Notes on Intersection Properties of simple random walk

We will now see that there is a very interesting difference between 4 and 5 dimensions for SRW. (We already saw a very important difference between 2 and 3 dimensions).

It is not hard to see (I will discuss this in class) that in 1 or 2 dimensions, if we run to independent rw's, then the two walkers meet each other infinitely many times while in 3-d, this is false. The reason why this is essentially true is the following. If you look at the difference of these two walkers, the difference is "essentially" doing a ordinary rw (check that this is not strictly true) and hence in 1 and 2 dimensions, the difference returns to 0 i.o. but not so in 3 or more dimensions. But, the difference returning to 0 i.o. is the same thing as the two walkers meeting i.o. and so that proves what we wanted.

Let's now ask a different question. Let's run 2 independent rw's in d dimensions and look at the corresponding trajectories (the places where the random walkers visit) and see if these two trajectories intersect in infinitely many places. Of course, in 1 or 2 dimensions, we are recurrent and so these trajectories are everything and so they certainly intersect in infinitely many places. Now consider $d \geq 3$. We know that the 2 walkers meet each other only finitely many times but this does NOT necessarily mean that their trajectories meet only finitely many times.

The important difference to realize here is that if the 2 trajectories intersect i.o., this means that there are infinitely many places which BOTH walkers visit but they need not visit these locations at the same time and so they need not meet i.o.

So it is conceivable that, say in $d = 3$, the trajectories intersect i.o. even though the walkers meet each other finitely many times.

Make sure you understand this difference and therefore agree that the previous scenario is at least possible!!

It turns out that not only is it possible but it is true and there is (again) interesting dimension dependent behavior.

Theorem: *In 3 and 4 dimensions, if we run to independent rw's, then with prob 1, the trajectories intersect i.o. while in $d \geq 5$, this fails.*

We want to give a "more or less" complete proof of this which will rely on one fact which we won't prove. We will only consider $d \geq 3$ here since we already saw that in the recurrent case, this is trivial.

The fact we need is the asymptotic behavior of the so-called Green's function which will now be explained.

Definition:*Let $G_d(x)$ denote the expected number of visits to location x for a rw in d dimensions.*

Note, since we are restricting to $d \geq 3$, the number of visits to a location is finite and therefore $G_d(x)$ is also (why?).

The proposition which we will need but will not prove is the following.

Proposition: $G_d(x) \asymp |x|^{2-d}$ *in the sense that there are positive constants*

C_1 and C_2 such that

$$C_1|x|^{2-d} \leq G_d(x) \leq C_2|x|^{2-d} \forall x \neq 0$$

where $|x|$ is the sum of the absolute values of the different coordinates of x (which is not the usual notion of length but equivalent to it in some sense).
Note that $G_d(x) \to 0$ as $|x| \to \infty$. Why is this reasonable??

Note first that

$$G_d(x) = P[x \text{ is reached }] G_d(0)$$

and hence we also have that

$$P[x \text{ is reached }] \asymp |x|^{2-d}.$$

Let Z be the random variable which is the number of locations which BOTH random walkers visit. Our question is of course whether Z is infinite or not.

What we can check pretty easily is that Z has an infinite expected value in dimensions 3 and 4 but not in $d \geq 5$. We first do this.

Let $\{S'_n\}_{n\geq 0}$ and $\{S''_n\}_{n\geq 0}$ denote the two independent walks and we have

$$Z = \sum_{x\in\mathbf{Z}^d} \mathbf{1}_{\{x\in\{S'_n\}_{n\geq 0}\cap\{S''_n\}_{n\geq 0}\}}.$$

Hence

$$E[Z] = \sum_{x\in\mathbf{Z}^d} P[x \in \{S'_n\}_{n\geq 0}\cap\{S''_n\}_{n\geq 0}] = \sum_{x\in\mathbf{Z}^d} P[x \in \{S'_n\}_{n\geq 0}]P[x \in \{S''_n\}_{n\geq 0}]$$

$$\asymp \sum_{0\neq x\in\mathbf{Z}^d} (|x|^{2-d})^2 = \sum_{n\geq 1} A_n(|n|^{2-d})^2$$

where A_n is the number of points x in the lattice with $|x| = n$.

You should check that A_n is essentially n^{d-1} in which case the above sum becomes

$$\sum_{n\geq 1} n^{d-1}(|n|^{2-d})^2 = \sum_{n\geq 1} n^{3-d}$$

which is finite if $d \geq 5$ and infinite if $d = 3$ or 4.

We have now shown that in $d \geq 5$, $E[Z] < \infty$ which implies that Z is finite a.s. and so a.s. the two trajectories intersect only finitely many times. This proves the theorem in the case $d \geq 5$.

Now, we deal with $d = 3$ and 4. We have seen that in these cases $E[Z] = \infty$. Does this imply that Z is infinite? Not immediately at all. (You should think about an example of a random variable which has an infinite expectation even though it is always finite!!)

