A rewriting of an argument of Benjamini, Kalai and Schramm: noise sensitivity for critical percolation and other functions with 'short' randomized algorithms

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February 24, 2012

Abstract

This is more or less just a rewriting of a theorem by Benjamini, Kalai and Schramm. A main point is to emphasize and make explicit that their proof of noise sensitiviy for crossings in critical percolation applies much more generally. Sequences of monotone Boolean functions for which there are randomized algorithms which determine the function but reveal the value of most fixed bits with small probability are noise sensitive. A more minor point is to do some of the arguments in a little more detail in order for this note to be suitable for a graduate class in probability.

1 Introduction

1.1 Influences and noise sensitivity

In [1], it was proved that crossings of percolation in a square are noise sensitive. The argument uses very little about percolation (only the Russo Seymour Welsh Theorem) and applies (with the same proof) to much more general situations. We explain this more general result here which has close connections to theoretical computer science.

We let $[n] := \{1, 2, \ldots, n\}$. A Boolean function is a function from $\{0,1\}^{[n]}$ into $\{0,1\}$ or $\{\pm 1\}$. The influence of the ith bit on such a function f, denoted by $\mathrm{I_i(f)}$, is defined to be the probability (w.r.t. to the other bits) that changing the value of the ith bit changes the value of f. The total influence, $I(f)$ is defined to be $\sum_i \mathrm{I_i(f)}$.

A key result proved in [1] is the following.

Theorem 1.1. *If $\{f_n\}$ is a sequence of Boolean functions on $\{0,1\}^{[n]}$ such that*

$$\Pi(f_n) := \sum_i \mathrm{I_i(f_n)}^2 \to 0 \quad \textit{as } \mathrm{n} \to \infty,$$

then $\{f_n\}$ is noise sensitive.

It is (pretty) elementary (using the easily verified fact that for monotone functions the ith influence is $|\hat{f}(\{i\})|$ and some standard descriptions of noise senstivitiy) that for monotone functions the converse is true. (The parity function shows that the converse cannot in general be true without the monotonicity assumption.)

The proof of Theorem 1.1 rested on some inequalities of Talagrand but it was also explained in [1] that under some stronger condition than $\Pi(f_n) \to 0$, one could prove the above result without these inequalities of Talagrand. I won't write what this stronger assumption is but suffice it to say that it is a weaker assumption than assuming that $\Pi(f_n) \leq C/\log(n)^\alpha$ for some fixed constants C and α. It turns out from this argument (but this is more implicit than explicit in the paper) that if one assumes even further that $\Pi(f_n) \leq C/n^\alpha$ for some fixed constants C and α, then the argument becomes even simpler (although still uses the Bonami-Beckner inequality and Fourier analysis). In addition, under the last condition $\Pi(f_n) \leq C/n^\alpha$, they showed that the spectrum up to $c\log(n)$ goes to 0 for some small enough constant c.

This is all we say about the above result and in particular we will *not* indicate at all the proof of this result. However, we will crucially use it later (but only under the assumption that $\Pi(f_n) \leq C/n^\alpha$ for some fixed constants C and α). Other than that, this note is supposed to be more or less self-contained.

Throughout this note, C will denote a arbitrary constant whose value will change from line to line.

1.2 A simple remark

It is clear (and an elementary exercise) to check that noise sensitive sequences are asymptotically uncorrelated with noise stable sequences. (Just look at the picture of their Fourier spectrums). In fact, one easily can prove the following result (exercise) which just uses the fact that the majority functions are noise stable.

Proposition 1.2. *Given a subset $K \subseteq [n]$, let M_K be the Boolean function on $\{0,1\}^{[n]}$ which is just majority on the bits in K. (This function is 1 when there are more 1's than 0's in K, is -1 when there are more 0's than 1's in K and is 0 if a tie.) If $\{f_n\}$ is noise sensitive, then*

$$\lim_{n\to\infty} \sup_{K\subseteq[n]} E[f_n M_K] = 0.$$

One can ask for the converse of the above proposition. The answer is trivially no since if f_n is $\chi_{\{1,2\}}$ for each n, then f_n is uncorrelated with every M_K since the former is an even function and the latter is odd. Perhaps if we also assume monotonicity, there is a converse. This is the result of the next subsection.

1.3 Weakly correlated with majority implies "noise sensitivity" for monotone functions.

The main result here is the following.

