Classical game theory, lecture 13

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Setting: The sets of possible moves for player I and II are

$$\mathrm{I} : \{1, \ldots, r\} \quad \mathrm{II} : \{1, \ldots, s\}$$

and the pay-off is a matrix of vectors, $M_{ij} \in \Re^n$.

Recall that if p is a probability measure on $\{1, \ldots, s\}$ then $R(p) \subset \Re^n$ is the convex hull of the r points given by $\sum_{j=1}^{s} m_{ij} p_j$. (If II uses p then no matter what I does, the expected payoff will lie in $R(p)$).

Blackwell's theorem: *Let $S \subset \Re^n$ be closed. For all x let $c(x)$ be a closest point in S to x. Assume that $\forall x \notin S, \exists p(x)$ such that $R(p(x))$ and x are on opposite sides of the hyper-plane through $c(x)$ and $\perp$ to $\overline{xc(x)}$, then S is approachable.* That S is approachable means that there exists a strategy for II such that no matter what I does, $P(d(\bar{x}_n, S) \to 0) = 1$, with $\bar{x}_n$ being the running average.

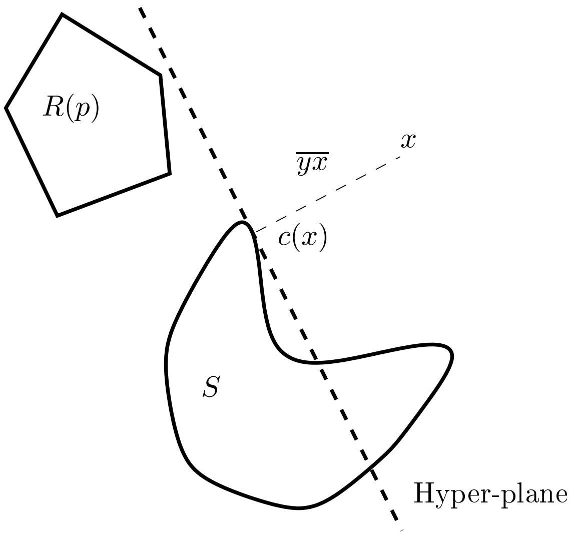

Figure 1: Separation of $R(p)$ and x in Blackwell's theorem

In particular, it then follows from the bounded convergence theorem that $\mathsf{E}[d(\bar{x}_n, S)] \to 0$. This last statement is all we will use, and all we will prove.

Proof: We construct a strategy for II which gives the desired result. The strategy is as follows:

- At time 0 do anything.
- If $\bar{x}_{n-1} \in S$ do anything.
- Otherwise, choose a distribution p on $1, \ldots, s$ such that it satisfies the property in the assumption, with $x = \bar{x}_{n-1}$.

Now we have to prove that this strategy really works. Case three above implies that no matter what I does,

$$< \mathsf{E}[x_{n+1}|x_1, \ldots, x_n] - c(\bar{x}_n), \bar{x}_n - c(\bar{x}_n) > \leq 0, \quad n \geq 1$$

holds. This is trivial if $\bar{x}_n \in S$ since then we have $\bar{x}_n - c(\bar{x}_n) = 0$.

Now, let $\delta_n := (d(\bar{x}_n, S))^2$ We now have

$$\delta_{n+1} \leq \|\bar{x}_{n+1} - c(\bar{x}_n)\|^2 = \|\bar{x}_{n+1} - \bar{x}_n + \bar{x}_n - c(\bar{x}_n)\|^2$$
$$= \|\bar{x}_{n+1} - \bar{x}_n\|^2 + \|\bar{x}_n - c(\bar{x}_n)\|^2 + 2 < \bar{x}_{n+1} - \bar{x}_n, \bar{x}_n - c(\bar{x}_n) >$$

It is trivial to check that $\bar{x}_{n+1} - \bar{x}_n = \frac{x_{n+1} - \bar{x}_n}{n+1}$ which implies that

$$2 < \bar{x}_{n+1} - \bar{x}_n, \bar{x}_n - c(\bar{x}_n) >= \frac{2}{n+1} < x_{n+1} - \bar{x}_n, \bar{x}_n - c(\bar{x}_n) >$$
$$= \frac{2}{n+1}(< x_{n+1} - c(\bar{x}_n), \bar{x}_n - c(\bar{x}_n) > + < c(\bar{x}_n) - \bar{x}_n, \bar{x}_n - c(\bar{x}_n) >).$$

