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Reduced branching processes with very heavy tails

(based on a joint paper with Andreas Lagerås, JAP 45 (2008) 190-200)

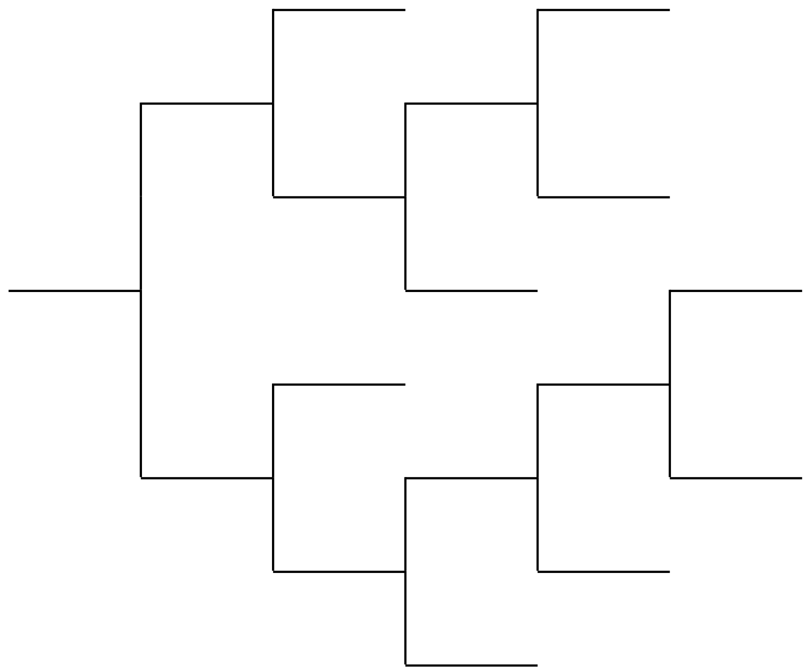
1. Coin-branching process
2. Reduced branching process
3. Markov branching process: $\mu = 1$, $\sigma^2 < \infty$
4. Heavy tail case: $\mu = 1$, $\sigma^2 = \infty$
5. Very heavy tails: $\mu = 1$, $\sigma^2 = \infty$, $\alpha = 0$
6. Simulations

1 Coin-branching process

What one can do with a bag full of coins? Simulate a branching process!

start with one particle $Z_0 = 1$ which produces 0 or 2 children

$Z_t =$ number of particles at generation t



Critical branching paradox:

$Z_t \to 0$ as $t \to \infty$ despite $E(Z_t) = 1$

Asymptotic results:

survival probability $Q_t = P(Z_t > 0) \sim \frac{2}{t}$

linear growth $(Z_t | Z_t > 0) \sim \frac{t}{2}\xi_1$, where $\xi_1 \sim \mathrm{Exp}(1)$

In other words

$$(Z_t Q_t | Z_t > 0) \to \xi_1$$

in full agreement with

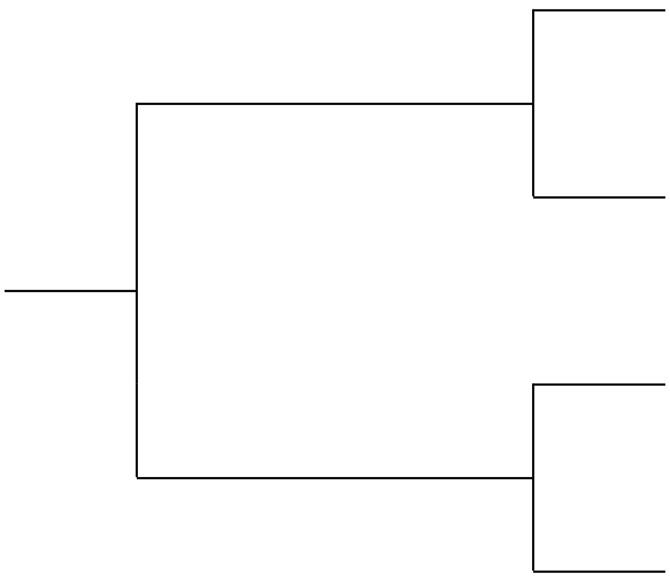
$$E(Z_t Q_t | Z_t > 0) = 1$$

2 Reduced branching process

For a given time of observation t consider historical time $0 \leq u \leq t$

$Z_t(u) =$ number of particles at time u

whose descendants are alive at time t



Zubkov (1975), Fleischmann-Z.Schultze (1977):