Theorem 1.3. *Let $\Lambda(f) := \max\{|E(fM_K)| : K \subseteq [n]\}$. There exists C such that for all $f : \{0,1\}^{[n]} \to 0,1$ which is monotone,*

$$\Pi(f) \leq C(\Lambda(f))^2(1 - \log(\Lambda(f)) \log n.$$

Remarks: (1). Since f is monotone, the FKG inequality tells us that $E(fM_K) \geq 0$.
(2). Theorem 1.3 states that if the maximum correlation of f_n with all majority functions goes to 0 slightly faster than $1/(\log n)^{1/2}$, then the sequence is noise sensitive provided the functions are monotone.
(3). While we do not discuss it here, if the definition of $\Lambda(f)$ is modified to be the maximum correlation of f with all weighted majority functions, then it was proved in [1] that for monotone functions, (the new) $\Lambda(f_n)$ going to 0 is necessary and sufficient for being noise sensitive.

1.4 Boolean functions with small revealment are noise sensitive

Theorem 1.3 (and part of its proof) will be key in proving the following result.

Theorem 1.4. *Let $\{f_n\}$ be a sequence of Boolean functions on $\{0,1\}^{[n]}$ mapping into $\{0,1\}$. Assume there is an integer B and constants C and δ so that the following holds. For all n, $[n]$ can be partitioned into at most B set $A_1^n, A_2^n, \ldots, A_{k_n}^n$ (so $k_n \leq B$) so that for each $i = 1, \ldots, k_n$, there exists a randomized algorithm $\mathcal{A}^{n,i}$ which queries the bits in $[n]$, one bit at a time (the bit chosen may depend on the outcome of the values of the earlier bits) and then stops at some point such that*
(i) when $\mathcal{A}^{n,i}$ stops, $f_n(\omega)$ is determined and
(ii) for all $j \in A_i^n$, the probability that $\mathcal{A}^{n,i}$ queries bit j is at most C/n^δ.
Then $\{f_n\}$ is noise sensitive.

1.5 Percolation is noise sensitive

Using Theorem 1.4, we will obtain the following theorem by finding appropriate algorithms.

Theorem 1.5. *If A_n is the event that there exists a crossing of an $n \times n$ square in 2-d critical percolation, then the sequence $\{A_n\}$ is noise sensitive.*

2 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The first lemma is a simple bound on tails of the binomial distribution.

Lemma 2.1. *There exist constants C_1, C_2 so that for all n and $\lambda \geq 0$, we have*

$$\frac{1}{2^n} \sum_{k \geq \frac{n+\lambda\sqrt{n}}{2}}^{n} \binom{n}{k}(2k-n) \leq C_1\sqrt{n}e^{-C_2\lambda^2}.$$

Remarks: Note that if X is a Binomial random variable with parameters n and $1/2$, then the left hand side above is just

$$E[(2X-n)I_{\{X\geq \frac{n+\lambda\sqrt{n}}{2}\}}].$$

Proof. We give only an outline. One can use the local central limit theorem to do this. However, we explain 'why' it is true by easily showing the inequality when X is replaced by a normal random variable with the same mean and variance as X. (This suggests, because of the CLT theorem, that the result is true but does not prove it.)

Assuming X is exactly $Z\sqrt{n}/2+n/2$ where Z is a standard normal, then $E[(2X-n)I_{\{X\geq \frac{n+\lambda\sqrt{n}}{2}\}}]$ becomes, after some easy algebra,

$$\sqrt{n}E[ZI_{\{Z\geq\lambda\}}].$$

The last expectation can be trivially computed exactly and it is $(1/\sqrt{2\pi})e^{-\lambda^2/2}$. □

We now need to define the influence of a variable on a function $f:\{0,1\}^{[n]}\to[0,1]$ which is monotone. Here $\mathrm{I_i(f)}$ is then defined to be

$$E(f|x_i=1)-E(f|x_i=0).$$

It is easy to check that if the image is $\{0,1\}$ AND the function is monotone, then this agrees with our earlier definition. As before, the total influence, $I(f)$ is defined to be $\sum_i \mathrm{I_i(f)}$.

