) - N(te))^2.
\]
These estimates and the decomposition (4.5) show that the asserted upper bound holds with

$$\rho_2(\varepsilon, t) = 2EN(N - N(te)).$$

On the other hand, decomposition (4.5) and the monotonicity of $\Psi$ imply

$$\Psi[f](t) \geq \Psi[f(1 - \varepsilon) + \cdot](te)$$

$$\geq \frac{1}{2} E(N(te)(N(te) - 1)(1 - \varepsilon)^N) \inf_{1 - \varepsilon \leq r \leq 1} f^2(rv).$$

Hence the asserted lower bound is valued with

$$\rho_1(\varepsilon, t, c) = \frac{1}{2} \{\sigma^2 - EN(te)(N(te) - 1)(1 - c)^N\}.$$

Both (4.3) and (4.4) follow from condition (3.1). \qed

5. Proof of the proposition

If $\mu_0 = \delta_0$, then

$$\langle f, \mu' \rangle = \sum_{j=1}^{N(t)} \langle f, \mu_j^{(1)} \rangle + \int_{t}^{\infty} f(u - t) dN(u),$$

where $\mu_j^{(1)}(du)$, $j = 1, \ldots, N$ are the daughter replicas of $\mu'(du)$. This decomposition yields first

$$E_0 \langle f, \mu' \rangle = \int_{0}^{\infty} E_0 \langle f, \mu_{-u}^{(1)} \rangle dA(u) + \int_{t}^{\infty} f(u - t) dA(u),$$

and second (owing to the independency of the daughter processes)

$$1 - Q(t,f) = E \prod_{j=1}^{N(t)} (1 - Q(t - \tau_j, f)) \exp \left( - \int_{t}^{\infty} f(u - t) dN(u) \right).$$

The renewal equation (5.1) reveals that the measure $E_0 \mu'(du)$ is the residual lifetime distribution corresponding to the lifetime distribution function $A(t)$. This fact implies the upper bound

$$Q(t,f) \leq \|f\|$$

and the convergence (cf. (1.4) and (3.3))

$$E_0< g_n, \mu^{(1)} > \to < g, M^{(1)} >, \quad n \to \infty.$$ (5.4)

Relation (5.2) gives the non-linear renewal equation

$$Q(t,f) = \int_{0}^{t} Q(t - u,f) dA(u) + C(t,f) - \Psi[Q(\cdot,f)](t),$$
where
\[ C(t, f) = E\left( 1 - \exp \left( - \int_t^\infty f(u - t) \, dN(u) \right) \right) \prod_{j=1}^{N(t)} (1 - Q(t - \tau^j, f)). \]

In terms of the renewal function
\[ U(t) = \sum_{k=0}^\infty A^*k(t) \]
we have
\[ Q(t, f) = \int_0^t (C(t - u, f) - \mathcal{P}[Q(\cdot, f)](t - u)) \, dU(u). \]

In the end of this section we demonstrate that
\[ n \int_0^{T_n} C(T_n - u, g_n) \, dU(u) \Rightarrow \langle g, M^t \rangle, \quad n \to \infty, \quad t \geq 0. \] (5.5)
under the hypotheses of the proposition. Thus
\[ nQ(T_n, g_n) = \langle g, M^t \rangle - n \int_0^{T_n} \mathcal{P}[Q(\cdot, g_n)](T_n - u) \, dU(u) + \rho_3(t, n) \]
and \( \rho_3(t, n) \to 0 \) as \( n \to \infty. \)

On the other hand, by the definition of the operator \( V^t \)
\[ V^t g(0) = \langle g, M^t \rangle - \int_0^t (V^{t-u} g(0))^2 \, du. \]

To deduce the convergence (3.4) from these two non-linear integral equations, we have to overcome the differences between the integrals involved. Condition (1.3) ensures a regular variation of the renewal function:
\[ U(t) \sim \frac{\sin \pi \beta}{\pi(1 - \beta)} t^\beta L^{-1}(t), \quad t \to \infty. \]

