Classical game theory, lecture 12

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Recall the value $V_n(p)$. Since we are interested in

$$\frac{V_n(p)}{n}$$

we might as well divide by n one step earlier, in the sense of thinking of our pay-off after n games to be the average of our n pay-offs rather than the sum. (Note that by average we mean time average, not expected value, expectation is taken later).

Theorem: *The value*

$$W(p) := \lim_{n\to\infty} \frac{V_n(p)}{n}$$

exists and we have that $W(p)$ is the concavification of $V(p)$ (from now on denoted $\operatorname{con}(V)$).

Proof: First part,

$$\lim_{n\to\infty} \frac{V_n(p)}{n} \geq \operatorname{con}(V)(p).$$

We have three cases

1 $\operatorname{con}(V)(p) = V(p)$, which occurs when $V(p)$ is concave (this is ** from last time).

2 $\operatorname{con}(V)(p) = pV(1) + (1-p)V(0)$ (this is case * from last time)

3 The rest of the cases, illustrated by Figure 1

In the last case, we have

$$\operatorname{con}(V)(p) = rV(q_1) + (1-r)V(q_2)$$

where

$$p = rq_1 + (1-r)q_2 \text{ and } q_1, q_2, r \in (0,1).$$

Here r is the fraction of the distance between q_2 and q_1 where p is located, or in other words

$$r = \frac{q_2 - p}{q_2 - q_1}$$

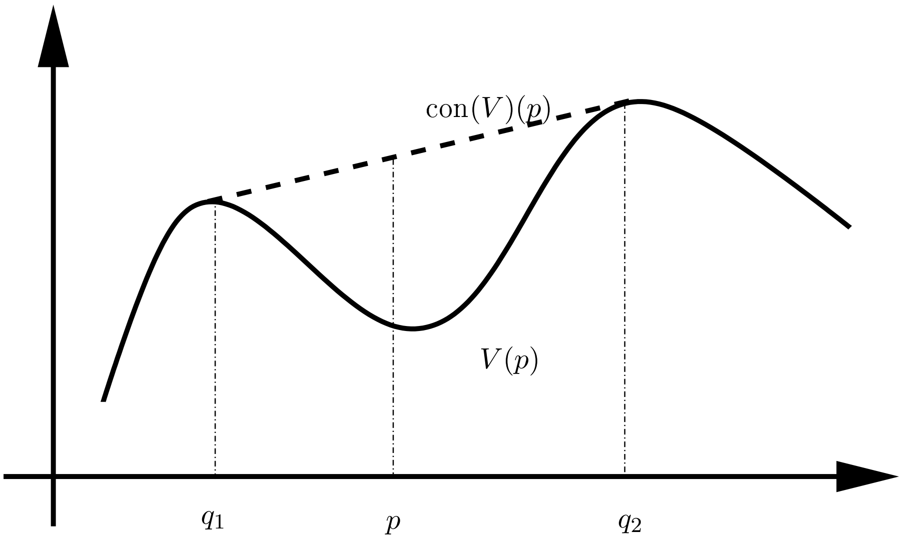

Figure 1: A concavification of a function

Since con(V) is the minimum of all linear functions $f \geq V$ (the proof of this is left as an exercise) this takes care of the rest of the cases.

Example: We now do case 3 on the example from the last time. Here I determines the strategy by a two-step procedure.

First let X, Y, Z be independent random variables with properties

$$P(X=1)=r, \quad P(X=0)=(1-r)$$
$$P(Y=H)=q_1, \quad P(Y=T)=(1-q_1)$$
$$P(Z=H)=q_1, \quad P(Z=T)=(1-q_1)$$

now let

$$W=\begin{cases} Y & \text{if } X=1 \\ Z & \text{if } X=0 \end{cases}$$

with W being the coin determining the game. We now have

$$P(W=H)=rq_1+(1-r)q_2=p$$

Now I does the following :

- if $X=1$ choose an optimal strategy for the game $q_1A+(1-q_1)B$ and play it independently each time.
- if $X=0$ choose an optimal strategy for the game $q_2A+(1-q_2)B$ and play it independently each time.

The only knowledge used by I is the value of X, and this is hence the only information which may be leaked to II. Now I:s expected gain is

$$P(X=1)(nV(q_1))+P(X=0)(nV(q_2)=n(rV(q_1)+(1-r)V(q_2))=n\text{con}(V)(P).$$

Now we go over to the second half of the proof, to prove

$$\lim_{n\to\infty} \frac{V_n(P)}{n} \le \text{con}(V)(P)$$

We hence need a good strategy for player II.