Lemma 2.2. *There exists a constant C so that for all n and for all $f:\{0,1\}^{[n]}\to[0,1]$ which is monotone,*

$$I(f)\leq C\sqrt{n}E(fM_n)(1+\sqrt{-\log(E(fM_n))}).$$

Proof. Let $\overline{f}(k)$ be the average of f on the set $\{\sum x_i=k\}$; this is $\binom{n}{k}^{-1}\sum_{|x|=k}f(x)$. It is easy to see that

$$E(fM_n)=2^{-n}\sum_{k>n/2}\binom{n}{k}[\overline{f}(k)-\overline{f}(n-k)].$$

On the other hand,

$$I(f)=2^{-n}\sum_x\sum_j|f(x)-f(x^j)|$$

where x^j is x flipped at j. If f is now monotone, this is

$$2^{-n}2\sum_{(y,w):y\leq w,|w|=|y|+1}(f(w)-f(y)).$$

This is the same as

$$2^{-n+1}\sum_x f(x)|x|-f(x)(n-|x|)$$

since each x comes up as a w in the previous sum $|x|$ times and as a y $n-|x|$ times. This simplifies to

$$
\begin{aligned}
&2^{-n+1}\sum_x f(x)(2|x|-n)\\
&=2^{-n+1}\sum_{k=0}^{n}\binom{n}{k}\overline{f}(k)(2k-n)\\
&=2^{-n+1}\sum_{k>n/2}\binom{n}{k}[\overline{f}(k)-\overline{f}(n-k)](2k-n).
\end{aligned}
$$

Given $\lambda>0$, let $k(\lambda)=\frac{n+\lambda\sqrt{n}}{2}$. We have

$$
\begin{aligned}
I(f)/2 &= 2^{-n}\sum_{k>n/2}^{k(\lambda)}\binom{n}{k}[\overline{f}(k)-\overline{f}(n-k)](2k-n)+2^{-n}\sum_{k>k(\lambda)}^{n}\binom{n}{k}[\overline{f}(k)-\overline{f}(n-k)](2k-n)\\
&\le \lambda\sqrt{n}E[fM_n]+C_1\sqrt{n}e^{-C_2\lambda^2}
\end{aligned}
$$

by Lemma 2.1. Setting

$$\lambda=(1/C_2^{1/2})\sqrt{-\log E[fM_n]},$$

the claim is obtained. $\square$

For $K\subseteq[n]$, we now let $I_K(f)=:\sum_{k\in K}I_k(f)$.

Corollary 2.3. *There exists $C>0$, such that if $f:\{0,1\}^{[n]}\to[0,1]$ is monotone, then for all $K\subseteq[n]$,*

$$I_K(f)\le C\sqrt{|K|}E(fM_K)\left(1+\sqrt{-\log(E(fM_K))}\right).$$

Proof. Assume that $K=\{1,\ldots,m\}$. For $z\in\{0,1\}^m$, let

$$f_K(z)=E[f|\omega=z\text{ on }K]=(1/2^{n-m})\sum_{y\in\{0,1\}^{\{m+1,\ldots,n\}}}f(zy).$$

Here zy means the obvious concatenation of z and y. It is easy to check that f_K is monotone since f is. Next, it is an easy exercise to check that $I(f_K)=I_K(f)$. Next, $E(fM_K)=E(f_KM_K)$; to see this, note f_K is the conditional expectation of f onto the bits in K and M_K is measurable w.r.t. these bits.

Using the above and Lemma 2.2, we then obtain

$$
\begin{aligned}
I_K(f)=I(f_K)&\le C\sqrt{|K|}E(f_KM_K)\left(1+\sqrt{-\log(E(f_KM_K))}\right)\\
&=C\sqrt{|K|}E(fM_K)\left(1+\sqrt{-\log(E(fM_K))}\right).
\end{aligned}
$$

$\square$

Lemma 2.4. *If $c_1 \geq c_2 \geq \cdots \geq c_n > 0$, then*

$$\max\{\sum a_i^2 : a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 : \forall k, \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} c_i\} = \sum c_i^2.$$

We first give the idea of the proof. Existence of a maximum follows from compactness. The key point is that the function x^2 is convex. This has the easy consequence that

$$\max\{a^2 + b^2 : 0 \leq a, b, a + b \leq c\} = c^2.$$

In other words, you should take one term as high as possible. This is "why" the result is true but the proof takes a number of steps since we have to worry about "boundary conditions".

Proof. Compactness implies there exists a maximum $a_1' \geq \cdots \geq a_n'$. Let $G := \{k : \sum_{i=1}^{k} a_i' = \sum_{i=1}^{k} c_i\}$.

Claim 1: G is nonempty.
subproof: If not, lift all the a_i''s by a small ϵ.

