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# SKELETONS OF NEAR-CRITICAL BRANCHING PROCESSES

Nearly critical branching processes

Yule process approximation for the supercritical skeleton

Skeletons for the critical branching processes

BGW processes with mutation

Probability of escape from extinction

Mutation skeleton and Time to escape

Bienaymé-Galton-Watson process  $\{Z(n)\}_{n \geq 0}$

starts from one particle  $Z(0) = 1$

particles reproduce independently

Single type case

common offspring number distribution  $p_k = \mathbb{P}(Z(1) = k)$

with mean  $m = \sum k p_k$  and variance  $\sigma^2$

Probability generating function $f(s) = \sum_{k=0}^{\infty} p_k s^k$
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Extinction probability  $q$ :

$m < 1$ , *subcritical case*, extinction with probability  $q = 1$

$m = 1$ ,  $p_1 \neq 1$ , *critical case*, extinction w.p.  $q = 1$

$m > 1$ , *supercritical case*, escape from extinction w.p.

$$\alpha = 1 - q, \quad q = f(q), \quad 0 \leq q < 1$$

## NEARLY CRITICAL BRANCHING PROCESSES

Nearly critical case

triangular array  $\{Z^{(\epsilon)}(n)\}$  as  $\epsilon \searrow 0$

convergence to the critical reproduction law  $p_k^{(\epsilon)} \rightarrow p_k$

with mean one and variance  $\sigma^2 \in (0, \infty)$

Key assumption: $\frac{m^{(\epsilon)}-1}{\epsilon} \rightarrow \gamma \in (-\infty, \infty)$
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Haldane's type approximation formula (Haldane's sieve)

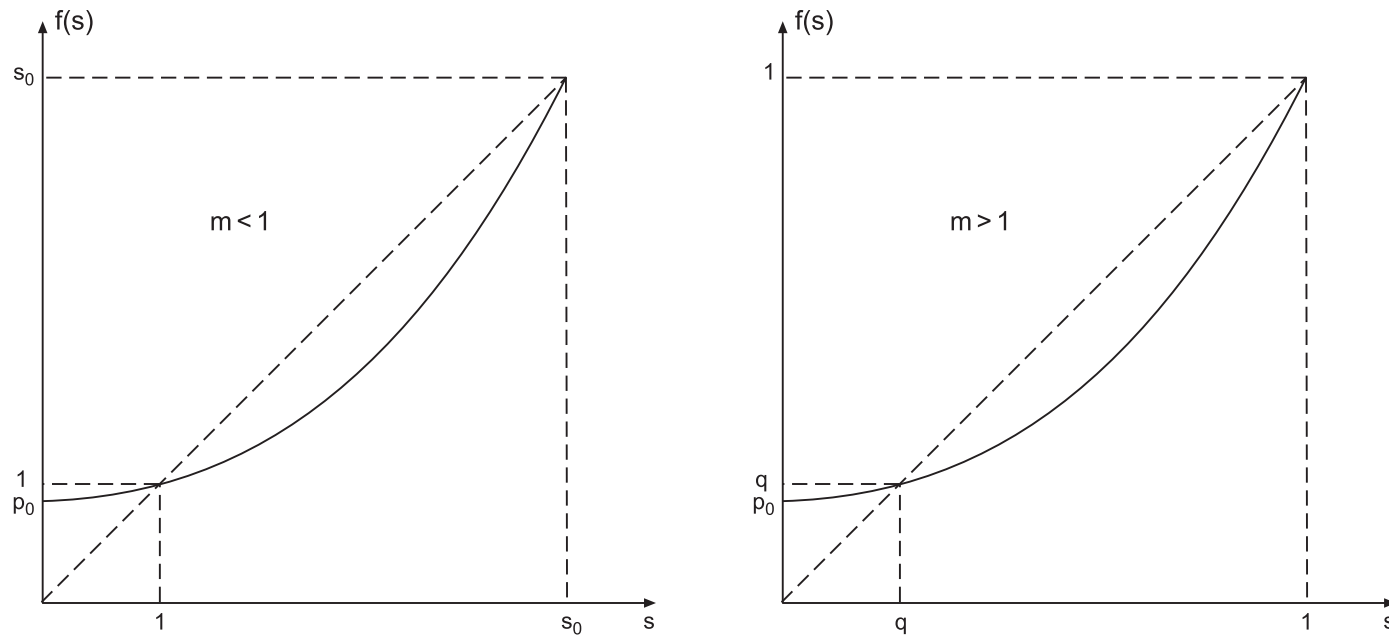
$$\mathbb{P}^\epsilon(\text{escape from extinction}) \sim c_0 \epsilon$$

where

$$c_0 = \begin{cases} 0, & \text{if } \gamma \leq 0 \\ \frac{2\gamma}{\sigma^2}, & \text{if } \gamma > 0 \end{cases}$$

## SUPERCritical SKELETON

The subcritical and supercritical pgfs are drawn in different scales.