Since all elements M_{ij} are bounded, this also implies

$$\|\bar{x}_{n+1} - \bar{x}_n\|^2 \leq \frac{c}{(n+1)^2}$$

for some c. Hence we have that

$$\delta_{n+1} \leq \frac{c}{(n+1)^2} + \delta_n(1 - \frac{2}{n+1}) + \frac{2}{n+1} < x_{n+1} - c(\bar{x}_n), \bar{x}_n - c(\bar{x}_n) > \tag{1}$$

This holds no matter what I and II do. Now let us take the expected value of equation (1) conditionally on $x_1, \ldots, x_n$, and remember that for any function $f(x_1, \ldots, x_n)$ the conditional expected value is given by

$$\mathsf{E}[f(x_1, \ldots, x_n)|x_1, \ldots, x_n] = f(x_1, \ldots, x_n)$$

since $f(x_1, \ldots, x_n)$ is known when $x_1, \ldots, x_n$ are known. Taking the expected conditional value of (1)hence gives

$$\mathsf{E}[\delta_{n+1}|x_1, \ldots, x_n] \leq \frac{c}{(n+1)^2} + \delta_n(1 - \frac{2}{n+1}) + \frac{2}{n+1} < \mathsf{E}[x_{n+1}|x_1, \ldots, x_n] - c(\bar{x}_n), \bar{x}_n - c(\bar{x}_n) > .$$

We now assume that II does an intelligent choice of p given $x_1, \ldots, x_n$ (p is chosen to fulfill the assumption in Blackwell's theorem) and hence that

$$< \mathsf{E}[x_{n+1}|x_1, \ldots, x_n] - c(\bar{x}_n), \bar{x}_n - c(\bar{x}_n) > \leq 0.$$

Taking the expected value of this (no conditioning) gives

$$\mathsf{E}[\delta_{n+1}] \leq \frac{c}{(n+1)^2} + \mathsf{E}[\delta_n](1 - \frac{2}{n+1}).$$

We want to show that $\mathsf{E}[\delta_n] \to 0$, and we would be done if the following lemma was true

Lemma: Let $\{a_n\}_{n=2}^{\infty}$ be a sequence such that $0 \le a_n \le c$ and $a_n \le \frac{c}{n^2} + a_{n-1}(1 - \frac{2}{n}) \forall n \ge 2$. Then $a_n \to 0$.

The lemma follows from $\prod_{n=1}^{\infty}(1 - \frac{1}{n}) = 0$, and the proof is left to the reader as an exercise.

Finally we will apply Blackwell's theorem to prove that

$$\lim_{n\to\infty} \frac{V_n(p)}{n} \le \mathrm{concav}(V)(p)$$

Idea: There are 2 pay-off matrices, each one having been possibly chosen, II doesn't know which. The two possible pay-offs may be seen as a vector valued game with matrix $m_{ij} = \{(a_{ij}, b_{ij})\} \in \Re^2$. The point is that Blackwells theorem will allow II to take control of the vector (a_{ij}, b_{ij}) enough to guarantee that I does no better than the concavification $(V)(p)$ in the long run. Assume that A is chosen with probability p, that is we play the game

$$pA + (1-p)B.$$

Let L_p be the linear function such that $L_p \ge \mathrm{concav}(V)$ and $L_p(p) = \mathrm{concav}(V)(p)$.

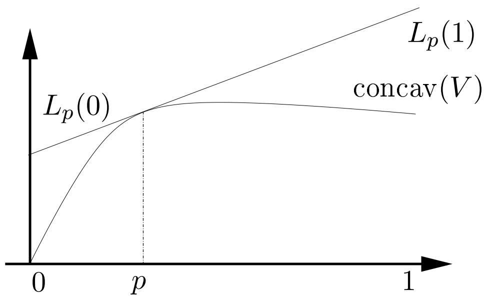

Figure 2: Illustration of L_p

Let S be $S := \{(x,y); x \le L_p(1), y \le L_p(0)\}$ and $S(\epsilon) := \{(x,y); x \le L_p(1) + \epsilon, y \le L_p(0) + \epsilon\}$