$$(Z_t(tx)|Z_t > 0) \to (R_1(x), 0 \leq x < 1)$$

The limit reduced branching process $R_1(x)$ is described by a simple stick-breaking and splitting algorithm

a particle born at time x dies at time $U(x, 1)$

and splits in two new particles

3 Markov branching process: $\mu = 1, \sigma^2 < \infty$

Continuous time branching process:

each particle lives Exp(1) time and then splits in ν particles

Critical case:

offspring number ν has mean $\mu = 1$ and variance σ^2

General asymptotic results

$Q_t = P(Z_t > 0) \sim \frac{2}{\sigma^2 t}$

$(Z_t Q_t | Z_t > 0) \to \xi_1$

$(Z_t(tx) | Z_t > 0) \to (R_1(x), 0 \le x < 1)$

4 Heavy tail case: $\mu = 1$, $\sigma^2 = \infty$

New parameter $0 < \alpha < 1$ regulates the tail of the offspring number distribution:

$$P(\nu > k) \sim c \cdot k^{-1-\alpha}, k \to \infty$$

More general asymptotic results, Zolotarev (1957):

$Q_t \sim b \cdot t^{-1/\alpha}$

polynomial growth $(Z_t Q_t | Z_t > 0) \to \xi_\alpha$

where $E(e^{-\lambda \xi_\alpha}) = \frac{\lambda}{(1+\lambda^\alpha)^{1/\alpha}}$

Yakymiv (1980): weak convergence of reduced BP

$$(Z_t(tx)|Z_t > 0) \to (R_\alpha(x), 0 \le x < 1)$$

The limit reduced branching process $R_\alpha(x)$:

the same stick-breaking mechanism

but a more general splitting rule

$$P(\nu_\alpha = k) = \frac{(1+\alpha)\Gamma(k-1-\alpha)}{k!\Gamma(1-\alpha)}$$

for $k = 2, 3, \ldots$

$$E(\nu_\alpha) = 1 + 1/\alpha$$

5 Very heavy tails: $\mu = 1$, $\sigma^2 = \infty$, $\alpha = 0$

Zubkov's condition with new parameter $0 < \beta < \infty$

$$P(\nu > k) \sim c \cdot k^{-1}(\ln k)^{-1-\beta}, k \to \infty$$

Asymptotic results, Nagaev-Wahtel (2007), Lagerås-Sagitov (2008):

$\ln \frac{1}{Q_t} \sim b \cdot t^{1/(1+\beta)}$

exponential growth $(\ln Z_t | Z_t > 0) \sim a t^{\beta/(1+\beta)^2} \eta_\beta$

where $P(\eta_\beta > x) = e^{-x^{1+\beta}}$

Very heavy tails critical branching paradox:

$(Z_t Q_t | Z_t > 0) \to 0$ despite $E(Z_t Q_t | Z_t > 0) = 1$

the max of $\beta/(1+\beta)^2$ is reached at $\beta = 1$

Lagerås-Sagitov (2008):

$$(Z_t(tx)|Z_t > 0) \to (R_{0,\beta}(x), 0 \le x < 1)$$

The new limit reduced branching process $R_{0,\beta}(x)$:

- the stick-breaking rule $(\tau, 1-\tau)$ depends on β

  Zubkov (1975): $\tau \sim Beta(1, \frac{\beta}{1+\beta})$ with mean $\frac{1+\beta}{1+2\beta}$