Supercritical case:  $Z(n) = Z_+(n) + Z_-(n)$

minus-particles are those whose lineages die out

plus-particles are those whose lineages survive forever

## SUPERCritical SKELETON

The skeleton for the supercritical branching process was defined in

*O'Connell (1993) Yule process approximation of a branching process*

as the branching process  $\{Z_+(n)\}$  formed by particles whose lineages survive forever conditional on the escape event.

Consider the nearly critical case

$$\frac{m^{(\epsilon)} - 1}{\epsilon} \rightarrow \gamma \in [0, \infty)$$

If  $\gamma > 0$ , then given that the initial particle is a plus-particle

$$Z_+([t/\epsilon]) \rightarrow S(t), \quad \epsilon \rightarrow 0$$

where  $S(t)$  is the Yule process of rate  $\gamma$ .

If $\gamma = 0$ , the limit skeleton is trivial: $S(t) \equiv 1$
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## CRITICAL SKELETON

Another definition of the skeleton in the critical case:

*Sagitov (1997) Limit skeleton for critical CMJ branching processes*

Plus-particles are those with  $\geq N$  descendants

$$P(Z_+(0) = 1) \sim \sqrt{\frac{2}{\pi\sigma^2 N}}, \quad N \rightarrow \infty$$

Given that the initial particle is a plus-particle:  $Z_+(0) = 1$

$$Z_+([t\sqrt{N}]) \rightarrow S(t), \quad N \rightarrow \infty$$

where  $S(t)$  is the critical Yule process of rate  $\frac{\sigma}{\sqrt{2\pi}}$ :

after an  $\text{Exp}(\frac{\sigma}{\sqrt{2\pi}})$  life length either 0 or 2 offspring

As  $\sigma \rightarrow 0$ , the limit skeleton becomes trivial:  $S(t) \equiv 1$

