Serik Sagitov (June 2006, Sozopol)

Size-biased branching processes with overlapping generations

(based on a paper in progress together with P.Jagers)

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1 Size-biased Galton-Watson process

The Galton-Watson process $\{Z(n)\}_{n\geq 0}$ is a Markov chain describing

- consecutive generation sizes of a population of (mortal) particles
- where particles reproduce independently with a common reproduction law $p_k = P(\nu = k)$, $k \geq 0$ with mean $m = E(\nu)$.

Size-biased GWP: $\widehat{\mathrm{GW}}$P $=$ an immortal particle $+$ GW$\hat{\mathrm{I}}$

GWI $=$ GWP with immigration

- an iid inflow of particles
- it is easier to study the limit behavior of GWI than that of GWP.

Size-biased reproduction law $\hat{p}_k = \frac{kp_k}{m},\ k \geq 1.$

- independent offspring numbers $\nu_1, \ldots, \nu_N$, where $N \gg 1$
- pick a particle from $S_N = \nu_1 + \ldots + \nu_N$ particles, what is the probability that it belongs to a family of size k?
- the number of offspring particles from families of size k

$$\nu_1 1_{\{\nu_1 = k\}} + \ldots + \nu_N 1_{\{\nu_N = k\}} = k \cdot \mathrm{Bin}(N, p_k)$$

- the proportion of offspring particles from families of size k is

$$\frac{k \cdot \mathrm{Bin}(N, p_k)}{S_N} \approx \frac{kNp_k}{Nm} = \frac{kp_k}{m}$$

The immortal particle has the size-biased offspring distribution

- one of the offspring is the immortal particle itself,
  other offspring are mortal particles

- the joint distribution of the offspring number k and the label j
  assigned to the immortal daughter

$$\hat{p}_{k,j} = 1_{\{1 \le j \le k\}} m^{-1} p_k$$

Lyons-Pemantle-Peres [5] approach

- two measures on a tree space: P for GWP and $\hat{P}$ for $\widehat{\mathrm{GW}}$P

- the Radon-Nikodym derivative $d\hat{P}/dP = m^{-n}Z(n)$ on the sigma
  algebra generated by the trees stopped at height n

- translate the easier results on GWI to the GWP

2 CMJ process in discrete time

Age-dependent reproduction law

- an individual can give birth several times during its life
- time unit is a year rather than a generation
- the joint distribution of the offspring number and the ages at birth

$$p_k(n_1, \ldots, n_k),\ k \geq 0,\ 1 \leq n_1 \leq \ldots \leq n_k < \infty$$

- the marginal offspring distribution of the offspring number

$$p_k = \sum_{1 \leq n_1 \leq \ldots \leq n_k < \infty} p_k(n_1, \ldots, n_k)$$

Individual life probability space $(\Omega_0, \mathcal{A}_0, P_0)$

- an element $\omega_0 \in \Omega_0$ is a vector

$$\omega_0 = (n_1, \dots, n_k),\ 1 \leq n_1 \leq \dots \leq n_k < \infty,\ k \geq 0$$

- $P_0(\omega_0) = p_k(n_1, \dots, n_k)$
- the offspring number $\nu = \nu(\omega_0) = k$
- consecutive ages at birth $\tau_i = \tau_i(\omega_0)$

$$\tau_i = n_i,\ i = 1, \dots, k;\ \tau_i = \infty,\ i > k$$

- the joint distribution of the offspring number and the ages at birth

$$P_0(\nu = k, \tau_1 = t_1, \dots, \tau_k = t_k) = p_k(t_1, \dots, t_k)$$

$\mathbb{I} = \cup_{k=0}^{\infty} \mathbb{N}^k$ is the set of all conceivable individuals

- (0) stands for the progenitor
- $(1), (2), (3), \ldots$ progenitor's daughters in the order of appearance
- $(1,1), (1,2), \ldots, (2,1), \ldots$ the granddaughters, and so on