The cleanest way to do this is to use Blackwell's theory of vector valued games, where a vector valued game is described below:

Consider 2 players with strategy-space

$$\{1, \dots, r\} \text{ and } \{1, \dots, s\}$$

The pay-off of the game is a matrix of vectors,

$$m_{ij} \in \Re^n, \forall 1 \le i \le r, 1 \le j \le s$$

OBS! We think of m_{ij} as the "vector" payment from II to I if i and j are chosen. Note that for $n = 1$ this is a zero sum game, but for $n = 2$ it does not correspond to the bimatrix of a non-zero sum game, as all components are paid by II to I. We now play this vector-valued game forever.

Let $x_k \in \Re^n$ be the payoff at time k and let the running average be

$$\bar{x}_k := \frac{1}{n}\sum_{k=1}^{n} x_k.$$

Question: Given a closed set $S \subset \Re^n$, can II choose a strategy (for the repeated game) which guarantees that $P(d(\bar{x}_n, S) \to 0) = 1$ regardless of what I does? If this is possible, then S is said to be *approachable*. Here $d(\bar{x}_n, S)$ is the distance from $\bar{x}_n$ to the closest point in S, and we know that at least one closest point exists, as S is closed.

Examples: For $n = 1$, an example where the answer is yes, may be

$$S = (-\infty, a], a \ge v$$

where v is the value of the zero-sum game, and a case where the answer is no may be

$$S = (-\infty, a], a < v$$

In the first case, we know that regardless of how good or bad I plays, player II can always force the average payoff not to exceed v . In the second case, if player I plays optimally , player II can not push the average payoff below v.

Now fix a probability distribution

$$p_j \ge 0, j \in \{1, \dots, s\}, \sum_{j=1}^{s} = 1,$$

Each element in $1, \dots, r$ now gives an element in $\Re^n$ by $\sum_{j=1}^{s} m_{ij}p_j$ as i varies (these points are the expected value of the game if player II chooses the mixed strategy p and player I chooses the pure strategy i). Now let $R(p)$ be the convex hull of these points. $R(p)$ is now the set in which player I may place the expected value of the game, given that player II constantly uses the strategy p.

Theorem: (Blackwell) *Let $S \in \Re^m$ be closed. Assume that for each $x \notin S$ there exists a probability distribution p such that the following property holds: with y being a closest point to x in S, the hyper-plane trough y with normal vector $\overline{yx}$ separates $R(p)$ from x. Then S is approachable.* The separation property is illustrated in Figure 2.

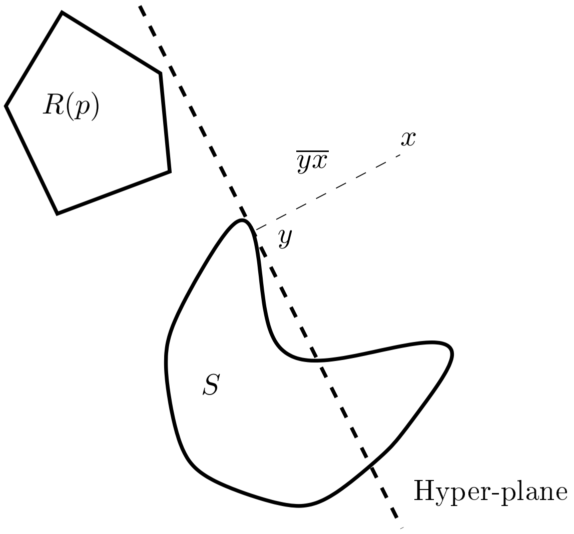

Figure 2: Separation of $R(p)$ and x

In particular we have that $\lim_{n\to\infty} \mathsf{E}[d(\bar{x}, S)] = 0$. We will only prove this last statement, as it is the only we will use.

Remark: As II wants to control $\bar{x}_n$ the only relevant information in the first n games is $\bar{x}_{n-1}$.

Proof: We construct a strategy for II which gives the desired result. The strategy is as follows:

- At time 0 do anything.
- If $\bar{x}_{n-1} \in S$ do anything.
- Otherwise, choose a distribution p on $1, \ldots, s$ such that it satisfies the property in the hypothesis, with $x = \bar{x}_{n-1}$.