Game theory
Lecture 10

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Consider the game

$$\begin{pmatrix} (8,8) & (1,2) \\ (2,1) & (0,0) \end{pmatrix}.$$

Trivially we have that $(1,1)$ is the only Nash equilibrium, but if we play this game an infinite number of times with δ-discounting (where $\delta < 1$ is large), then there are other Nash equilibriums other than the obvious. One example is the following:

Player I's strategy: If $(1,1)$ has been played an even number of times then player I plays 1, otherwise 2.

Player II's strategy: If $(1,1)$ has been played an even number of times then player II plays 2, otherwise 1.

δ-discounting: weighting the n:th payoff by $(1-\delta)\delta^{n-1}$ where $0 < \delta < 1$.

Clearly player I won't unilaterally deviate. If player II does *not* unilaterally deviate her total payoff will be 2. If she deviates and plays 1 instead on the first move, she'll get less than or equal to

$$(1-\delta)8 + \sum_{k=1}^{\infty}(1-\delta)\delta^k \cdot 1 = 8 - 8\delta + \delta = 8 - 7\delta$$

so if $\delta > 6/7$ then the payoff will be strictly less than 2, since $8 - 7\delta < 8 - 7\frac{6}{7} = 2$.

Remark. The 8:s could be changed to any number and this would still be a Nash equilibrium for big enough δ.

Conclusion: Not all Nash equilibriums are "nice".

Repeated zero-sum games with partial information

Random games

Let A and B be two $n \times m$ matrices. Denote the game where A is played with probability p and B is played with probability $1-p$ with $pA+(1-p)B$. When you make your move you don't know which game was chosen.

Playing this game is essentially the same as playing the game with matrix C where $c_{ij} = pa_{ij} + (1-p)b_{ij}$.

Example 1.

$$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Example 2.

$$\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

The value of Example 1 is $1/4$ and the $(1/2, 1/2)$ is an optimal strategy for both players.

Again we play $pA + (1-p)B$, but now player I gets to know the coin before moving and player II doesn't. This is a new game where player II has m moves and player I has n^2 moves, where the move (i, i') means that player I plays i if A occurs and i' if B occurs.

Example 1 and 2 now becomes

$$\begin{array}{c} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} \begin{pmatrix} 1/2 & 0 \\ 1/2 & 1/2 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad \begin{array}{c} (1,1) \\ (1,2) \\ (2,1) \\ (2,2) \end{array} \begin{pmatrix} -1/2 & 0 \\ -1/2 & -1/2 \\ 0 & 0 \\ 0 & -1/2 \end{pmatrix}$$

which both are easily solved (by domination) and so the values are $1/2$ and 0 respectively.

Let A and B be two matrices again. Flip a coin to determine if we play A or B. We then play the game n times. (Note that we don't reflip the coin each time.) Player I is told the outcome of the coin flip and player II is not. As the n games are played payoffs are not informed, only the players moves.

Let V_n be the value of this game.

- For Example 2 it is clear that $V_n = 0$ for all n. (Observe that player I can reveal the information to player II without hurting himself.)

- For Example 1 we can see that

  - ⋆ $V_n \leq n/2$ since player II can play $(1/2, 1/2)$ each time.
  - ⋆ $V_n \geq n/4$ since player I can completely ignore his extra information and play $(1/2, 1/2)$ each time, which is the optimal strategy for $(1/2)A + (1/2)B$.

Note that if player II finds out what the coin flip was "the game is up"; compare this to the situation in Example 2.

Fact: $\lim_{n\to\infty} V_n/n = 1/4$.

Remark. In Example 2 player I can reveal all his extra information, but in Example 1 he plays optimal (for big n) if he totally ignores the extra information.

- The revealing information strategy (for player I) is simply to choose an optimal strategy for the game which arises every time, which gives $V_n \geq (n/2)V(A) + (n/2)V(B)$, where $V(A)$ and $V(B)$ are the values of A and B respectively.
- The non-revealing information strategy (for player I) is to choose an optimal strategy for the game $(1/2)A+(1/2)B$ each time. This gives $V_n \geq nV((1/2)A+(1/2)B)$.