Classical game theory, lecture 15

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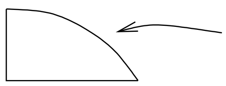
Example: I and II have to agree to split 30\$ ("burning" is allowed). If they fail they get nothing $(0,0)$. Assume I's utility for money is $u(x) = x$ (linear) and that II's utility for money is $u(x) = \sqrt{x}$. How does Nash predict the split will occur?

$v = (0,0)$. If x\$ is given to I, then the resulting utilities to the 2 players are $(x, \sqrt{x})$. This implies

$(0, \sqrt{30})$

F=  $f(x) = \sqrt{30-x}$

$(30, 0)$

Nash's solution is obtained by maximizing $y_1\sqrt{30-y_1}$ over $y_1 \in [0,30]$, which is maximized at $(20, \sqrt{10})$ (units of utility). This implies a money split of $(20, 10)$.

Aside:

Later we will analyse the following game:

$$\begin{pmatrix} (5,3) & (0,-4) \\ (0,0) & (3,6) \end{pmatrix}$$

We now assume linear utility functions. In connection to this game we will get into contracts and transferable utility. An example of a contract is "I plays 1, II plays 2, and I give II 2\$". You never break a contract! How do we analyse this game based on Nash's theory?

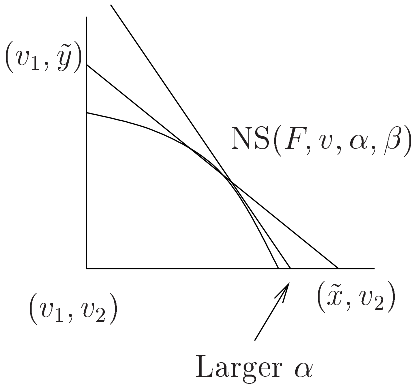
Can one really defend the Nash axioms for the bargaining problems? This question led to what was called the Nash program. Does there exist a bargaining game such that the (or a) NE for this game is (or approximates) the NS (Nash Solution)? If so, this would be an important defence of the Nash axioms. (The answer is of course yes.)

Let (F, v) be as in the Nash's bargaining problem.
definition: The Nash solution to (F, v) with bargaining powers $\alpha, \beta \quad (\alpha+\beta = 1,\ \alpha, \beta > 0)$ is the unique $(x_0, y_0) \in F$ maximizing

$$(x - v_1)^{\alpha}(y - v_2)^{\beta}.$$

This solution is labelled $\mathrm{NS}(F, v, \alpha, \beta)$. Since a function has the same maximum as its square, $\mathrm{NS}(F, v, \frac{1}{2}, \frac{1}{2})$ is our original Nash solution.

Lemma: (Calculus)
If $(x_0, y_0) = \mathrm{NS}(F, v, \alpha, \beta)$, then $(x_0, y_0) = \alpha(\tilde{x}, v_2) + \beta(v_1, \tilde{y})$, where $\tilde{x}$ is the x coordinate of the point where the tangent to F at (x_0, y_0) hits the point $y = v_2$, and conversely $\tilde{y}$ is the y coordinate of the point where the tangent to F at (x_0, y_0) hits the point $x = v_1$.

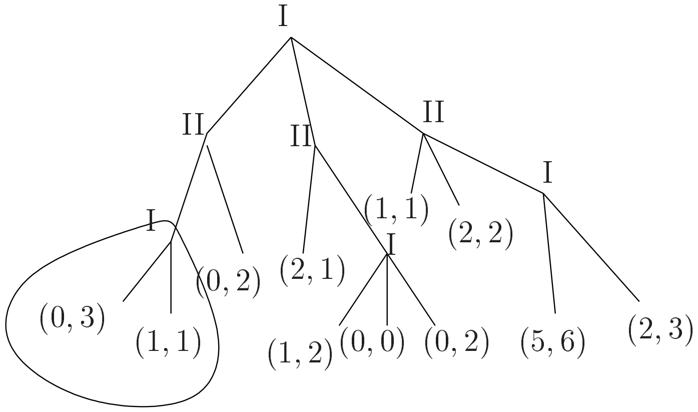


(The larger α is, the more money player I will get.)

Ex. If F is the area under $y = 1 - x$, then $\mathrm{NS}(F, (0,0), \alpha, \beta) = (\alpha, \beta)$. Observe that $\mathrm{NS}(F, v, \alpha, \beta)$ satisfies 4 of the 5 Nash axioms, not symmetry.

