$M_{10} = M(9) \subseteq S_{\mathbb{F}_9 \cup \{\infty\}} = S_{10}$

$$= \begin{cases} x \mapsto \dfrac{ax+b}{cx+d} & ad-bc = \square \\ x \mapsto \dfrac{ax^3+b}{cx^3+d} & ad-bc \neq \square \end{cases}$$

Thm: $\exists\ M_{11} \subseteq S_{11}$ which is sharp 4-transitive

Pf: We show M_{10} has a trans. ext to $M_{11} \subseteq S_{\mathbb{F}_9 \cup \infty \cup w}$. By "trans ext thm", we need to find $x \in \mathbb{F}_9 \cup \{\infty\}$, $g \in M_{10}$, $h \in S_{11}$ s.t.

1. $gx \neq x$. 2. $hw \neq w$. 3. $h^2 \in M_{10}$
4. $(gh)^3 \in M_{10}$. 5. $h(M_{10})_x h = (M_{10})_x$

$x = \infty$, $gx = 1/x$

$h = (w\ \infty)(\pi\ \pi^2)(\pi^3\ \pi^7)(\pi^5\ \pi^6)$

$= (w\ \infty) \quad x \mapsto \pi^2 x + \pi x^3$

1-4 ✓ $\quad \mathbb{F}_9 = \mathbb{Z}_3[x]/_{x^2+x-1}$

$\pi = x$.

$h(M_{10})_\infty h \in (M_{10})_\infty$

NTV $f \in (M_{10})_\infty \Rightarrow hfh \in M_{10}$.

$$f(x) = \begin{cases} \pi^{2i} x + \alpha \\ \pi^{2i+1} x^3 + \alpha \end{cases}$$

Just 1st case $f(x) = \pi^{2i}x + \alpha$

$h \circ f \circ h\,(x) =$

$$(\pi^{2i+4} + \pi^{6i+4})x + (\pi^{2i+3} + \pi^{6i+7})x^3 + \pi^2\alpha + \pi\alpha^3.$$

Cases: i even, i odd. $\pi^8 = 1$

$i = 2j$

$$\pi^{4j+3} + \pi^{12j+7} = \pi^{4j+3}(1 + \underbrace{\pi^4}_{=-1}) = 0.$$

$$\pi^{4j+4} + \pi^{(12j+4)} = 2\pi^{4j+4}$$

$$= (-1)(\pi^{2j+2})^2$$

$$= \square$$

Thm $\exists$ a sharp 5-trans s.g ~~of~~ M_{12} of S_{12} which is a trans ext. of M_{11}. ($\Rightarrow$) $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95040$

Pf. Apply trans ext. thm to M_{11} (a perm. group on ~~F_q~~ $F_q \cup \infty \cup \varpi$.)

We add Ω. $\Rightarrow F_q \cup \infty \cup \omega \cup \Omega$.

Let $x = \omega$. Let h be as in prev. thm. This plays the role of g in ext. thml

Let $K = (\omega\,\Omega)(\pi\,\pi^3)(\pi^2\,\pi^6)(\pi^5\,\pi^7)(1)(\pi^4)(0)(\infty)$

(1-5) ✓. 1. $h(\omega) \neq \omega$. 2. $K(\Omega) \neq \Omega$.

3. $K^2 = 1$ 4. $(hK)^3 = 1$.

$hK = \dfrac{3 \text{ cycles} + 1 \text{ cycles}}{\text{cycle class rep}}$

$(a+b)^3 = a^3 + b^3$

5. $K(M_{11})_\omega K \subseteq (M_{11})_\omega$.

M_{10} NTS $f \in (M_{11})_\omega \Rightarrow KfK \in M_{11}$

$K = (\omega, \Omega)\ (x \to x^3)$ ✓

$$f(x) = \begin{cases} \dfrac{ax+b}{cx+d} & ad - bc = \square \\ \dfrac{ax^3+b}{cx^3+d} & ad - bc = \cancel{\square} \end{cases}$$

$K \circ f \circ K$ 2 cases. Do only case 1

$$x \xrightarrow{K} x^3 \xrightarrow{f} \frac{ax^3+b}{cx^3+d} \xrightarrow{K}$$

$\dfrac{a^3x + b^3}{c^3x + d^3}$, need det $= \square$

det $= a^3d^3 - c^3 ~~ad~~ b^3$

$= (ad - bc)^3 = \square^3 = \square$

case 2: ~~d~~ similar $\square$

Thm: Let $G \leq S_n$ nontriv. + k-trans.
(1) If $k=4$, then $G = M_{11}, M_{12}$ or one of 2 further groups $M_{23} \leq S_{23}$ or $M_{24} \leq S_{24}$.
(2) If $k=5$, $G = M_{12}$ or M_{24}
(3) $k \geq 6 \Rightarrow$ can't happen.
Much harder than Jordan.
needs CFSG.