The CMJ process is defined on the product probability space

$$(\Omega, \mathcal{A}, P) = \prod_{x \in \mathbb{I}} (\Omega_x, \mathcal{A}_x, P_0)$$

- $(\Omega_x, \mathcal{A}_x, P_0)$ is a copy of $(\Omega_0, \mathcal{A}_0, P_0)$
- an outcome $\omega_x \in \Omega_x$ describes a potential life $(n_{x1}, \ldots, n_{xk_x})$
- an element $\omega = \{\omega_x\}_{x \in \mathbb{I}} \in \Omega$ contains information about all conceivable individuals, but only some of them will be recruited

3 The size-biased CMJ process

The size-biased life distribution found by Jagers and Nerman [4]

$$\hat{p}_{k,j}(n_1, \ldots, n_k) = 1_{\{1 \le j \le k\}} e^{-\alpha n_j} p_k(n_1, \ldots, n_k)$$

α is the *Malthusian parameter*: $\sum_{n=1}^{\infty} e^{-\alpha n} m_n = 1$

- $m = \sum_{n=1}^{\infty} m_n$ where $m_n =$ mean offspring number at age n
- $\alpha = 0$ if $m = 1$, $\alpha > 0$ if $m > 1$, and $\alpha < 0$ if $m < 1$
- in the GW case $\alpha = \ln m$, all $n_i = 1$, and

$$\hat{p}_{k,j} = 1_{\{1 \le j \le k\}} m^{-1} p_k$$

Enhanced individual life space $(\hat{\Omega}_0, \hat{\mathcal{A}}_0)$

- an element $\hat{\omega}_0 \in \hat{\Omega}_0$ is a vector

  $$\hat{\omega}_0 = (j, n_1, \ldots, n_k),\ 0 \le j \le k,\ 1 \le n_1 \le \ldots \le n_k < \infty,\ k \ge 0$$

- if $j = 0$, then the individual is mortal, otherwise the individual is immortal and j specifies its immortal daughter

- put $\gamma = \gamma(\hat{\omega}_0) = j$

Two probability measures P_0 and $\hat{P}_0$ on $(\hat{\Omega}_0, \hat{\mathcal{A}}_0)$

$$\begin{aligned} P_0(\gamma = j, \nu = k, \tau_1 = n_1, \ldots, \tau_k = n_k) &= p_k(n_1, \ldots, n_k) 1_{\{j=0\}} \\ \hat{P}_0(\gamma = j, \nu = k, \tau_1 = n_1, \ldots, \tau_k = n_k) &= \hat{p}_{k,j}(n_1, \ldots, n_k) \end{aligned}$$

The regeneration age τ_γ of the immortal individual

- $\hat{P}_0(\tau_\gamma = n) = e^{-\alpha n} m_n$
- pick an individual in an old supercritical population and follow backwards in time its lineage, then $\tau_\gamma \approx$ the age at childbearing
- average age at birth for the mortal individuals

$$\beta = \hat{E}_0(\tau_\gamma) = E_0\left(\tau_1 e^{-\alpha \tau_1} + \ldots + \tau_\nu e^{-\alpha \tau_\nu}\right)$$

The size-biased distribution of the offspring number

$$\hat{P}_0(\nu = k) = E_0(\xi \cdot 1_{\{\nu = k\}})$$

- where $\xi = e^{-\alpha \tau_1} + \ldots + e^{-\alpha \tau_\nu}$ [$= \frac{\nu}{m}$ in the GW case]
- $E_0(\xi) = \sum_{n=1}^{\infty} e^{-\alpha n} m_n = 1$

Jagers [3] approach for constructing multitype branching processes based on Tulcea's theorem.