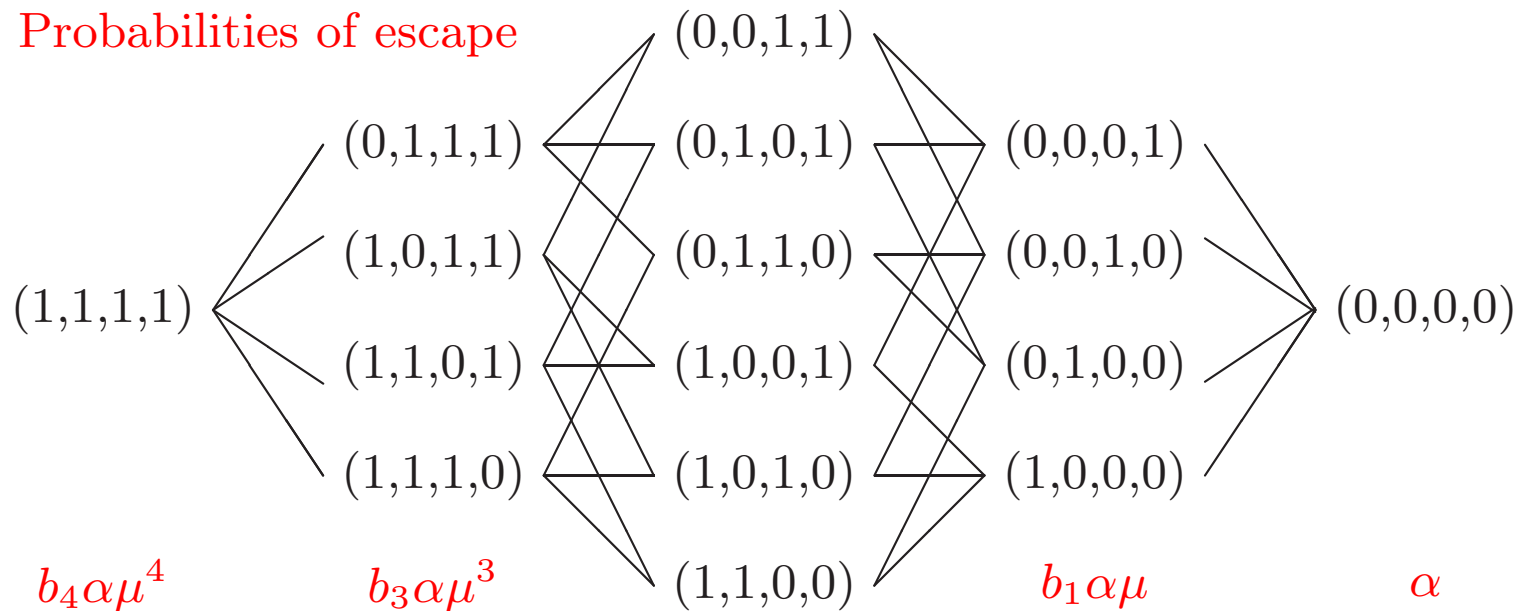
## BGW PROCESSES WITH MUTATIONS

Motivated by

*Iwasa et al (2004) Evolutionary dynamics of invasion and escape*

we have studied the SUBCRITICAL case with escape mutants in

*Sagitov and Serra (2009) Multitype BGW processes escaping extinction*



## ONE MUTATION STEP

Important special case: two types of particles

wild type particles and mutants escaping extinction w.p.  $\alpha$

Conditional mutation probability

$$\mu \cdot a_k = \mathbb{P}(\text{mutation of a particle born to a family of } k \text{ children})$$

Assume  $\sum k p_k a_k = 1$ , then

$\mu$  is the total mutation rate per particle per generation

Key questions:

asymptotical probability of escape as  $\mu \rightarrow 0$

the point process of mutation times

time to escape  $T_\alpha =$  time to the first escape mutant

A small prob. of back mutations does not change asymp. results



## MUTATION SKELETON

Wild type particles form a two-type BGW process  $(Z_+(n), Z_-(n))$ :

plus-particles that have a mutation in their line of descent

minus-particles that have no mutation in their line of descent

$$\mathbb{P}^\mu(Z_+(0) = 1) = \mathbb{P}^\mu(\text{mutation}) \sim \frac{\mu}{1-m}$$

$$\mathbb{P}^\mu(Z_-(0) = 1) = \mathbb{P}^\mu(\text{no mutation})$$

BGW process  $\{Z_+(n)\}$  will be called the mutation skeleton.

In the subcritical case with one mutation step

mutation skeleton is approximated by a BGW process with 0

or 1 offspring w.p.  $1 - m$  and  $m$

Time to escape:  $P(T_\alpha < \infty) \sim \frac{\mu\alpha}{1-m}$

$(T_\alpha | T_\alpha < \infty)$  has asymptotically  $\text{Geom}(1 - m)$  distribution

Several mutation steps: sum of a random number of geometric r.v.

## NEARLY CRITICAL CASE

In the ongoing joint project with M.C.Serra we study the nearly critical case:  $p_k^{(\mu)} \rightarrow p_k$  with mean  $m^{(\mu)} \rightarrow 1$ .

$$\text{Key assumption: } \frac{m^{(\mu)} - 1}{\sqrt{\mu}} \rightarrow \gamma \in (-\infty, \infty)$$

For a given escape probability  $\alpha$  for a mutant particle

$$\mathbb{P}^\mu(\text{escape}) \sim c_\alpha \sqrt{\mu}$$

where

$$c_\alpha = \frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}$$

There are two mutually exclusive ways to escape

r-escape: reproductive success of the wild-type particles

m-escape: escape solely through mutation

## PROBABILITY OF ESCAPE

Haldane's sieve

$$\mathbb{P}^\mu(\text{r-escape}) \sim c_0 \sqrt{\mu}, \quad c_0 = \begin{cases} 0, & \text{if } \gamma \leq 0 \\ \frac{2\gamma}{\sigma^2}, & \text{if } \gamma > 0 \end{cases}$$

Thus (notice the symmetry between the sub- and supercritical cases):

$$\mathbb{P}^\mu(\text{m-escape}) \sim (c_\alpha - c_0) \sqrt{\mu}, \quad c_\alpha - c_0 = \frac{\sqrt{\gamma^2 + 2\alpha\sigma^2} - |\gamma|}{\sigma^2}$$

If  $\alpha = 1$ , then

$$\{\text{escape}\} = \{\text{r-escape}\} \cup \{\text{mutation}\} = \{\text{mutation}\}$$

and therefore for any  $\alpha$

$$\mathbb{P}^\mu(\text{mutation}) \sim c_1 \sqrt{\mu}$$

$$\mathbb{P}^\mu(\text{mutation without r-escape}) \sim (c_1 - c_0) \sqrt{\mu}$$

## MUTATION SKELETON APPROXIMATION

Two offspring distribution pdfs:

$$\mathbb{E}_-^\mu \left( s^{Z_+(1)} u^{Z_-(1)} \right) = f(u) + o(1)$$

$$\begin{aligned} \mathbb{E}_+^\mu \left( s^{Z_+(1)} u^{Z_-(1)} \right) &= (1 - \lambda_1 \sqrt{\mu}) s (f_1(u) + o_u(1)) \\ &\quad + \lambda_1 \sqrt{\mu} \{ (1 - p)\psi(u) + ps^2 f_2(u) \} + o(\sqrt{\mu}) \end{aligned}$$