Mathieu groups $M_{11}, M_{12}, M_{23}, M_{24}$
5th group is $M_{22} \leq S_{22}$.
3 transitive.

$$PSL(3,4) \leq M_{22} \leq M_{23} \leq M_{24}$$
$\leq S_{21}$ ($\underbrace{\text{2-trans sharp}}$), $\leq S_{22}$, $\leq S_{23}$, $\leq S_{24}$

Linear groups. $|F| = q$
F = Field
$GL(n,F) = \{$invert linear maps from $F^n \to F^n\}$
$SL(n,F) = \{ \ldots \det = 1\}$
$PGL(n,F) = GL(n,F)/\text{center}$ $\overset{Ex}{=} GL(n,F)/\{\text{scalar matrix } cI\}$
$PSL(n,F) = \dfrac{SL(n,F)}{\text{center}} = \ldots \{cI\}$ $c^n = 1$

$|GL(n,F)| = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})$.

~~$SL(n,F)$~~
$GL(n,F) \longrightarrow F^*$
$M \longmapsto \det M$.
Kernel $= SL(n,F)$
$\Rightarrow |SL(n,F)| = \dfrac{|GL(n,F)|}{|F^*| = q-1}$

$|PGL(n,F)| = \dfrac{|GL(n,F)|}{|F^*|}$.
$= |SL(n,F)|$

$|PSL(n,F)| = \dfrac{|SL(n,F)|}{\gcd(n,q-1)}$

$|PGL(2,9)|$ $(= L(9))$
$= \dfrac{GL(2,9)}{8} = \dfrac{(81-1)(81-9)}{8} = 720$

$PSL(3,4) = \dfrac{|SL(3,4)|}{\gcd(3,4-1)} = \dfrac{|SL(3,4)|}{3} = \dfrac{|GL(3,4)|}{9}$
$= \dfrac{(64-1)(64-4)(64-16)}{9} = \ldots 20{,}160$

We will see $PGL(2,9) \subseteq S_{10}$

and $PSL(3,4) \subseteq S_{21}$

Motivation for defn

$GL(n,F)$ acts trans. on F^n.

But $|q| \geq 3$ not primitive

since 1-d ss. $\setminus\{0\}$ are blocks

Reduce blocks to pts.

$\Rightarrow$ PF^n = projective space

$$|PF^n| = (q^n - 1)/q-1$$

$GL(n,F)$ acts PF^n.

Problem: not faithful

Thm: Kernel of this action is scalar matrices.

$\Rightarrow$ $PGL(n,F)$ acts faithful on PF^n $\Rightarrow$ $PGL(n,F) \subseteq S_{PF^n}$.

$n=2$ $|F|=9$. $PF^n = \frac{9^2-1}{8} = 10$

$\Rightarrow PGL(2,9) \subseteq S_{10}$

Same thm for special linear groups

$PSL(n,F)$ acts faith. on PF^n.

$PSL(3,4) \subseteq S_{|PF^n|} \cong S_{21}$

$n=3$
$|F|=4$
$= \frac{4^3-1}{3} = 21$

pf Kernel = scalar matrix

$\supseteq$ ✓

$\subseteq$ assume $M \in GL(n,q)$
$\in$ Kernel of
$GL(n,q)$ acting on PF^n.

~~Thm~~ ~~PSL~~ $v_1 - v_n$ basis

$Mv_i = \alpha_i v_i$ NTS α_i's same.

$M(v_1+v_2) = c(v_1+v_2)$ $c \neq 0$.

$= \alpha_1 v_1 + \alpha_2 v_2$

$v_1 v_2$ lin indep $\Rightarrow \alpha_1 = \alpha_2$.

$\square$