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Problem set 3

for the course on

Markov chains and mixing times

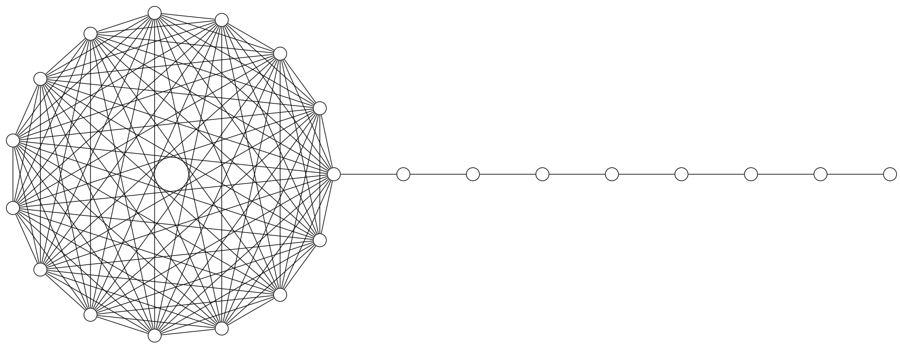
Exercise 13. Consider the simple random walk (SRW) on the complete b-ary tree of height k, having $n = \frac{b^{k+1}-1}{b-1}$ vertices.

(i) If 0 denotes the root and L the set of leaves, reduce the walk to a birth-and-death chain to get a good bound on $\mathbb{E}_0(\tau_L)$, the expected time to hit a leaf, starting at the root.

(ii) Use Moon's formula to calculate $\mathbb{E}_l(\tau_0)$, the expected time it takes to get from a fixed leaf l to the root.

Putting both parts together show that the time it takes to get from the root to the set of leaves and back is in expectation less than $2n$ in the binary case.

Exercise 14 (SRW on a lollipop graph).*



A lollipop graph consisting of a K_{15} and a joined chain of further 8 vertices.

(i) Consider a lollipop graph consisting of a chain on $\frac{n}{2}$ vertices joined to a $K_{n/2}$. Show that the (expected) hitting time t_{hit} is $\Theta(n^3)$.

(ii) Modify the fraction of nodes in the chain and the complete graph to maximize the hitting time. What do you get as leading term?

Exercise 15 (Connection between diameter and conductance).*

Consider a time-reversible Markov chain (on a finite state space S) which is aperiodic and irreducible. Let G be the corresponding Markov chain graph and denote by d_G its diameter, by $\Phi_\star$ the bottleneck ratio. With the notation $\pi_{\min} := \min_{i \in S} \pi(i)$ and $\pi(A) = \sum_{i \in A} \pi(i)$, show the following:

(i) If we start in state i and let A_t be the set of nodes at distance t or less in G from i, then

$$\pi(A_t) \geq \min\{\tfrac{1}{2}, \pi(i)\,(1+\Phi_\star)^t\}.$$

(ii) Using part (i), show that

$$d_G \leq 1 + \frac{c\,\log(1/\pi_{\min})}{\Phi_\star},$$

where c is a universal constant.

Exercise 16 (Commute time for SRW on a graph).

Consider the SRW on a connected graph with n vertices. For $i \neq j$, let T_{iji} denote the commute time from i to j and back, i.e. $\min\{t \geq \tau_j,\, X_t = i\}$ given $X_0 = i$ and write T_{ii} for τ_i^+ given $X_0 = i$.

(i) If the vertices i and j have the same degree, then the probability that the SRW starting in i visits j before returning to i equals the probability that the SRW starting in j visits i before returning to j.

(ii) The probability that SRW from i visits j before returning to i is equal to

$$\mathbb{P}(T_{ii} = T_{iji}) = \frac{\mathbb{E}T_{ii}}{\mathbb{E}T_{iji}} = \frac{1}{\pi(i)\,\mathbb{E}T_{iji}}.$$

(iii) For a regular graph, i.e. all vertices have the same degree, show that

$$\mathbb{E}T_{iji} \geq n.$$

Exercise 17 (A general bound for hitting times applied).

(i) Consider a Markov chain on n states, $n \in \mathbb{N}$. Show that the hitting time t_{hit} is at least $\frac{n-1}{4}$ by looking at the set of states a random walk encounters up to time $2\,t_{\text{hit}}$.

(ii) Use part (i) to show that for some properly chosen states x,y, the expected time until two independent simple random walks on the d-dimensional torus $\mathbb{Z}_n^d$ – started at x and y respectively – meet is of order n^d, which is the number of nodes.

Exercise 18 (A time-consuming way to shuffle a deck of cards).

(i) Consider an irreducible MC. For $A \subseteq S$, define the hitting time

$$\tau_A = \inf\{t \geq 0,\, X_t \in A\}.$$

Derive the lower bound

$$t_{\text{mix}} \geq \max_{u,A} \left(\pi(A) - \tfrac{1}{4}\right) \mathbb{E}_u(\tau_A).$$

(ii) Consider the lazy version of the transposition shuffle in which we pick an adjacent pair of cards uniformly in every step, i.e. given the deck of n cards as a stack, in each round we do the following:
First we flip a fair coin. If it comes up heads we do nothing this round, if it comes up tails we pick a pair of adjacent cards uniformly at random and transpose the two.

Show that the mixing time of this MC is at least of order n^3.

Turn in your solutions before March 19, 2014.