Hence the distinction between the expression under the differential sign is removed by the choice of the time scale \( T \):
\[ n^{-1} U(T_n) \Rightarrow 2 \sigma^2 \, t^\beta, \quad n \to \infty. \] (5.6)

Applying (5.6), we get
\[ nQ(T_n, g_n) - V^t g(0) = \rho_4(t, n) + n^{-1} \int_0^t \left[ \frac{\sigma^2}{2} (V^{t-y} g(0))^2 \right. \]
\[ - n^2 \mathcal{P}[Q(\cdot, g_n)](T(t - y)) \left. \right] dU(T_y) \]
with \( \rho_4(t, n) \to 0 \) as \( n \to \infty. \) (5.7)
Lemma 4.1 together with the upper bounds (5.3) and (3.2) allow us to replace
\( \Psi[Q(\cdot, g_n)] \) by \( \frac{1}{2} \sigma^2 Q^2(\cdot, g_n) \):
\[
\frac{\sigma^2}{2} \sup_{1 - \varepsilon < t < 1} Q^2(Ttv, g_n) - \left( \frac{\lambda}{n} \right)^2 \rho_1 \left( e, Tt, \frac{\lambda}{n} \right) \leq \Psi[Q(\cdot, g_n)](Tt)
\]
\[
\leq \frac{\sigma^2}{2} \sup_{1 - \varepsilon < t < 1} Q^2(Ttv, g_n) + \left( \frac{\lambda}{n} \right)^2 \rho_2(e, Tt).
\]
This double-sided estimate shows that the absolute value of the integral from (5.7) does not exceed
\[
\frac{\sigma^2}{2n} \int_{y-\varepsilon}^{t-\varepsilon} \sup_{1 - \varepsilon < t < 1} |n^2 Q^2(T(t - y)v, g_n) - (V^*(t - y)v g(0))^2| dU(Ty)
\]
plus an expression \( \rho_5(e, t, n) \) complying with
\[
\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \sup_{0 < t < t_0} \rho_5(e, t, n) = 0.
\]
We conclude that
\[
\sup_{0 < t < t_0} |nQ(Tt, g_n) - V^*g(0)| \leq \rho_6(e, n) + 2\lambda t_0^2 \sup_{0 < t < t_0} |nQ(Tt, g_n) - V^*g(0)|
\]
with \( \rho_6(e, n) \to 0 \) as first \( n \to \infty \) and then \( \varepsilon \to 0^+ \). This inequality leads directly to (3.4) if \( t_0^2 < (2\lambda)^{-1} \). When the interval \([0, t_0]\) is large, it has to be splitted in sufficiently small intervals beforehand.

This completes the proof of the proposition.

Proof of (5.5). The difference
\[
C_1(t, f) = \left| C(t, f) - \int_t^\infty f(u - t) dA(u) \right|
\]
does not exceed
\[
E \left[ \exp \left( - \int_t^\infty f(u - t) dN(u) \right) - 1 + \int_t^\infty f(u - t) dN(u) \right] \times E \left( 1 - \exp \left( - \int_t^\infty f(u - t) dN(u) \right) \right) \left( 1 - \prod_{j=1}^{N(t)} (1 - Q(t - \tau_j,f)) \right).
\]
Using the upper bound (5.3), we get
\[
C_1(t, f) \leq 2 \| f \|^2 EN(N - N(t)).
\]
Hence condition (3.2) yields
\[
nC_1(Tt, g_n) \leq 2\lambda^2 n^{-1} EN(N - N(Tt)).
\]
This and (5.6) imply
\[ n \int_0^{T_t} C_1(T_t - u, g_n) \, dU(u) \Rightarrow 0, \quad n \to \infty. \]

It remains to note that
\[ \left| n \int_0^{T_t} \left( C(T_t - u, g_n) \, dU(u) - E_0 \langle g_n, \mu^{T_t} \rangle \right) \right| \leq n \int_0^{T_t} C_1(T_t - u, g_n) \, dU(u) \]
and that the convergence (5.4) holds uniformly in \( t \in [0, t_0] \) for any finite \( t_0 \).

