Homework 1: Due February 6

(a). You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn.

(b) However, you should write up the final solutions yourselves.

(c) Solutions should (of course) always be correct, clear and concise.

(d) Please also write down with whom you have worked to figure out the problems.

(e) Feel free (obviously) to come and discuss things with me.

1. Show, using countable additivity (and without using the reflection principle), that for every $\epsilon > 0$, there exists an M such that

$$P(|B_t| \geq M \text{ for some t } \in [0,1]) < \epsilon.$$

2.(a). Explain why the definition of BM determines the finite dimensional joint distributions (meaning the joint distribution of $B_{t_1}, B_{t_2}, \ldots, B_{t_k}$ for any finite set of times $t_1, t_2, \ldots, t_k$).

(b). By filling in the details of what I briefly mentioned in the first class, give an example of a process (with an explanation of why it works) which has the same finite dimensional joint distributions as Brownion Motion but such a.s., the path is not a continous function of t.

(c) [OPTIONAL] Improve the example (b) so that the path is nowhere continuous.

3. Show that

$$P(|B_t| = 1 \text{ for some } t \geq 0) = 1.$$

4. Show, using countable additivity, that for all $\epsilon > 0$, there exists $\delta > 0$ so that

$$P(B_t \text{ hits } -\delta \text{ before } 1\ ) \geq 1 - \epsilon.$$

5. Let $I = [a, b]$.

(a). Are the random variables

$$\max_{s \in I} |B_s - B_a|$$

and

$$\max_{s \in I} |B_s - B_b|$$

equal?

(b). Do they have the same distribution?

6. Show that for all $\alpha < 1/2$,

$$P(\limsup_{t \to 0} \frac{B_t}{t^\alpha} = 0) = 1.$$

Hint: First show that

$$P(\limsup_{n \to \infty} \frac{B_{\frac{1}{n}}}{(\frac{1}{n})^\alpha} = 0) = 1$$

and then use the fact (without proving it as we will prove it later) that

$$P(\max_{0 \leq t \leq b} |B_t| \geq a) \leq 4P(B_b \geq a).$$

7. Show that

$$\{\frac{B_t}{t^\alpha} \leq K \ \ \forall t \in (0,1]\}$$

is measurable.

Hint: You need to show that this event can be expressed in terms of just countable many random variables (rather than in terms of uncountable many random variables as it is defined).