A measure $\hat{P}$ for the size-biased CMJ process is defined on

$$(\hat{\Omega}, \hat{\mathcal{A}}) = \prod_{x \in \mathbb{I}} (\hat{\Omega}_x, \hat{\mathcal{A}}_x)$$

- individual life spaces $(\hat{\Omega}_x, \hat{\mathcal{A}}_x)$ are copies of $(\hat{\Omega}_0, \hat{\mathcal{A}}_0)$
- assign the immortal measure $\hat{P}_0$ to the progenitor's life $(\hat{\Omega}_0, \hat{\mathcal{A}}_0)$
- given $\hat{\omega}_0 = (j, n_1, \ldots, n_k)$ assign $\hat{P}_0$ to $(\hat{\Omega}_j, \hat{\mathcal{A}}_j)$
- given $\hat{\omega}_j = (j', n'_1, \ldots, n'_{k'})$ assign $\hat{P}_0$ to $(\hat{\Omega}_{jj'}, \hat{\mathcal{A}}_{jj'})$ and so on
- assign the mortal measure P_0 to $(\hat{\Omega}_x, \hat{\mathcal{A}}_x)$ off the immortal lineage

4 CMJ trees

A CMJ tree defined on $(\Omega, \mathcal{A}, P) = \prod_{x\in\mathbb{I}} (\Omega_x, \mathcal{A}_x, P_0)$

- $\omega_x \in \Omega_x$ describes a potential branch at vertex x
- let $\sigma_x = \sigma_x(\omega)$ stand for the height of the vertex x: $\sigma_0 = 0$ and

$$\sigma_{xi} = \sigma_x + n_{xi}, \text{ given } \omega_x = (n_{x1}, \dots, n_{xk_x})$$

- the set of branches $T(\omega) = \{\omega_x\}_{x:\sigma_x<\infty}$ builds a CMJ tree
- the tree $T(\omega)$ is independent of $\{\omega_x\}_{x:\sigma_x=\infty}$
- the CMJ tree $[T]_n = \{\omega_x\}_{x:\sigma_x\le n}$ stopped at level n

$$P([T]_n = t) = p_k(n_1, \dots, n_k) \prod_{i:\, n_i \le n} P([T]_{n-n_i} = t^{(i)})$$

A measure on the tree space induced by $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{P})$

- $\hat{\omega}_x = (j, \omega_x)$ describes a potential branch of a tree with an immortal lineage
- $\hat{T}(\hat{\omega}) = \{\hat{\omega}_x\}_{x:\sigma_x<\infty}$ defines a CMJ tree $T(\hat{\omega}) = \{\omega_x\}_{x:\sigma_x<\infty}$ with an immortal lineage $\{j, j', \ldots\}$
- the stopped tree $[\hat{T}]_n = \{\hat{\omega}_x\}_{x:\sigma_x\leq n}$ has a discrete distribution satisfying the recursion

$$\hat{P}([\hat{T}]_n = \hat{t}) = \hat{p}_{k,j}(n_1, \ldots, n_k)\hat{P}([\hat{T}]_{n-n_j} = \hat{t}^{(j)}) \prod_{i\neq j:\, n_i\leq n} P([T]_{n-n_i} = t^{(i)})$$

Radon-Nikodym derivative

- Nerman's martingale $Y_n = \sum_{x \in I_n} e^{-\alpha \sigma_x}$
  the set of leaves $I_n \subset \mathbb{I}$ of the stopped tree $[T]_n$