Claim 2: (i). If $k \in G$ and $k+1 \notin G$, then $a_{k+1}' < a_k'$. (ii). If $k \notin G$ and $k+1 \in G$, then $a_{k+1}' > a_{k+2}'$.
subproof of (i): $k \in G$ and the constraint at $k-1$ implies $a_k' \geq c_k$. $k \in G$ and $k+1 \notin G$ implies $a_{k+1}' < c_{k+1}$. But $c_{k+1} \leq c_k$.
subproof of (ii): $a_{k+1}' > c_{k+1} \geq c_{k+2} \geq a_{k+2}'$. $k \notin G$ and $k+1 \in G$ implies the first inequality and $k+1 \in G$ implies the last inequality.

Claim 3: $a_n' > 0$.
subproof: Otherwise, we could take all the 0's and lift a little maintaining the constraints (using the c_i's are positive).

Claim 4: If $b_1 \geq \cdots \geq b_{k+m} \geq \delta > 0$, then

$$\sum_{i=1}^{k+m} b_i^2 < \sum_{i=1}^{k} (b_i + \epsilon/k)^2 + \sum_{i=k+1}^{k+m} (b_i - \epsilon/m)^2$$

for small ϵ.
subproof: Consider first order in ϵ on the right hand side. It is positive unless all the b_i's are equal and then in that case the second order in ϵ does the job.

Claim 5: $n \in G$.
subproof: Otherwise, there is $k \in G$ with $k < n$ such that $k+1, \ldots, n \notin G$. Claim 2(i) implies $a_{k+1}' < a_k'$. Now we can lift $a_{k+1}', \ldots, n$ a little maintaining the constraints.

Finally, we let k be the maximum element not in G. Claim 5 gives that $k < n$. Let ℓ be the maximum element (if any) in G which is less than k. Claim 2(i) gives that $a_{\ell+1}' < a_\ell'$. Claim 2 (ii) gives that $a_{k+1}' > a_{k+2}'$ (if $k+1 < n$). We can lift $a_{\ell+1}', \ldots, a_k'$ by $\varepsilon/(k-\ell)$ and lower a_{k+1}' by ϵ. Using Claim 3, for small ϵ, the constraints are maintained and Claim 4 implies the sum of the squares increases. If there is no element in G less than k, modify the above by lifting $a_1', \ldots, a_k'$ instead. $\square$

Proof of Theorem 1.3. Assume without loss of generality that $I_1(f) \geq \ldots \geq I_n(f)$. Corollary 2.3 implies (using that $x(1+\sqrt{\log(1/x)})$ is increasing for small x) that

$$\sum_{i=1}^{k} I_i(f) \leq C\sqrt{k}(\Lambda(f))(1+\sqrt{-\log(\Lambda(f))}).$$

Choose $c_1, \ldots, c_n$ so that for each k, we have

$$\sum_{i=1}^{k} c_i = C\sqrt{k}(\Lambda(f))(1+\sqrt{-\log(\Lambda(f))})$$

Since $\sqrt{x}$ is concave, the c_i's are weakly decreasing. Lemma 2.4 then gives that

$$\Pi(f) \leq \sum_{k=1}^{n} C^2(\Lambda(f))^2(1+\sqrt{-\log(\Lambda(f))})^2(\sqrt{k}-\sqrt{k-1})^2$$

$$\leq \sum_{k=1}^{n} C^2(\Lambda(f))^2(1+\sqrt{-\log(\Lambda(f))})^2(1/k) \leq C_1(\Lambda(f))^2(1-\log(\Lambda(f)))\log n.$$

□

3 Proof of Theorem 1.4

Remark:
We will assume for simplicity that the algorithms are deterministic in that no exterior randomness is used; this is the case in the application of Theorem 1.4 to percolation crossings. The proof can be easily adapted to randomized algorithms. The main modifications in the proof is that x in the proof below should then be a function of ω, ω', z and the exterior randomness (rather than just a function of ω, ω' and z) and that when one conditions on ω, ω', one should also condition on the information of the exterior randomness that one has obtained at the completion of the randomized algorithm.

We first give the idea of the proof of the theorem. For $K \subseteq A_i^n$, the ith algorithm does not hit so many points in K and so f should be fairly uncorrelated with M_K which implies by Lemma 2.2 that I_K is not so large; this is exactly the key lemma. Then, as in the proof of Theorem 1.3, we obtain that $\Pi(f_n)$ is small yielding noise sensitivity.