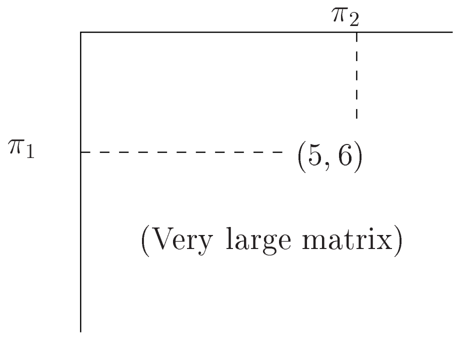
Aside:

Games in extended form.
This is a game tree:



The game proceeds by going down the tree. At the leaves the game ends and we must specify a pay–off. We can use backwards induction to predict which way we must go to get a maximal pay–off. Call the resulting strategies (π_1, π_2). A strategy for I is a specification of what he would do at every node which has an I on it.

An extended game can be converted to a normal game:

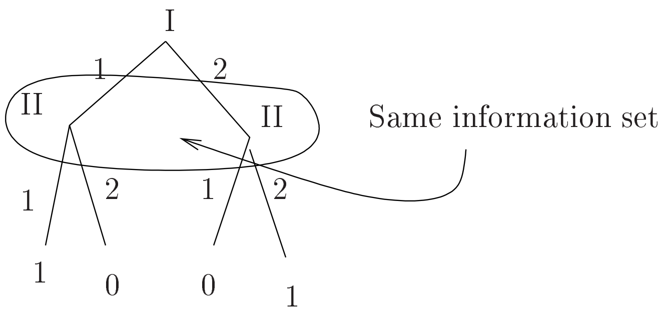


In our example we have given a specific pair of strategies (π_1, π_2) which gives pay–off $(5, 6)$. It can be verified that $(5, 6)$ is a pure Nash Equilibrium (PNE).

Can you represent every game in extended form? Since not every game has a PNE (e. g. most 2×2–zero sum games) it follows that not every game comes from a game in extended form, but if we modify the definition of a game in extended form, then every game can be represented in extended form. Consider the zero sum game:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To represent this in extended form we must use information sets:



Consider the pair (π_1', π_2) where π_2 is as above and π_1' is exactly as π_1 except that in the bottom left of the tree, I chooses $(0, 3)$ rather than $(1, 1)$. Is (π_1', π_2) a NE, although it seems stupid? YES!! (But it is undesirable.)

Definition: (π_1, π_2) is a *sub-game perfect pure Nash equilibrium* (SGPNE) if in every subtree it is a NE.

(π_1, π_2) is a SGPNE (the original strategies constructed).
(π_1', π_2) is not.

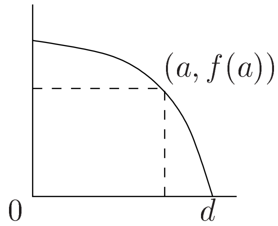
(End of discussion of games in extended form.)

Rubensteins bargaining model
(Long aside which eventually will be reconnected to NS)

Consider a convex set $F \subseteq R^2$, with disagreement point (dp) $(0, 0)$.

$$F = \{(x, y) : x \in [0, d],\ y \geq 0,\ y \leq f(x)\}$$

where f is a strictly decreasing continuous concave function which is 0 at d.



We don't have to think about utilities, we can think of (x, y) as meaning I gets x\$ and II gets y\$ and utility is linear.

Let $\delta_1, \delta_2 \in (0, 1)$. δ_1 is the discount factor for player I, which means that 1\$ today is worth δ_1\$ tomorrow to player I.
(small $\delta \Leftrightarrow$ in a rush to make a deal)
(large $\delta \Leftrightarrow$ less in a rush)

Game proceeds as follows:

Time 0 I chooses a pareto point (x_0, y_0) on the upper boundary as an offer.
II accepts or rejects. If accepts; that is the pay–off.

Time 1 Otherwise II rejects and makes a counter–offer (x_1, y_1) (all offers are on the boundary of F), and I accepts of rejects. If I accepts, the po is $(\delta_1 x_1, \delta_2 x_2)$.

Time 2 If I rejects, he offers (x_2, y_2). If II accepts, the po will be $(\delta_1^2 x_2, \delta_2^2 y_2)$ etcetera...
$\vdots$

If they never agree the pay–off is $(0, 0)$.

Main theorem
This game, G, has a SGPPNE and all SGPPNEs yield the same po which is going to be $(a, f(a))$ (first offer actually accepted) characterised by $f(a) = \delta_2 f(a\delta_1)$.

In addition this equation has a unique solution. Finally, in the related game H, where II makes the first offer, if $(f^{-1}(b), b)$ is the pay–off for the SGPPNE, then $f(a)) = b\delta_2$, and $b = f(a\delta_1)$.