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Problem set 1

for the course on

Markov chains and mixing times

Exercise 1 (Functions of Markov chains are not necessarily Markov chains).
Let $(X_t)_{t\in\mathbb{N}_0}$ be a 3-state Markov chain with transition probability matrix P and state space $\{1,2,3\}$. Find a mapping

$$f : \{1,2,3\} \to \{1,2\}$$

and a matrix P such that $(f(X_t))_{t\in\mathbb{N}_0}$ is not a Markov chain.

Exercise 2. Consider a Markov chain $(X_t)_{t\in\mathbb{N}_0}$ with finite state space $S = \{1,\dots,n\}$. Recall that two states i and j are said to communicate – written as $i \sim j$ – if there exist $s,t > 0$ such that

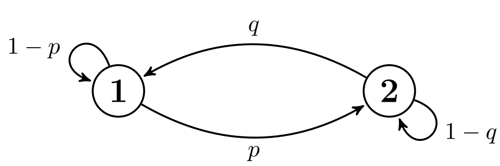
$$p_{ij}^{(t)} = \mathbb{P}[X_t = j \mid X_0 = i] > 0 \quad \text{and likewise} \quad p_{ji}^{(s)} > 0.$$

(i) Show that the relation $\sim$ is symmetric and transitive. Give an example to justify that it is not an equivalence relation in general.

(ii) If we extend it by adding $i \sim i$ for all states i to the relation, $\sim$ becomes an equivalence relation which then partitions S into classes of communicating states. Show that the states of any such communicating class are either all essential or all inessential and have the same period.

(iii) Explain why at least one state must be essential. How many classes of communicating states does an irreducible Markov chain have?

Exercise 3. Consider a 2-state Markov chain with transition probabilities as sketched on the right.



(i) For $p,q \in [0,1]$, calculate a stationary distribution π for this Markov chain.

(ii) Calculate the two eigenvalues $\lambda_1 \geq \lambda_2$ of the transition matrix P.

(iii) Assume now that $p,q \in (0,1)$, which makes the chain irreducible and aperiodic. Calculate P^t, the t-step transition matrix, and use it to verify that λ_2 controls the speed of convergence of P^t towards stationarity, i.e. $\Pi = \begin{pmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{pmatrix}$.

Exercise 4. In this exercise you are asked to give another direct proof of the uniqueness of the stationary distribution corresponding to a finite irreducible Markov chain – this time without using the eigenspace argument you saw in class.

Instead, consider two stationary distributions π_1,π_2 and a state i that maximizes the fraction $\frac{\pi_1(i)}{\pi_2(i)}$ (why is this always well-defined?). Verify that states j, from which i is accessible, must satisfy $\frac{\pi_1(j)}{\pi_2(j)} = \frac{\pi_1(i)}{\pi_2(i)}$ in order to establish the claim.

Exercise 5. Show that the set of stationary distributions for an n-state Markov chain (considered as a set of vectors) forms a polyhedron in $\mathbb{R}^n_{\geq 0}$ with one vertex for each essential communicating class.

To shorten the argument, you may use Proposition 1.25 without proving it.

Exercise 6 (Dynamic urn).
Assume that we have an urn with four differently colored balls. In one step, we draw one ball – which has color α say – draw a second ball, color this second ball with color α and throw both back into the urn. In each drawing all the balls currently contained in the urn are equally likely to be drawn.

Let $(X_t)_{t\in\mathbb{N}_0}$ denote the color composition in the urn, only accounting for the number of balls having the same color. Then $(X_t)_{t\in\mathbb{N}_0}$ is a Markov chain with starting state (1,1,1,1). The other states are (2,1,1), (2,2), (3,1) and (4), the last one being absorbing. Sketch the Markov chain graph including the corresponding transition probabilities and compute the expected number of steps until the urn contains nothing but balls of the same color.

Turn in your solutions during the lecture on February 5, 2014.