Key notation:  $\lambda_\alpha = \sqrt{\gamma^2 + 2\alpha\sigma^2}$ , in particular  $\lambda_1 = \sqrt{\gamma^2 + 2\sigma^2}$

Three *size-biased* distributions describing the number of siblings to ...

- $f_1(s) = f'(s) = \sum k p_k s^{k-1}$  ... one sampled particle
- $f_2(s) = \frac{f''(s)}{f''(1)} = \frac{1}{\sigma^2} \sum k(k-1) p_k s^{k-2}$  ... two sampled siblings
- $\psi(s) = \sum k p_k a_k s^{k-1}$  ... a mutant

## MUTATION SKELETON APPROXIMATION

Mutation skeleton with the time scale  $\mu^{-1/2}$  is approximated by the continuous time BP:

- exponentially distributed life times with parameter  $\lambda_1$
- 2 daughters w.p.  $p = \frac{1}{2} + \frac{\gamma}{2\lambda_1}$
- 0 daughters w.p.  $1 - p = \frac{1}{2} - \frac{\gamma}{2\lambda_1}$  (a mutation event)

Numbers of minus-offspring are given by the size-biased pgfs

$$f_1(s), \quad f_2(s), \quad \psi(s)$$

If  $\gamma > 0$ , the mutation skeleton survives forever with probability

$$\frac{2\gamma}{\lambda_1 + \gamma} = \frac{c_0}{c_1} = \lim_{\mu \rightarrow 0} \mathbb{P}^\mu(\text{r-escape}|\text{mutation})$$

Let  $T_\alpha$  be the time to escape and put

$$X(t) = \lim_{\mu \rightarrow 0} \mathbb{P}_+^\mu(T_\alpha > t/\sqrt{\mu})$$

Due to the branching property of the mutation skeleton

$$X(t) = e^{-\lambda_1 t} + (1-p)(1-e^{-\lambda_1 t})(1-\alpha) + p \int_0^t X^2(t-u)\lambda_1 e^{-\lambda_1 u} du$$

where  $p = \frac{1}{2} + \frac{\gamma}{2\lambda_1}$ . Multiply by  $e^{\lambda_1 t}$  and take the derivative over  $t$

$$X'(t) + \lambda_1 X(t) = (1-p)(1-\alpha)\lambda_1 + \lambda_1 p X^2(t)$$

Substitution  $Z(t) = \lambda_1 p \left( \frac{1}{2p} - X(t) \right)$  gives

$$Z'(t) = \frac{\gamma^2 + 2\alpha\sigma^2}{4} - Z^2(t) \text{ with } Z(0) = -\frac{\gamma}{2}$$

After solving this ode we derive

$$1 - X(t) = \frac{c_\alpha}{c_1} \cdot \frac{1 - e^{-\lambda_\alpha t}}{1 + \frac{\lambda_\alpha + \gamma}{\lambda_\alpha - \gamma} e^{-\lambda_\alpha t}}$$

with

$$\lambda_\alpha = \sqrt{\gamma^2 + 2\alpha\sigma^2}, \quad c_\alpha = \frac{\gamma + \lambda_\alpha}{\sigma^2}$$