6. Convergence of the finite-dimensional distributions

Let
\[ t_i \in \mathbb{R}_+, \quad f^i \in C_b^+, \quad f^i_n(u) = n^{-1} f^i(u/T), \quad i = 1, 2, \ldots \]
The log-Laplace transform of \( \mu^i \)
\[ W^i f^1(u) = -\log E\{ e^{-\langle f^i, \mu^i \rangle} | \mu_0 = \delta_u \}, \]
where \( \delta_u \) is the Dirac measure concentrated at the point \( u \), because the Proposition complies with
\[ nW^{T_t} f^1_n(0) \Rightarrow V^i f^1(0), \quad n \to \infty. \quad (6.1) \]
The convergence (6.1) and the formula
\[ W^i f^1(u) = \begin{cases} W^{1-n} f^1(0), & 0 \leq u < t, \\ f^1(u - t), & u \geq t, \end{cases} \quad (6.2) \]
yield
\[ nW^{T_t} f^1_n(T_t) \Rightarrow V^i f^1(t), \quad n \to \infty. \]

Using the Proposition once again, we get
\[ nW^{T_t} (f^2_n + W^{T_t} f^1_n)(0) \Rightarrow V^i (f^2 + V^i f^1)(0), \quad n \to \infty. \]
This, in turn, implies
\[ nW^{T_t} (f^2_n + W^{T_t} f^1_n)(T_t) \Rightarrow V^{i2} (f^2 + V^i f^1)(t), \quad n \to \infty. \]

Acting along this scheme, we obtain for \( p = 1, 2, \ldots \)
\[ nW^{T_t} (f^p_n + \cdots + W^{T_t} (f^2_n + W^{T_t} f^1_n) \cdots)(T_t) \Rightarrow V^{i^p} (f^p + \cdots + V^{i2} (f^2 + V^i f^1) \cdots)(t), \quad n \to \infty. \]
This and condition (1.2) yield the asserted convergence of p-dimensional distributions:

\[ E \exp \left\{ - \frac{1}{n} \sum_{i=1}^{p} \int_{0}^{\infty} f_i(u) \mu_{T_{i+}} + \ldots + t_i(T \, du) \right\} \]

\[ = E \exp \left\{ - \left< W_{t} f_n + \ldots + W_{t} f_n + W_{t} f_n \right>, \mu^{0} \right\} \]

\[ \rightarrow E \exp \left\{ - \left< V_{t} f_n + \ldots + V_{t} f_n + V_{t} f_n \right>, \pi^{0} \right\} \]

as \( n \rightarrow \infty \). \( \square \)

7. Tightness

Here we use the approach of Section 7 of Dawson and Fleischmann (1988).

Fix some \( t_0 > 0 \) and \( f \in \mathbb{C}_b^+ \). By a criterion of Roelly-Coppoletta (1986) it suffices to show that the family

\[ \left\{ \eta^{n}(t) = \frac{1}{n} \int_{0}^{\infty} f(u) \mu_{T_{i}}(T \, du), \ t \geq 0, \ n = 1, 2, \ldots \right\} \]

is tight in \( \mathcal{D}(\mathbb{R}^+, \mathbb{R}^+) \). This is true if

\[ \eta^{n}(\tau_n + b_n) - \eta^{n}(\tau_n) \rightarrow 0, \ n \rightarrow \infty, \]

in distribution, where \( b_n \) are positive constants converging to zero as \( n \rightarrow \infty \) and each \( \tau_n \in [0, t_0] \) is a stopping time of the process \( \eta^{n}(\cdot) \) with respect to the usual filtration (Aldous (1978)).