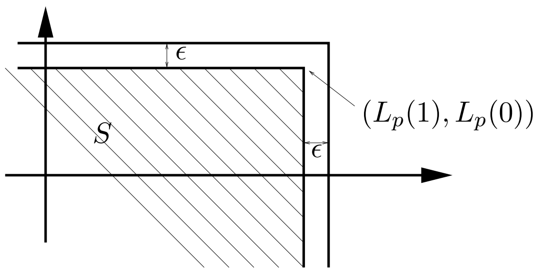

Figure 3: Definition of S and $S(\epsilon)$

In a moment we will verify that S satisfies our assumption in Blackwell's theorem. Blackwells theorem now implies that there exists a strategy for II such that no matter what I does, $\mathsf{E}[d(\bar{x}_n, S)] \to 0$. Hence, $\forall \epsilon > 0, \exists N$ such that

$$P(\bar{x}_n \notin S(\epsilon)) < \epsilon, \quad \forall n \ge N$$

If c is such that all pay-offs are at most c, then the expected pay-off to I for $n \geq N$ is at most

$$cP(\bar{x}_n \notin S(\epsilon)) + (L_p(1) + \epsilon)P(\bar{x}_n \in S(\epsilon) \cap A \text{ chosen})$$
$$+(L_p(0) + \epsilon)P(\bar{x}_n \in S(\epsilon) \cap B \text{ chosen}).$$

Since we have that

$$p - \epsilon \leq P(\bar{x}_n \in S(\epsilon) \cap A \text{ chosen}) \leq P(A \text{ chosen}) = p$$
$$1 - p - \epsilon \leq P(\bar{x}_n \in S(\epsilon) \cap B \text{ chosen}) \leq P(B \text{ chosen}) = 1 - p$$
$$P(\bar{x}_n \notin S(\epsilon)) \leq \epsilon$$

we get that the expected pay-off to I is at most

$$L_p(1)p + L_p(0)(1 - p) + c'\epsilon = \text{concav}(V)(p) + c'\epsilon.$$

with c' being constant.

We now must verify that S satisfies the assumption in Blackwell's theorem.

We need to verify that for all lines $\tilde{L}$ through $(L_p(1), L_p(0))$ there exists a probability distribution p such that $R(p) \subset$ "left of the line" $\tilde{L}$.

We sum this up: If $\tilde{L}$ is a vertical line, then p is given by the optimal strategy for player II for game A. In this case, no matter what I does, the x coordinate of the pay-off would be at most $V(A) = V(1) \leq \text{concav}(V)(1) \leq L_p(1)$. The horizontal case is treated in the same fashion.

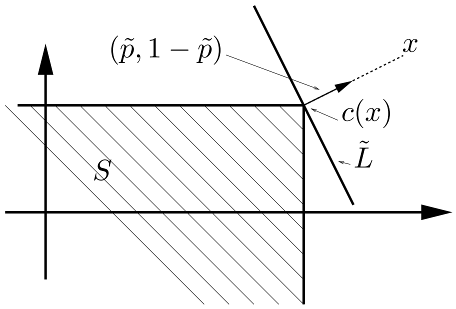

Figure 4: Illustration of $\tilde{L}$.

For the rest of the cases, choose $\tilde{p}$ such that $(\tilde{p}, 1 - \tilde{p}) \perp \tilde{L}$.

In this case, let p be an optimal strategy for the game $\tilde{p}A + (1 - \tilde{p})B$. We now have to show that $R(p)$ is located to the left of $\tilde{L}$. Let $i \in \{1, \ldots, r\}$. Let (u, v) (depending on p and i) be the corresponding pay-off. We need to show that

$$< (u, v) - (L_p(1), L_p(0)), (\tilde{p}, 1 - \tilde{p}) > \leq 0$$

which is equivalent to

$$u\tilde{p} + v(1 - \tilde{p}) \leq \tilde{p}L_p(1) + (1 - \tilde{p})L_p(0)$$

By the definition of p as an optimal strategy for the game $\tilde{p}A + (1 - \tilde{p})B$, we have that the left hand side $\leq V(\tilde{p})$. Further we have, from the definition of L_p that

$$V(\tilde{p}) \leq \text{concav}(V)(\tilde{p}) \leq L_p(\tilde{p}) = \tilde{p}L_p(1) + (1 - \tilde{p})L_p(0).$$