- the immortal leaf $y \in I_n$ defines the stopped immortal lineage

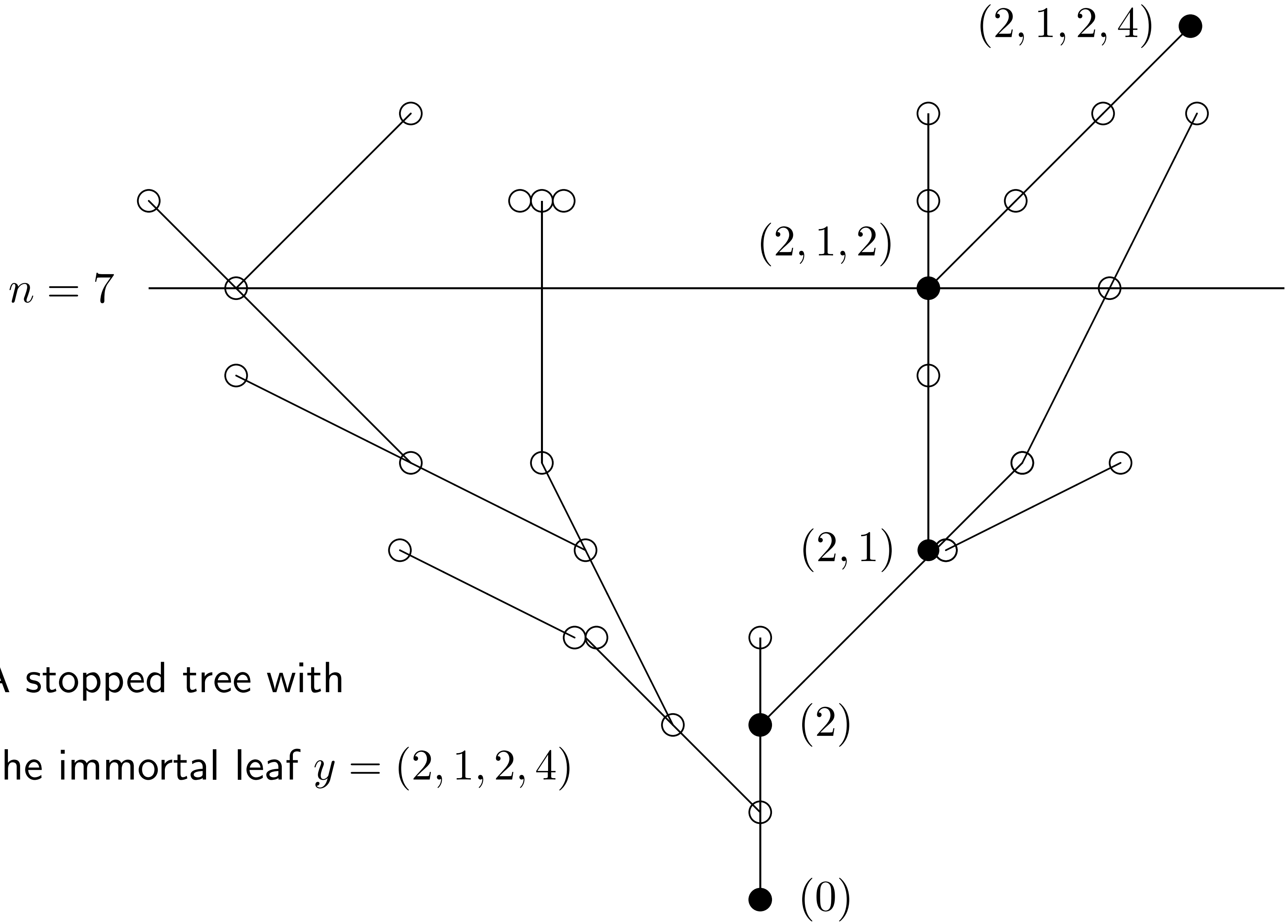
- the two recursions for the stopped trees imply

$$\hat{P}([\hat{T}]_n = (t, y)) = e^{-\alpha \sigma_y} \cdot P([T]_n = t)$$

- summing over possible immortal lineages

$$\hat{P}([T]_n = t) = Y_n \cdot P([T]_n = t)$$

- $\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{B}_n} = Y_n$ on the σ-algebra $\mathcal{B}_n \subset \mathcal{A}$ generated by the complete lives of all individuals born up to time n: even if x is born at n we still include her future in $\mathcal{B}_n$



A stopped tree with

the immortal leaf $y = (2, 1, 2, 4)$

5 $x \log x$ condition

Individual contributions $\chi_x(n) = \chi(n, w_x)$ by $x \in \mathbb{I}$.