The following lemma is key. We prove it later.

Lemma 3.1. *Assume the conditions of Theorem 1.4 (except we don't need to make the monotonicity assumption). There exists a constant C_1 so that for all n, for all $i = 1, \ldots, B$ and for all $K \subseteq A_i^n$, we have*

$$E[f_n M_K] \leq C_1(\log n)/n^{\delta/3}.$$

Proof of Theorem 1.4. The above lemma together with Corollary 2.3 (and a tiny computation) implies there is a constant C so that for all n, for all $i = 1, \dots, B$ and for all $K \subseteq A_i^n$,

$$I_K(f_n) \le C\sqrt{|K|}(\log n)^{3/2}/n^{\delta/3}.$$

Since we partition $[n]$ into at most B sets, we have that there exists a constant C_2 so that for all n and for all $K \subseteq [n]$, we have that

$$I_K(f_n) \le C_2\sqrt{|K|}(\log n)^{3/2}/n^{\delta/3}.$$

Now we use the proof method of Theorem 1.3. We assume without loss of generality that $I_i(f_n)$ is nonincreasing in i. We have from the last inequality that for each k

$$\sum_{j=1}^{k} I_j(f_n) \le C_2\sqrt{k}(\log n)^{3/2}/n^{\delta/3}.$$

As in the proof of Theorem 1.3, $\sum_{j=1}^{n} I_j(f_n)^2$ cannot be any larger than when equality holds in the above for all k. Hence

$$\Pi(f_n) \le \sum_{j=1}^{n}(C_2(\log n)^{3/2}/n^{\delta/3})^2((\sqrt{k} - \sqrt{k-1}))^2$$

$$\le C_3[(\log n)^3/n^{2\delta/3}]\log n = C_3(\log n)^4/n^{2\delta/3} \le C_4/n^{\delta/2}.$$

Now apply Theorem 1.1. □

Before starting on the proof of Lemma 3.1, we state without proof two elementary probability facts without proof.

Lemma 3.2. *If $\{S_k\}$ is simple random walk, then for all m and a, we have that*

$$P(S_k \ge a \text{ for some } k \in \{1, \dots, m\}) \le 2e^{-a^2/2m}$$

Lemma 3.3. *There exists a constant C so that if $\{S_k\}$ is simple random walk, then for all r and α, we have that*

$$P(|S_r| \le \alpha) \le C\alpha/\sqrt{r}$$

Proof of Lemma 3.1 Fix n, $i \in \{1, \dots, B\}$, $K \subseteq A_i^n$ and consider the algorithm $\mathcal{A}^{n,i}$. Let ω, ω' and z be independent with ω uniform in $\{0,1\}^{[|K|]}$, ω' uniform in $\{0,1\}^{[n-|K|]}$ and z uniform in $\{0,1\}^{[n]}$. Using these, we will choose a uniform configuration x from $\{0,1\}^{[n]}$ as follows. We run $\mathcal{A}^{n,i}$. When it chooses a bit in K to query, the value that we assign to that bit is the first bit of ω not yet used. When it chooses a bit not in K to query, the value that we assign to it is the first bit of ω' not yet used. Finally assign all bits not yet assigned using z. This final assignment is called x and it is clearly uniform. Since the algorithm determines f_n, we have that $f_n(x)$ is measurable with respect to ω and ω'.

Let V (for visited) be the random set of bits queried by $\mathcal{A}^{n,i}$. By assumption (ii) in Theorem 1.4, we easily obtain

$$E[|V \cap K|] \le |K|C/n^{\delta}.$$

Letting $A_1 := \{|V \cap K| \geq |K|/n^{2\delta/3}\}$, Markov's inequality yields that

$$P(A_1) \leq C/n^{\delta/3}.$$

Next, let

$$A_2 := \{\exists j \in [1, |K|/n^{2\delta/3}] : |\sum_{i=1}^{j} \omega_i - j/2| \geq \sqrt{|K|/n^{2\delta/3}} \log n\}.$$

Lemma 3.2 and a computation yields that

$$P(A_2) \leq C/n^{\delta}.$$

Now, let

$$Q := \{|K \cap V| < |K|/n^{2\delta/3}\} \cap \{|\sum_{i=1}^{|K \cap V|} \omega_j - |K \cap V|/2| < \sqrt{|K|/n^{2\delta/3}} \log n\}.$$

