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Problem set 2

for the course on

Markov chains and mixing times

Exercise 7 (Reversible chains).

(i) Assume that we have a Markov chain with transition matrix P, such that there exist a positive function f and a non-negative function g on S with

$$p_{ij} = f(i)\,g(j) \quad \text{for all } i,j \in S.$$

Show that such a Markov chain is reversible and derive the corresponding stationary probability vector. Why is the stationary distribution unique in this case? How many steps will it take to reach equilibrium?

(ii) Can 2-state Markov chains be non-reversible? Exhibit a 3-state Markov chain with $p_{ij} > 0$ for all $i,j \in S$ which is not reversible.

For the next exercise you will need the following corollary to the Convergence Theorem (Thm. 4.9):
A finite Markov chain $(X_t)_{t\in\mathbb{N}_0}$ which is irreducible and aperiodic 'forgets' its initial state and its distribution converges to equilibrium in the sense that

$$\lim_{t\to\infty} p_{ij}^{(t)} = \lim_{t\to\infty} \mathbb{P}[X_t = j \mid X_0 = i] = \pi(j)$$

for all states i,j.

Exercise 8.* Let $(X_t)_{t\in\mathbb{N}_0}$ be a finite irreducible Markov chain having stationary distribution π and further $N(i,t)$ denote the (random) number of visits of state i among $X_1,\dots,X_t$. Without using Proposition 1.14 or Theorem 4.16, show that

$$\tfrac{1}{t}\,\mathbb{E}[N(i,t)] \to \pi(i) \quad \text{as } t \to \infty$$

and that $\frac{1}{t}\,N(i,t) \to \pi(i)$ in probability. Note that Theorem 4.16 actually implies that this latter convergence holds almost surely.

Hint: Write $N(i,t)$ as a sum of indicator variables and bound its variance.

Exercise 9. You are given two probability measures μ and ν on a finite set S. Bob is required to flip a fair coin (which you cannot see the result of) and if the coin is heads, he must give you an element of S which has distribution μ and if the coin is tails, he must give you an element of S which has distribution ν (independently chosen of the coin toss). Based upon what you end up receiving, your job is to try to guess if the coin was heads or tails and to maximize the probability that you are correct. Of course you can be correct with probability $\frac{1}{2}$ by just always guessing heads but you want to do better than this.

(i) Show that if $\|\mu - \nu\|_{\mathrm{TV}} \geq \delta$, then there is a strategy which gives a probability of being correct which is at least $\frac{1}{2} + \frac{\delta}{2}$.

(ii) Show that if $\|\mu - \nu\|_{\mathrm{TV}} \leq \delta$, then the maximal probability of being correct is at most $\frac{1}{2} + \frac{\delta}{2}$.

In conclusion, total variation measures the degree to which you can statistically tell apart two distributions.

Exercise 10. Let P be the transition matrix of a finite Markov chain $(X_t)_{t\in\mathbb{N}_0}$ with stationary distribution π and starting distribution $\mu = \mathcal{L}(X_0)$. Show that the total variation distance of the distribution of X_t, i.e. $\mu\, P^t$, to π is non-increasing with t, i.e.

$$\|\mu\, P^t - \pi\|_{\mathrm{TV}} \geq \|\mu\, P^{t+1} - \pi\|_{\mathrm{TV}} \text{ for all } t \in \mathbb{N}_0.$$

Explain how this implies that $d(t) = \max_{i\in S}\|P^t(i,\cdot\,) - \pi\|_{\mathrm{TV}}$ is non-increasing.

Exercise 11.

(i) Consider an aperiodic irreducible finite Markov chain having stationary distribution π and the property that there exists a state i and a set $A \subseteq S$ with $\pi(A) = \sum_{i\in A}\pi(i) > \frac{1}{4}$, and $d(i,A) \geq \Delta$, where $d(i,A)$ is the shortest path distance from i to any node in A in the (directed) Markov chain graph. Show that $t_{\mathrm{mix}} \geq \Delta$.

(ii) Use this to obtain lower bounds for the mixing time for a lazy random walk on the hypercube $\mathbb{Z}_2^d$ and on the torus $\mathbb{Z}_m^d$, $m > 2$. How sharp a bound can you get for the hypercube arguing this way? For what combinations of m and d can you prove, using (i), that the chain is not rapidly mixing?

Exercise 12.** Show that there exists a finite state Markov chain so that for two of its states i and j,

$$\lim_{t\to\infty}\|P^t(i,\cdot\,) - P^t(j,\cdot\,)\|_{\mathrm{TV}} > 0$$

but there exists a coupling of the Markov chain starting respectively at i and at j, $(X_t,Y_t)_{t\in\mathbb{N}_0}$ so that $T := \inf\{t : X_t = Y_t\}$ is finite with probability 1.

Note that this tells you that condition (5.2) is very essential in the statement of Theorem 5.2.

Turn in your solutions during the lecture on February 21, 2014.