Jagers' population sum

$$X_n = \sum_{x:\sigma_x \le n} \chi_x(n - \sigma_x)$$

has different distributions under the probability measures P and $\hat{P}$

- X_n is adapted to the filtration $\mathcal{B}_n$ and $\left.\frac{d\hat{P}}{dP}\right|_{\mathcal{B}_n} = Y_n$
- thus for any positive Borel function g

$$\hat{E}[g(X_n)] = E[Y_n g(X_n)]$$

In the supercritical case

$$(Y_n, e^{-\alpha n} X_n) \to (W, cW),\ n \to \infty$$

and therefore

$$\hat{E}(e^{-\lambda e^{-\alpha n} X_n}) \to E(W e^{-\lambda c W})$$

The limit W is not degenerate provided the $\xi \log \xi$ condition:

$$E(\xi \log \xi) < \infty,$$

where $\xi = e^{-\alpha \tau_1} + \ldots + e^{-\alpha \tau_\nu}$ and $E(\xi \log \xi) = \hat{E}(\log \xi)$

Under $\hat{P}$ the weak convergence $e^{-\alpha n} X_n \to c\hat{W}$ holds with $\hat{E}(e^{-\lambda \hat{W}}) = E(W e^{-\lambda W})$

Since $\nu \geq \xi \geq \nu e^{-\alpha\tau_\nu}$, condition $(\xi \log \xi)$ follows from

$$E(\xi \log \nu) < \infty$$

(a condition arising in the subcritical case) which in turn is weaker than

$$E(\nu \log \nu) < \infty$$

which is equivalent to $(\xi \log \xi)$ given $\tau_\nu \leq$ const.

In general $(\xi \log \xi)$ is not equivalent to $(\nu \log \nu)$

Counterexample: the Sevastyanov model with $\tau_1 = \ldots = \tau_\nu = \tau$

- $\xi = e^{-\alpha\tau}\nu$, $\beta = E(\tau\xi)$, and $E(\xi \log \xi) = E(\xi \log \nu) - \alpha\beta$
- if $\beta < \infty$ condition $(\xi \log \xi)$ becomes equivalent to $(\xi \log \nu)$
- $\tau = \nu$, $E(\nu \log \nu) = \infty$, $E(e^{-\alpha\nu}\nu^2) < \infty \Rightarrow \beta < \infty$ and $(\xi \log \nu)$

In general if $\alpha > 0$ and $\beta < \infty$ (supercritical Malthusian case), then $(\xi \log \xi)$ is equivalent to $(\xi \log \nu)$

A sketch of the proof

- we verify that if $\beta < \infty$ and $\hat{E}(\log \nu) = \infty$, then $E(\xi \log \xi) = \infty$
- turn to the total number of individuals X_n born up to time n
- since $\beta < \infty$, the size-biased process is bounded from below by a GWI process with iid immigration distributed as $(\nu - 1)$ under $\hat{P}$
- given $\hat{E}(\log \nu) = \infty$ we modify the argument in Section 3 of [5] to show $e^{-cn} X_n \to 0$ for the ordinary CMJ process whatever is $c > 0$
- according to [4] it follows that condition $(\xi \log \xi)$ does not hold

6 Further questions

1. In the supercritical case: if $\beta < \infty$ is necessary for the equivalence $(\xi \log \xi) \Leftrightarrow (\xi \log \nu)$?

2. In the critical case obtain alternative proofs for the asymptotic results: classical and Vatutin's discrete theorems

3. In the subcritical case obtain an alternative proof for the limit theorem under condition $(\xi \log \nu)$

4. Alternative proofs for the reduced CMJ processes

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