Note Q is measurable with respect to ω, ω' and that $Q^c \subseteq A_1 \cup A_2$ and hence

$$P(Q^c) \leq C/n^{\delta/3}.$$

Now

$$|E[f_n M_K]| \leq |E[f_n I_{Q^c} M_K]| + |E[f_n I_Q M_K]|.$$

The first term is at most $P(Q^c) \leq C/n^{\delta/3}$. The second term is

$$|E[f_n I_Q M_K]| = |E\left[E[f_n I_Q M_K \mid \omega, \omega']\right]|$$

$$= |E\left[f_n E[I_Q M_K \mid \omega, \omega']\right]| \leq E\left[|E[I_Q M_K \mid \omega, \omega']|\right]$$

We claim $(\omega, \omega') \in Q$ implies that

$$|E[M_K \mid \omega, \omega']| \leq C_9 \log n / n^{\delta/3}. \quad (1)$$

This would then give us that $|E(f_n M_K)| \leq C \log n / n^{\delta/3}$, which is the desired result.

Note in the continuation that terms such as $|K \cap V|$ are now no longer random since we have conditioned on (ω, ω') with respect to which $|K \cap V|$ is measurable. Returning to prove (1), the reason this is true is essentially because M_K is only affected by $(\omega, \omega') \in Q$ if the sum of the other bits in K is closer to its mean than $\sqrt{|K|/n^{2\delta/3}} \log n$ but we make this more precise as follows. Note that $|\sum_{i=1}^{|K \cap V|} \omega_j - |K \cap V|/2| < \sqrt{|K|/n^{2\delta/3}} \log n$ implies that the difference (in absolute value) between the number of 1's and 0's in $K \cap V$ is at most $2\sqrt{|K|/n^{2\delta/3}} \log n$. Let W be the difference between the number of 1's and 0's in $K \backslash (K \cap V)$ and we let

$$U = \{|W| > 2\sqrt{|K|/n^{2\delta/3}} \log n\}.$$

Note that U is independent of (ω, ω') and from this, it is easy to see by symmetry that

$$E[M_K I_U \mid \omega, \omega'] = 0.$$

It follows that

$$|E[M_K \mid \omega,\omega']| = |E[M_K I_{U^c} \mid \omega,\omega']| \leq P(U^c) \tag{2}$$

where the independence of U and (ω,ω') is used again in the last inequality.

Now, using Lemma 3.3 for the first inequality below, we obtain

$$P(U^c) \leq C\sqrt{|K|/n^{2\delta/3}} \log n \left(1/\sqrt{|K\backslash(K\cap V)|}\right)$$

$$\leq C\sqrt{|K|/n^{2\delta/3}} \log n \left(1/\sqrt{|K|(1-n^{-2\delta/3})}\right) \leq C\log n/n^{\delta/3}$$

and hence by (2) that $|E[M_K \mid \omega,\omega']| \leq C\log n/n^{\delta/3}$ when $(\omega,\omega') \in Q$ as desired. qed

4 Proof of Theorem 1.5

Proof. We simply apply Theorem 1.4. We take $B = 2$. We take A_1^n to be the right hand side of the box including the center line and take A_2^n to be the left hand side of the box not including the center line.

We consider the following algorithm $\mathcal{A}^{n,1}$. Order all the edges arbitrarily. Let V_1 be the set of vertices on the left side. Choose the first (according to our arbitrary ordering) edge from V_1 to V_1^c and query that edge. If the edge is on, add the vertex of the edge which was in V_1^c to V_1. If not, don't. Continue looking at edges (in order) from V_1 to V_1^c which have not been checked before. (V_1 is then sort of a growing cluster.) Stop when we hit the right side (in which case, we know there is a crossing) or when there are no further edges to check (in which case, we know there is no crossing).

This algorithm clearly determines $f(x)$. If j in is A_1^n, then if j is queried, there there is necessarily an open path from j to distance $n/2$ away. However, RSW (with a little standard stuff in percolation) tells us that the latter event has probability bounded above by C/n^δ for some C and δ. □

Acknowledgment. Thanks to Erik Broman for comments and corrections on an earlier version of these notes.

References

[1] Benjamini, I., Kalai, G. and Schramm, O. Noise sensitivity of Boolean functions and applications to percolation. *Inst. Hautes Études Sci. Publ. Math.* **90**, (1999), 5–43.