Classical game theory, lecture 14

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notes converted to latex by Sverker Lundin

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Nash's axiomatic solution to the bargaining problem

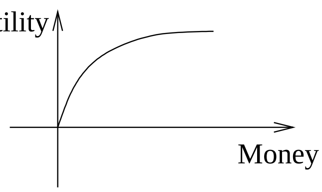
How do we split resources (money) between two people? Nash had a theory for how this will work. Consider the following example:

> Bob and Nancy are given a dollar to split. If they don't agree, they get nothing. They are both broke. Bob's son needs (crucially) an operation which costs 60 cents. Nancy's son needs an operation which costs 40 cents. It's obvious what they will do.

Here things that are difficult to describe mathematically determine how the money will be split. Due to this kind of "extraneous factors", it is hard to see a general solution to the problem.

Nash's answer: You don't split money, but you split utility.

Utility theory: Consider a situation where you can choose between getting one million dollars for certain, and a situation where you get two million dollars with a probability slightly higher than 50% and nothing with the remaining probability. We would all choose the first alternative. This tells us that happiness (utility) is not a linear function of money (but probably more like the square root of money):



The idea of **Morgenstern – Von Neuman**'s theory of utility is that people try to maximize their expected utility. I.e. If Ω is the set of all objects in the world, each person has a utility function $u : \Omega \to R$ such that if m_1, m_2 are two probability measures on Ω, you like m_1 as much as m_2 if and only if

$$\int u dm_1 \geq \int u dm_2.$$

If u is a utility function for a person, then $c_1 u + c_2$ $(c_1 > 0)$ is also a utility function. On the other hand, if u and v are two utility functions for a person, then there exists c_1, c_2 such that $u = c_1 v + c_2$ (this requires a proof).

Consider a situation where you can chose between these two games:

A	millions	0	1	5
	prob.	0	1	0

B	millions	0	1	5
	prob.	0	1	0

And then a situation where you can chose between these two games:

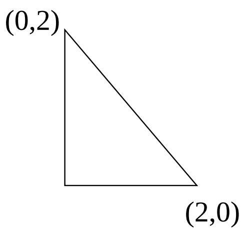
C	millions	0	1	5
	prob.	0.89	0.11	0

D	millions	0	1	5
	prob.	0.9	0	0.1

It turns out that the combination of prefering A to B and D to C is inconsistent with the existence of a utility function. People that make these choices don't "have" a utility function.

Nash's bargaining problem: Given a compact convex set $F \subseteq R^2$ and a point $v \in F$, where a point $(x, y) \in R^2$ represents an allocation of resources where I receives utility x and II receives utility y.

F is the set of possible utility pairs, given what we have to allocate. Say we have 1$ to split (with the possibility to throw some of it away), and that we get nothing if we cannot reach an agreement regarding how to split it. If $u(x) = 2x$ for both, then F would be:



The assumption of convexity is reasonable since if $(x, y), (\tilde{x}, \tilde{y}) \in F$, $0 < p < 1$, then we can do the following allocation: with probability p, take (x, y), and with probability $(1 - p)$, take $(\tilde{x}, \tilde{y})$. From linearity it follows that $p(x, y) + (1 - p)(\tilde{x}, \tilde{y}) \in F$.

The point v is the "disagreement point". This is the utility pair resulting if the players cannot reach an agreement. In this case, $v = (0, 0)$.

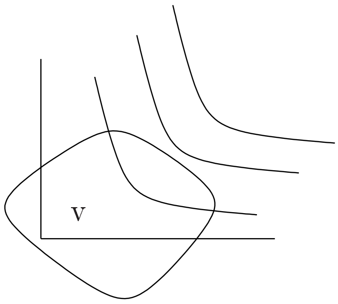
Problem: Given (F, v) as above, find $\Phi(F, v) \in F$ which is a reasonable predictive solution of the bargaining problem. Nash proved that there exists a unique solution to this, satisfying five axioms.

