Game theory, lecture 1*:
Zero-sum games

(Finite zero-sum games)

((Finite, 2 persons zero-sum games))

(((Finite, 2 persons, zero-sum games in strategic/normal form)))

Those notes should be complementary to Thomas S. Ferguson's game theory notes. Please refer to this for more details.

▷ **Example:** Alice asks Bob to play the following game: she puts either 10$ or 20$ in an envelope and then Bob tries to guess the amount.
If he is right, he keeps the envelope, otherwise, Alice gets the envelope back, and Bob gives her 15$.
Should Bob accept?

The context

⟶▷ *2 players games*: We are interested in "games" between 2 players, usually called I and II. (I is masculine and II feminine to make it easier to distinguish.)

⟶▷ *finite*: They both have a finite set of "moves" or "strategies": X is the set of moves for I and Y is the set of moves for II.

I and II chose their move simultaneously (*i. e.* without knowing the other player's move).

⟶▷ *0-sum*: There is a "payoff function" $A : X \times Y \to \mathbb{R}$: if I and II respectively chose x and y, then II pays $A(x, y)$\$ to I.

This is called 0-sum because the total amount of money doesn't change: what II looses, I gains.

◁ **Remark:** The payoff function is nothing but a matrix. We'll use $a_{i,j}$ to denote $A(i, j)$. I choses a row, and II choses a column.

▷ **Example:** $A = \begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}$
This example is easy: I will always chose row 1 (no matter what II does, I gains more), and II will always chose column 1 (no matter what I does, II loses less).
The "value" of the game (not defined yet) should be $a_{1,1} = 3$.

▷ **Example:** $A = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$
I will always chose row 1 (same reason as above). Because of this, II should always chose column 1, to minimize her loss. The value should thus be 3.
Note that II assumes that I plays intelligently. If I were to chose 2, II's strategy isn't good anymore.

▷ **Example:** a little more difficult: $A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 2 & 4 \\ -3 & 0 & 1 \end{pmatrix}$
The entry $a_{2,2} = 2$ has a certain stability: neither I nor II can gain by unilateral deviation from this point. This is the best choice for both of them, and the value should be $2 = a_{2,2}$

---

* Course given by Jeff Steif (steif@math.chalmers.se), notes written by Pierre Hyvernat (hyvernat@cs.chalmers.se)

◁ **Remark:** Note that the above game matrix can be reduced by *dominance*: I will never choose row 3 (because row 1 or 2 both guarantee better gain). Knowing that, II will always choose column 2 (because it minimizes the loss, no matter what), and so, I should always choose row 2, to maximize his gain. The best moves are the same as above, and so is the value.
This is not always the case that to every saddle point correspond such dominating strategies: see for example $A = \begin{pmatrix} -2 & -1 & 3 \\ 1 & 0 & 2 \\ 2 & -2 & -3 \end{pmatrix}$ ($a_{2,2} = 0$ is the saddle point.)

Let's formalize the above phenomenon:

◦ **Definition:** Let $A = (a_{i,j})_{i \in X, j \in Y}$ be a (game) matrix. An entry $a_{i,j}$ is called a saddle point if:

- $a_{i,j} = \max\limits_{k \in X} a_{k,j}$;
- and $a_{i,j} = \min\limits_{k \in Y} a_{i,k}$;

We argued above the the value of the game should be the value of such a saddle point (if it exists). This is sound as shown by the following proposition (lemma).

◇ **Proposition:** Suppose A is a game matrix indexed by $X \times Y$, suppose $a_{i,j}$ and $a_{i',j'}$ are two saddle points; then $a_{i',j}$ and $a_{i,j'}$ are also saddle points, and all four are equal.

proof: exercise! Simple.

However, not every game has a saddle point. See for example $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

Here are a few things we can say about this game:

- I can guarantee a minimal gain of 2, by playing 1;
- II can guarantee a maximal loss of 3, by playing 2.

This means that I is willing to give $2 - \varepsilon$ ($\varepsilon \geq 0$) to II to play and II is willing to accept $3 + \varepsilon$ from I to play. The value of the game should lie somewhere 2 and 3...

Formally:

◦ **Definition:** Let A be a game matrix,

- the *gain floor* of the game is $V_{\mathrm{I}}' = \max\limits_{i \in X} \min\limits_{j \in Y} a_{i,j}$; (this is the maximal gain I can guarantee himself)
- the *loss ceiling* of the game is $V_{\mathrm{II}}' = \min\limits_{j \in Y} \max\limits_{i/in X} a_{i,j}$. (this is the minimal loss II can guarantee herself)

The following is (almost) obvious:

◇ **Proposition:** For any (finite 0-sum game), $V_{\mathrm{I}}' \leq V_{\mathrm{II}}'$, and equality holds exactly when there is a saddle point.

When there is no saddle point (see the above example), what should we do?

▷ **Example:** the game of "pennies": I and II put a penny on the table. If both show the same face (head or tails), then II gets the 2 pennies, otherwise, I gets them.
$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$
$V_{\mathrm{I}}' = -1$ and $V_{\mathrm{II}}' = 1$. The best way for I to play is to chose the row randomly with probability $(1/2, 1/2)$. Then, no matter what II choses, the expected gain of I is 0.
Similarly, II should chose column randomly with probability $(1/2, 1/2)$, and her expected loss will be 0.
The value of this game should be 0, and the "optimal" strategies are represented by the strategies with probabilities $p = (1/2, 1/2)$ for I, and $q = (1/2, 1/2)$ for II.

◦ **Definition:** A *mixed strategy* on the set $X = \{x_1, \dots x_n\}$ is a probability distribution on X: $p = (p_1, \dots p_n)$, with $p_i \geq 0$ and $\sum p_i = 1$.
A *pure strategy* is a mixed strategy where all the p_i are 0, except one. (*i. e.* the player always choses a specific move)

Let's go back to the previous example: $A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$.

$\longrightarrow\triangleright$ from I's point of view:

suppose I choses 1 with probability p, and 2 with probability $(1-p)$. What is his expected gain?

- if II choses 1: $4p + (1-p) = 3p + 1$;
- if II choses 2: $2p + 3(1-p) = 3 - p$.

By choosing p so that the 2 above quantity are equal, I can guarantee an expected gain independent of II's choice:

$$\dots \ p = 1/2$$

By choosing 1 with probability $1/2$ and 2 with probability $1/2$, I has an expected gain of $5/2$, whatever II does.

$\longrightarrow\triangleright$ from II's point of view

suppose II choses 1 with probability q (and 2 with probability $(1-q)$), what is her expected loss?

- if I choses 1: $4q + 2(1-q) = 2q - 2$;
- if I choses 2: $q + 3(1-q) = 3 - 2q$.

By choosing a q equalizing the above, II can guarantee an expected loss independent of what I does:

$$\dots \ q = 1/4$$

By choosing 1 with probability $1/4$ and 2 with probability $3/4$, II has an expected loss of $5/2$, independently of I's move.

It seems logical to call $5/2$ the *value* of this game.

▷ **Example:** The first game with Alice and Bob has matrix $A = \begin{pmatrix} -10 & 15 \\ 15 & -20 \end{pmatrix}$.
What is its value? What are optimal mixed strategies for I and II?