Lemma 7.1. Denote

\[ F_n(b_n) = E \exp \{ - r \eta^{n}(\tau_n) - s \eta^{n}(\tau_n + b_n) \}, \ r, s \geq 0. \]

Conditions (3.1), (1.3) and (1.2) imply

\[ F_n(b_n) - F_n(0) \rightarrow 0, \ n \rightarrow \infty. \]

Proof. By the strong Markov property of the process \( \{ \mu^t, t \geq 0 \} \)

\[ F_n(b_n) = E \exp \{ - \left< f_n + s W_{T_1} f_n, \mu_{T_1} \right> \}, \] (7.1)

where \( f_n(u) = n^{-1} f(u/T) \). For \( t_1 > 0 \) and a natural \( n \) introduce the probability \( P_n \) by

\[ P_n(B) = P(B; \mu^{0}(\mathbb{R}^+) \leq n t_1). \] (7.2)

According to (7.1)

\[ |F_n(b_n) - F_n(0)| \leq P \{ \mu^{0}(\mathbb{R}^+) > n t_1 \} + s E_n \left< |W_{T_1} f_n - f_n|, \mu_{T_1} \right>. \]

The last expectation does not exceed

\[ n t_1 \sup_{0 \leq u \leq T_{t_1}} |W_{T_1} f_n(u) - f_n(u)| + n^{-1} \| f \| E_n \mu_{T_1}((T_{t_1}, \infty)). \] (7.3)
Owing to (6.1), (6.2) and uniform continuity of the function \( f(\cdot) \), the first summand in (7.3) tends to zero as \( n \to \infty \). Estimate the second summand in (7.3) using the evident inequality

\[
\mu^{T_n}(T(t_2, \infty)) \leq \mu^{T_0}(T(t_2 - t_0, \infty)).
\]

As a result, we get

\[
\lim_{n \to \infty} \sup_i |F_n(b_n) - F_n(0)| \leq P(\pi^0(\mathbb{R}_+) \geq t_1) + s \|f\| E\pi^0(t_2 - t_0, \infty) I(\pi^0(\mathbb{R}_+) \leq t_1).
\]

The RHS converges to zero as first \( t_2 \to \infty \) and then \( t_1 \to \infty \). \( \square \)

Take an arbitrary subsequence of \( \{n\} \). It suffices to find a further subsequence \( \{n_k\} \) ensuring

\[
\eta^n(\tau_{n_k} + b_{n_k}) - \eta^n(\tau_{n_k}) \to 0, \quad k \to \infty,
\]

in distribution (Theorem 2.3 from Billingsley (1968)). Take a subsequence \( \{n_k\} \) guaranteeing the weak convergence

\[
\eta^n(\tau_{n_k}) \to \eta, \quad k \to \infty.
\]

By Lemma 7.1 we have

\[
(\eta^n(\tau_{n_k}), \eta^n(\tau_{n_k} + b_{n_k})) \to (\eta, \eta), \quad k \to \infty,
\]

and (7.4) is valid.

It remains to verify the tightness of the sequence \( \{\eta^n(\tau_n)\} \). According to the definition of the process \( \eta^n(\cdot) \)

\[
P\{\eta^n(\tau_n) > t_1^2\} \leq P\left\{ \sup_{0 \leq t \leq t_0} \mu^{T_t}(\mathbb{R}_+) > nt_1^2 / \|f\| \right\}.
\]

The martingale property of the process \( \mu^t(\mathbb{R}_+) \) implies (cf. (7.2))

\[
P\left( \sup_{0 \leq t \leq t_0} \mu^{T_t}(\mathbb{R}_+) > nt_1^2 / \|f\| \right) \leq \|f\| / t_1.
\]

Thus

\[
P\{\eta^n(\tau_n) > t_1^2\} \leq P(\mu^0(\mathbb{R}_+) > nt_1) + \|f\| t_1^{-1},
\]

and the tightness of \( \{\eta^n(\tau_n)\} \) follows from the tightness of \( \{n^{-1}\mu^0(\mathbb{R}_+)\} \). \( \square \)

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