Game theory
Lecture 11

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Recall where we are.
Game: $pA + (1-p)B$. Let $V(p)$ be the value of this game and let $V_n(p)$ be the value of this game when player I knows which matrix is chosen, player II doesn't and it is played n times where moves, but not payoffs are revealed each game (and the "coin" is only flipped once, in the beginning). Observe that in general $V_1(p) \neq V(p)$.
Goal: Determine $\lim_{n\to\infty} V_n(p)/n$ and showing that it always exists.

Example (Example 1 from last lecture). $V_n(1/2) = 0$. This is achieved be player I by playing optimally for the matrix which occur. Therefore player I is willing to reveal complete information.

Example (Example 2 from last lecture). $V_n(1/2) \approx n/4$ (didn't prove; we proved that $n/4 \le V_n(1/2) \le n/2$). This is achieved by player I by playing optimally for the game $(1/2)A + (1/2)B$ each time (and thereby giving no information to player II since player I doesn't even need to know if A or B occurs).

In general, when $p = 1/2$:

- $\star$ player I can always guarantee himself $n(V(A)/2 + V(B)/2)$ by playing optimally with respect to the game which was chosen. (Revealing complete information)
- $\star\star$ player I can always guarantee himself $n(V(1/2)A + (1/2)B)$ by playing optimally with respect to the game $(1/2)A + (1/2)B$. (Revealing no information)

Example (Example 3).

$$A = \begin{pmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{pmatrix}$$

The game is $(1/2)A + (1/2)B$. $V_n(1/2) \ge 0$ since player I can simply choose strategy $\star$:

$$\begin{cases} 1 & \text{if } A \text{ is chosen} \\ 2 & \text{if } B \text{ is chosen.} \end{cases}$$

Also with strategy $\star\star$ player I can guarantee himself 0 since

$$\frac{1}{2}A + \frac{1}{2}B = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

has value 0. I.e, if player I does not look at which matrix is chosen player II simply chooses 3.

Actually $V_n(1/2) \geq n$ by the following strategy for player I. The matrix A is played if H (head) arises and B is played if T (tail) arises. Also, let X, Y, Z be independent random variables such that

$$\begin{aligned} &P(X=1) = P(X=0) = 1/2 \\ P(Y=H) = 3/4, \qquad &P(Y=T) = 1/4 \\ P(Z=H) = 1/4, \qquad &P(Z=T) = 3/4 \end{aligned}$$

and define

$$W = \begin{cases} Y & \text{if } X = 1 \\ Z & \text{if } X = 0 \end{cases}.$$

Clearly, $P(W = H) = P(W = T) = 1/2$. W will determine if we play A or B.

If $X = 1$ ($X = 0$) at that time we're playing

$$(3/4)A + (1/4)B = \begin{pmatrix} 3 & 1 & 1 \\ 3 & 1 & -1 \end{pmatrix}$$

$$\left((1/4)A + (3/4)B = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 3 & 1 \end{pmatrix} \right)$$

Then, after knowing the outcome of X, player I will use an optimal strategy for the game which is 1 (2), which guarantees a win of at least 1. I.e.,

$$\text{Player I does } \begin{cases} 1 & \text{if } X = 1 \\ 2 & \text{if } X = 0 \end{cases}.$$

Player I is willing to announce what X is. Player II can not obtain further information about W since player I does not use any further information.

Fix matrices A and B (of the same dimensions).

Theorem. *If $V_n(p)$ is defined as above then*

$$W(p) = \lim_{n \to \infty} \frac{V_n(p)}{n}$$

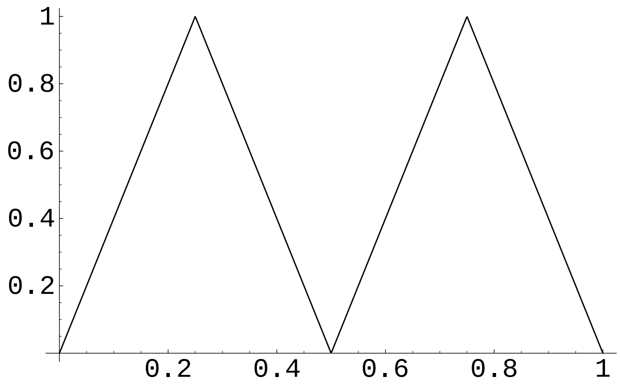
exists and W is the smallest concave function greater than or equal to V (called the "concavification" of V and denoted by concav(V)*).*

Remark. To determine $W(1/2)$ we need to look at different values of p.

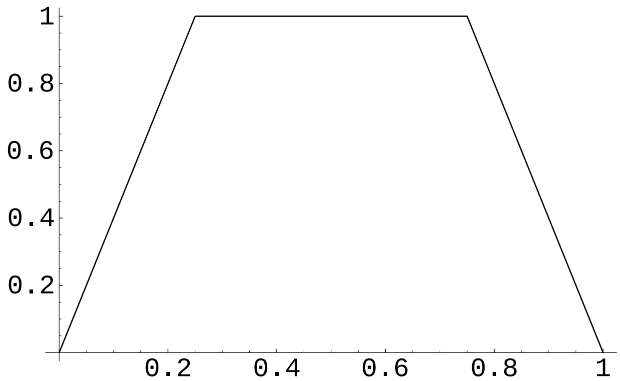
Example (Example 1). $V_n(p) = 0$ for all n. $V(p) = V(pA + (1-p)B) = p^2 - p$ and so $W(p) = 0$ for all p, since $V(0) = V(1) = 0$ and $V(p) < 0$ for all $0 < p < 1$.

Example (Example 2). $V(p) = p - p^2$ which is itself a concave function, so $W(p) = V(p) = p - p^2$ and so $W(1/2) = 1/4$.

Example (Example 3). $V(p)$ is



and so, by the theorem, $W(p)$ is



which shows that $W(1/2) = 1$.

Proof of the theorem. **Part 1:** Show $\liminf_{n\to\infty} V_n(p)/n \geq \text{concav}(V)(p)$.
There are 3 cases (the third one is proved next lecture):

1. When $\text{concav}(V)(p) = V(p)$. If player I uses strategy $\star\star$, i.e., he uses an optimal strategy for the game $pA + (1-p)B$ completely ignoring his information, he gain $nV(p)$, hence $V_n(p) \geq nV(p)$ and so $\liminf_{n\to\infty} V_n(p)/n \geq V(p)$.

2. When $\text{concav}(v)(p) = pV(1) + (1-p)V(0)$. If player I uses strategy $\star\star$, i.e., he plays optimally for A if A occurs and optimally for B if B occurs, then he gain $nV(A)$ if A occurs and $nV(B)$ if B occurs. Thus player I can guarantee himself
$$pnV(A) + (1-p)nV(B) = n\,\text{concav}(V)(p),$$
hence
$$\liminf_{n\to\infty} \frac{V_n(p)}{n} \geq \text{concav}(V)(p). \qquad \square$$