Homework 2: Due February 28

(a). You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn.

(b) However, you should write up the final solutions yourselves.

(c) Solutions should (of course) always be correct, clear and concise.

(d) Please also tell me with whom you have worked to figure out the problems.

Note: A part of problem 1 is exercise 2.13 in MP. But I want you to follow the outline below that I suggest instead.

1. (a). Look up what a martingale is and show that for all θ

$$e^{\theta B_t - \theta^2 t/2}$$

is a martingale.

Now assume the following facts (you can look these up).

(i). If $\{M_t\}$ is a martingale and T is an almost surely finite stopping time, then $\{M_{t \wedge T}\}$ is a martingale.

(ii) If $\{M_t\}$ is a (uniformly) bounded continuous martingale and T is an almost surely finite stopping time, then $E[M_T] = E[M_0]$. (This is one easy case of the "optional stopping theorem".)

(b). Let $a > 0$, $b \geq 0$ and $\tau = \inf\{t : B_t = a + bt\}$ and show that for $\lambda > 0$

$$E[e^{-\lambda\tau}] = e^{-a(b+(b^2+2\lambda)^{1/2})}.$$

Hint: Apply (ii) to the martingale given in part (a) with $\theta = b + (b^2 + 2\lambda)^{1/2}$. But do this carefully using truncations since τ is not a.s. finite. (Note the formula you get when $b = 0$.)

(c). Use (b) to show that the probability that τ is finite is e^{-ab}.

(d). Note that the above probability depends only on the product of a and b. Give an alternative argument using Brownian scaling (and none of the above) that the above probability depends only on the product ab (without however deriving the exact formulate for what it is in terms of ab).

2. Exercise 2.12 (b,c) in MP.

3. Recall two processes X_t and Y_t and versions of each other if

$$P(X_t = Y_t) = 1 \ \forall t$$

and we say that two processes X_t and Y_t and indistinguishable if

$$P(X_t = Y_t \ \forall t) = 1.$$

Show that if X and Y are versions of each other and that they each have right continuous paths for all ω, then X and Y are indistinguishable.

4. We have shown that $P(\{t : B_t = 0\}$ has no isolated points $) = 1$.

a. Is it true that for all real a, $P(\{t : B_t = a\}$ has no isolated points $) = 1$.

b. Is it true that $P(\{t : B_t = a\}$ has no isolated points $\forall a) = 1$.

5. The next problem is to get you warmed up for the important so-called second moment method which will be crucial to us later. Things are intentionally slightly vague.

Let X be a nonnegative integer valued random variable.

a. If the mean of X is very small, can you conclude that the probability that X is positive is very small? If yes, explain why. If no, give an example showing that the answer is no.

b. If the mean of X is very large, can you conclude that the probability that X is positive is very large? If yes, explain why. If no, give an example showing that the answer is no.

c. If your answer to b. is no, what extra conditions would allow you to make such a conclusion?