Where are we? Done with ~~[illegible]~~, M_{11}, M_{12}. Today do $M_{22,23,24}$.

Build by successive trans. extensions starting from $PSL(3,4)$

$$PSL(3,4) = SL(3,4)/\{cI : c \neq 0\} \qquad |\ \ | = 20{,}160.$$

$$PSL(3,4) \subseteq S_{21} = S_{P(\mathbb{F}_4^3)} \longrightarrow \text{set of 1-d ss. of } \mathbb{F}_4^3$$

$$|\ \ | = \frac{63}{3} = 21$$

$SL(3,4)$ action on $P(F_4^3)$,
Kernel $= \{cI : c \neq 0\}$

Thm $\forall n \geq 2$ $SL(n,F)$ or $PSL(n,F)$ is 2-trans. on $P(F_4^3)$

Pf Let v_1, v_2 be lin. indep.
$w_1 w_2$ - - - -

Extend to bases $v_1 v_2 \ldots v_n$, $w_1 \ldots w_n$.
$\exists M \in GL(n,F)$ s.t $M(v_i) = w_i$ $\forall i$
$\Rightarrow GL(n,F)$ 2-trans.

$M(\langle v_1 \rangle) = \langle w_1 \rangle \longrightarrow$
$M(\langle v_2 \rangle) = \langle w_2 \rangle$.

Let T: $w_1 \longrightarrow w_1/\mathrm{Det}M$
$w_2 \longrightarrow w_2$
$\vdots$
$w_n \longrightarrow w_n$

$T \circ M \langle v_1 \rangle = \langle w_1 \rangle$
$T \circ M \langle v_2 \rangle = \langle w_2 \rangle$
$\mathrm{Det}(T \circ M) = 1$

not sharp 2-tran since
$20{,}160 > 21 \cdot 20$

3 trans. ?
$20{,}160 > 21 \cdot 20 \cdot 19$
7920

$\exists$ matrix s.t
$\langle v_1 \rangle \longrightarrow \langle v_1 \rangle$
$\langle v_2 \rangle \longrightarrow \langle v_2 \rangle$
$\langle v_1 + v_2 \rangle \longrightarrow \langle v_3 \rangle$

$v_1 v_2 v_3$ Lin. indep

Thm $\exists$ $M_{22} \in S_{22}$ which is a t.e. of $PSL(3,4)$ which therefore is 3-trans and has size $22 \cdot |PSL| = 443{,}520$

Pf The 1-d ss. of F_4^3 are represented by $\{[xyz]\}$ $\langle x,y,z \rangle$.
Call this X. Extend to $X \cup \{\infty\}$
$x = [100]$ $g([xyz]) = [yxz]$
$h_1 = ([100], \infty)$ f_1
$f_1([u,v,w]) = [u^2 + vw, v^2, w^2]$
f_1 welldefined since 2-homo.
prob V $g \in PSL(3,4)$
$g = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\det g = -1 = 1$
char 2.

1. $g(x) \neq x$ $\quad g([100]) = [010] \neq [100]$.

2. $h([00]) = [100]$ ✓

3. $h^2 \in G$

$[u,v,w] \xrightarrow{h} [u^2+vw, v^2, w^2]$

$\xrightarrow{h} [(u^2+vw)^2 + v^2w^2, v^4, w^4]$

$= [u^4 + \underbrace{v^2w^2 + v^2w^2}_{=0}, v^4, w^4]$

$= [u, v, w]$

4. $h(gh)^3 \in G$. In fact $(gh)^3 = 1$.

5. $h_1 \, PSL(3,4)_{[100]} \, h_1 \subseteq PSL(3,4)|_{[100]}$

LHS fixes $[100]$. NTS

$K \in PSL(3,4)_{[100]} \Rightarrow h_1 K h_1 \in PSL(3,4)$

K rep. $\begin{bmatrix} 1 & * & * \\ 0 & a & b \\ 0 & c & d \end{bmatrix}$ $\quad ad - bc = 1$

claim $h_1 K h_2 = \begin{bmatrix} 1 & * & * \\ 0 & a^2 & b^2 \\ 0 & c^2 & d^2 \end{bmatrix}$

$\text{Det} \rightarrow \; = a^2d^2 - c^2b^2$

$= (ad - bc)^2 = 1.$

$\Rightarrow h_1 K h_2 \in PSL(3,4)$.

$M_{22} = \langle PSL(3,4), h_1 \rangle$ $\quad \square$

Thm: $\exists$ a 4 trans sg. $M_{23} \subseteq S_{23}$ which is a t.e. of M_{22} with size $23|M_{22}| = 10,200,960$.

Pf M_{22} 3-trans on $X \cup \infty \cup \{w\}$. Add w. $X \cup \{\infty\} \cup \{w\}$.

$x = \infty$. $g = h_1$ and $h = h_2$

$h_2 \quad (w, \infty)\, f_2$

$f_2([u,v,w]) = [u^2, v^2, \beta w^2]$

f_2 well defin β prim. elemt of F_4.

1. $g(x) \neq x \quad h_1(\infty) \neq \infty$ ✓
2. $h_2(w) \neq w$.
3. $h_2^2 = 1$

$[u,v,w] \to [u^2, v^2, \beta w^2] \to [u^4, v^4, \beta^3 w^4] = [u,v,w]$. You ✓

~~4 h~~ 4. $(h_1 h_2)^3 = 1$ ✓

5. $h_2 \, PSL(3,4) \, h_2 = PSL(3,4)$ you ✓. $\square$

Thm: M_{23} has a t.e. to $M_{24} \subseteq S_{24}$. Hence M_{24} is 5 trans. and $|M_{24}| = 24|M_{23}| = 244,823,040$

Pf M_{23} acts on $X \cup \{\infty\} \cup \{w\}$ Add Ω. $\Rightarrow X \cup \infty \cup w \cup \Omega$.

$x = w \quad g = h_2 \quad h = h_3 = (w, \Omega)\, f_3$

$f_3[u,v,w] = [u^2, v^2, w^2]$.

1. $g(w) \neq w$ 2. $h_2(\Omega) \neq \Omega$.

~~3. 4, 5~~ 3. $h_3^2 = 1$ 4. $(gh)^3 = (h_2 h_3)^3 = 1$

5. $h_3 M_{23_w} h_3 = M_{23_w}$

$\Rightarrow h_3 M_{22} h_3 = M_{22}$

4,5 ✓ yourself $\square$