To go from the fact that Z has an infinite expectation to the fact that it is infinite, we will need to use the "second moment method" which we briefly described last lecture and will come up often in the last part of this course on applications of probability to combinatorics.

To do this, let

$$Z_n = \sum_{x\in\mathbf{Z}^d} \mathbf{1}_{\{x\in\{S'_i\}_{0\leq i\leq n}\cap\{S''_i\}_{0\leq i\leq n}\}}$$

be the number points common to the 2 trajectories run for n units of time.

The KEY STEP will be to show

$$E[Z_n^2] \leq CE[Z_n]^2$$

for some constant C (which does NOT depend on n).

Let's see why this would do it for us. Of course, like in the 2nd moment method, we will use the Cauchy-Schwartz inequality which says that

$$E[XY] \leq E[X^2]^{\frac{1}{2}}E[Y^2]^{\frac{1}{2}}.$$

Now,

$$E[Z_n] = E[Z_n \mathbf{1}_{\{Z_n > \frac{1}{2}E[Z_n]\}}] + E[Z_n \mathbf{1}_{\{Z_n \le \frac{1}{2}E[Z_n]\}}] \le$$

by Cauchy-Schwartz

$$E[Z_n^2]^{\frac{1}{2}} P[Z_n > \frac{1}{2}E[Z_n]]^{\frac{1}{2}} + \frac{1}{2}E[Z_n]$$

and so

$$E[Z_n] \le 2E[Z_n^2]^{\frac{1}{2}} P[Z_n > \frac{1}{2}E[Z_n]]^{\frac{1}{2}}$$

and so

$$P[Z_n > \frac{1}{2}E[Z_n]] \ge \frac{E[Z_n]^2}{4E[Z_n^2]} \ge \frac{1}{4C}$$

which gives

$$P[Z > \frac{1}{2}E[Z_n]] \ge \frac{1}{4C}.$$

Let E_n be the event that $\{Z > \frac{1}{2}E[Z_n]\}$. We clearly have

$$E_1 \supseteq E_2 \supseteq \dots$$

and $P(E_n) \ge \frac{1}{4C}$ for each n and so

$$P(\cap_{n=1}^{\infty} E_n) \ge \frac{1}{4C}. \quad \text{(Why??)}$$

Now Z_n increases up to Z as $n \to \infty$ and so

$$\lim_{n\to\infty} E[Z_n] = E[Z] = \infty. \quad \text{(Why??)}$$

Therefore $\cap_{n=1}^{\infty} E_n$ implies that $Z = \infty$ finally giving us that

$$P(Z = \infty) \ge \frac{1}{4C}.$$

We have now seen that with positive probability, the 2 trajectories intersect i.o. In fact, it turns out that the 2 trajectories intersect i.o. with probability 1 but this requires some more advanced theorem which I won't do.

Can anyone come up with an easy argument that allows one to conclude from the fact that with positive probability, the 2 trajectories intersect i.o that they actually intersect i.o. with probability 1?

(I haven't thought about this and so I don't know if there is an "easy argument").

We are now left with the KEY STEP which is showing that

$$E[Z_n^2] \leq CE[Z_n]^2$$

for some constant C and all n. We first let $F_n(x)$ be the probability that x is visited during the first n steps of our random walk. We will also use the inequality

$$(a+b)^2 \leq 2(a^2+b^2)$$

Verify this last inequality as well as the first inequality below which might take a little thought.

$$E[Z_n^2] = E[\sum_{x\in\mathbf{Z}^d} \mathbf{1}_{\{x\in\{S_i'\}_{0\leq i\leq n}\cap\{S_i''\}_{0\leq i\leq n}\}} \sum_{y\in\mathbf{Z}^d} \mathbf{1}_{\{y\in\{S_i'\}_{0\leq i\leq n}\cap\{S_i''\}_{0\leq i\leq n}\}}] =$$
$$\sum_{x,y} P[x,y\in\{S_i'\}_{0\leq i\leq n}\cap\{S_i''\}_{0\leq i\leq n}] = \sum_{x,y} P[x,y\in\{S_i'\}_{0\leq i\leq n}]^2 \leq$$
$$\sum_{x,y}[F_n(x)F_n(y-x)+F_n(y)F_n(x-y)]^2 \leq \sum_{x,y} 2[F_n^2(x)F_n^2(y-x)+F_n^2(y)F_n^2(x-y)]$$
$$= \sum_{x,y} 4F_n^2(x)F_n^2(y-x) = 4\sum_x F_n^2(x)(\sum_y F_n^2(y-x)) = 4[\sum_x F_n^2(x)]^2 = 4E[Z_n]^2$$

Verify the last equality!

This shows that $E[Z_n^2] \leq 4E[Z_n]^2$ which is what we wanted to show (so C can be taken to be 4).