Nash's axioms for a solution to the bargaining problem:
There is a unique $\Phi(F, v) \in F$ satisfying: (where $v \geq w$ means that $v_i \geq w_i$ $\forall i$)

1. (Strong efficiency or Pareto optimality) If $x \geq \Phi(F, v)$, $x \in F$, then $x = \Phi(F, v)$.
2. $\Phi(F, v) \geq v$
3. (Scale invariance) $\forall \lambda_1, \lambda_2, \gamma_1, \gamma_2$, where $\lambda_1, \lambda_2 > 0$, let $G = \{(\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2) : (x_1, x_2) \in F\}$ i. e. G is the image of F under the mapping $(x_1, x_2) \to (\lambda_1 x_1 + \gamma_1, \lambda_2 x_2 + \gamma_2)$. Let $w = (\lambda_1 v_1 + \gamma_1, \lambda_2 v_2 + \gamma_2)$. Then $\Phi(G, w) = (\lambda_1 \phi_1(F, v) + \gamma_1, \lambda_2 \phi_2(F, v) + \gamma_2)$, where $\Phi(F, v) = (\Phi_1(F, v), \Phi_2(F, v))$.
4. (Independence of irrelevant altenatives) If $v \in G \subseteq F$ (both compact and convex) and if $\Phi(F, v) \in G$ then $\Phi(G, v) = \Phi(F, v)$.
5. (Symmetry) If $v_1 = v_2$ and if F is symmetric about $x = y$, then the allocation $\Phi(F, v) \in \{(x, y) : y = x\}$.

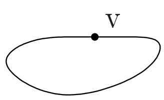
Moreover,
$$\Phi(F, v) \in \operatorname{argmax}(x_1 - v_1)(x_2 - v_2),\ x \in F,\ x \geq v \tag{1}$$

Draw hyperbolas:



Two situations:

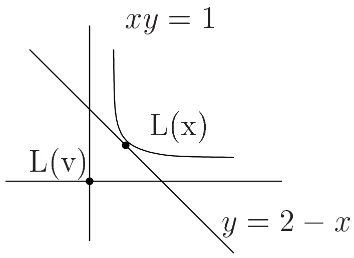
- Essential problem: $(\exists y \in F$ s. t. $y_1 > v_1,\ y_2 > v_2$.
- Inessential problem: Here 1 is zero, so we get many solutions.



If the problem is essential, the maximum is unique. The element $\operatorname{argmax}(x_1 - v_1)(x_2 - v_2)$ is easily seen to satisfy $1, 2, 4$ and 5, and (exercise) it satisfies 3. But we must prove uniqueness.

We now show that if $\Phi(F, v)$ satisfies 1 – 5, it is given by expression 1. Given (F, v) (essential), let $(x_1, x_2) = \operatorname{argmax}(\tilde{x_1} - v_1)(\tilde{x_2} - v_2)$. We want to show that $(x_1, x_2) = \Phi(F, v)$. Let
$$L : R^2 \to R^2;\ L(y) = \left(\frac{y_1 - v_1}{x_1 - v_1}, \frac{y_2 - v_2}{x_2 - v_2}\right).$$
Observe that if $z = L(y)$, then
$$z_1 z_2 = \frac{1}{x_1 - v_1} \frac{1}{x_2 - v_2} \cdot (y_1 - v_1)(y_2 - v_2).$$

Since (x_1, x_2) maximizes $(y_1 - v_1)(y_2 - v_2)$ over $\{(y_1, y_2) \in F\}$, $L(x)$ must maximize $\{z_1 z_2 : (z_1, z_2) \in L(F)\}$.



$L(v) = (0, 0)$. $L(x) = (1, 1)$. So $L(F)$ sits below the hyperbola. $L(F)$ is also convex, which implies that $L(F)$ sits below $y = -x + 2$.

Make a triangle S large enough to contain $L(F)$. $\Phi(S, 0) = (1, 1)$, by axiom 5. Axiom four implies that $\Phi(L(F), 0) = (1, 1)$. Finally by axiom three we have that $\Phi(F, L^{-1}(0)) = L^{-1}(1, 1)$ i. e., $\Phi(F, v) = (x_1, x_2)$.