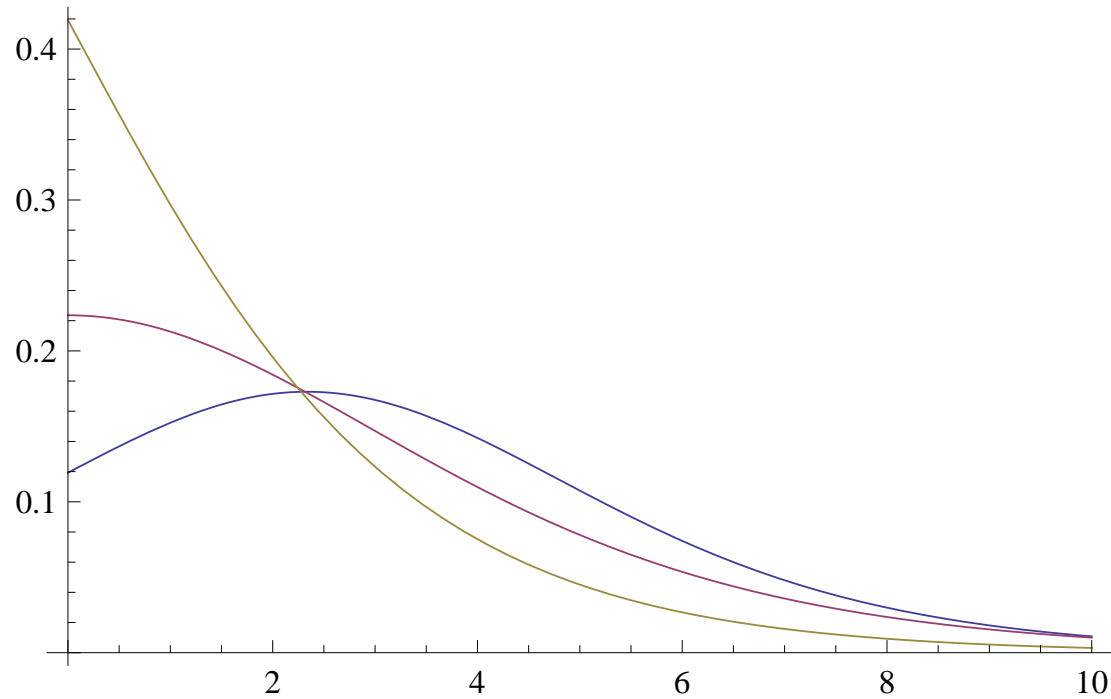
It follows

$$\mathbb{P}^\mu(T_\alpha \leq t/\sqrt{\mu}) \sim c_\alpha \sqrt{\mu} \cdot \frac{1 - e^{-\lambda_\alpha t}}{1 + \frac{\lambda_\alpha + \gamma}{\lambda_\alpha - \gamma} e^{-\lambda_\alpha t}}$$

Thus the *scaled time to escape*  $T_\alpha \sqrt{\mu}$  *conditioned on escape* has a limit distribution with density

$$\phi(t) = 2(\lambda_\alpha - \gamma)e^{-\lambda_\alpha t} \left( \frac{\lambda_\alpha}{\lambda_\alpha - \gamma + (\lambda_\alpha + \gamma)e^{-\lambda_\alpha t}} \right)^2$$

## TIME TO ESCAPE



Three density curves  $\phi(t)$ : subcritical, critical, and supercritical

if  $\gamma \geq 0$ , then  $\phi(t)$  has its maximum at  $t^* = \frac{1}{\lambda_\alpha} \ln \frac{\lambda_\alpha + \gamma}{\lambda_\alpha - \gamma}$

Parameter values:  $\gamma = \pm 0.3$ ,  $\alpha = 0.1$ ,  $\sigma = 1$ .



## TWO MUTATION STEPS

For a given escape probability  $\alpha$  for a mutant particle

$$\mathbb{P}^\mu(\text{escape}) \sim \sqrt{\mu} \cdot \frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2}$$

If mutants form a nearly critical BP with  $\alpha \sim c\sqrt{\mu}$ ,  
then

$$\frac{\gamma + \sqrt{\gamma^2 + 2\alpha\sigma^2}}{\sigma^2} \sim \begin{cases} \frac{2\gamma}{\sigma^2}, & \text{if } \gamma > 0 \\ \frac{\sqrt{2c}}{\sigma} \cdot \mu^{1/4}, & \text{if } \gamma = 0 \\ \frac{c}{|\gamma|} \cdot \sqrt{\mu}, & \text{if } \gamma < 0 \end{cases}$$

and

$$\mathbb{P}^\mu(\text{escape}) \sim \begin{cases} \frac{2\gamma}{\sigma^2} \cdot \sqrt{\mu}, & \text{if } \gamma > 0 \\ \frac{\sqrt{2c}}{\sigma} \cdot \mu^{3/4}, & \text{if } \gamma = 0 \\ \frac{c}{|\gamma|} \cdot \mu, & \text{if } \gamma < 0 \end{cases}$$

## SEVERAL MUTATION STEPS

Related work motivated by

*Y.Iwasa, F.Michor, M.A.Nowak (2004) Stochastic tunnels in evolutionary dynamics*

Moran model the same mutation rate for  $m$  consecutive mutations:

*J.Schweinsberg (2008) The waiting time for  $m$  mutations*

Varying mutation rates in the Moran model:

*R.Durrett, D.Schmidt, J.Schweinsberg (2009) A waiting time problem arising from the study of multi-stage carcinogenesis*

Thank You!