- the splitting rule is independent of β

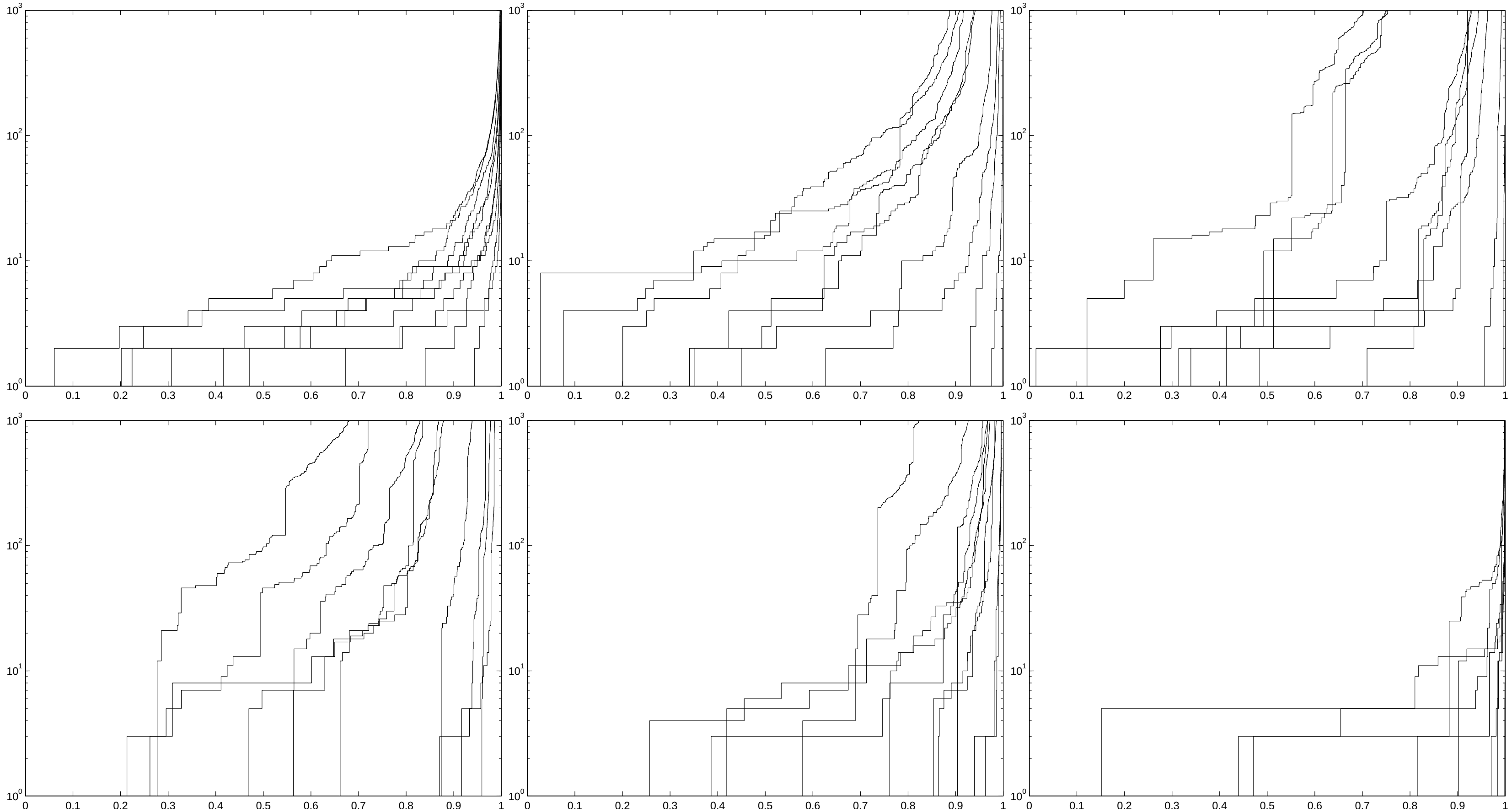
$$P(\nu_0 = k) = \frac{1}{k(k-1)}, \quad E(\nu_0) = \infty$$

6 Simulations

Six blocks with ten realizations each of the limit process for different parameter values

$\alpha = 1$	$\alpha = 0.3$	$\alpha = 0.1$
$\beta = 5$	$\beta = 1$	$\beta = 0.2$

For $\beta = 0.2$, some trajectories are too close to 1 to be visible. Time to the most recent common ancestor is the horizontal distance from the right end of the time interval $[0, 1)$ to the point where the trajectory leaves the state $1 = 10^0$.


10^3
10^2
10^1
10^0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1