Game theory, lecture 2*:
The minimax theorem

Some definitions

Let $A = (a_{i,j})$ be an $X \times Y$ indexed game matrix,

- **Definition:** the *lower value* of the game A is:

$$V_{\mathrm{I}} = \underset{(p_i)}{\mathbf{sup}} \ \underset{(q_j)}{\mathbf{inf}} \ \sum_{i,j} a_{i,j} p_i q_j$$

(where (p_i) and (q_j) range over all possible mixed strategies.)

  This is the maximal expected gain I can guarantee himself by using mixed strategies.

It is a simple exercise to check that $\mathbf{inf}_{q_j} \sum_{i,j} a_{i,j} p_i q_j = \mathbf{min}_j \sum_i a_{i,j} p_i$, and to deduce that $V_{\mathrm{I}} \geq V'_{\mathrm{I}}$.
It is also not difficult to see that the above **sup** is achieved for some (p_i). This justifies the following:

- **Definition:** an *optimal strategy* for I is a mixed strategy (p_i) s.t. $\mathbf{inf}_{(q_j)} \sum_{i,j} a_{i,j} p_i q_j = V_{\mathrm{I}}$.

Similarly, we define:

- **Definition:** the *upper value* of the game is

$$V_{\mathrm{II}} = \underset{(q_j)}{\mathbf{inf}} \ \underset{(p_i)}{\mathbf{sup}} \ \sum_{i,j} a_{i,j} p_i q_j$$

  This is the minimal loss II can guarantee using mixed strategies.

  An optimal strategy for II is a mixed strategy (q_j) s.t. $\mathbf{sup}_{(p_i)} \ \sum_{i,j} a_{i,j} p_i q_j = V_{\mathrm{II}}$.

Like above, $V_{\mathrm{II}} \leq V'_{\mathrm{II}}$.

The following is obvious:

Lemma: for every (finite, 2 person, 0-sum) game we have $V_{\mathrm{I}} \leq V_{\mathrm{II}}$

proof: exercise.

The minimax theorem

The following is not obvious:

- **Proposition:** (John von Neumann minimax theorem)
  For every (finite, 2 person, 0-sum) game we have:

$$V_{\mathrm{I}} = V_{\mathrm{II}}$$

First, some lemmas:

Lemma: (separating hyperplane theorem)
Suppose B is a closed convex subset of $\mathbb{R}^n$ and let $x \notin B$,
then there exist numbers $a_1, \ldots a_{n+1}$ s.t.

- $a_{n+1} = \sum_{i=1}^n x_i a_i$;
- for $y \in b$, $\sum_{i=1}^n y_i a_i > a_{n+1}$.

  ◁ **Remark:** this only says that there is a hyperplane passing through x s.t. B lies on one side of the hyperplane and doesn't touch it...

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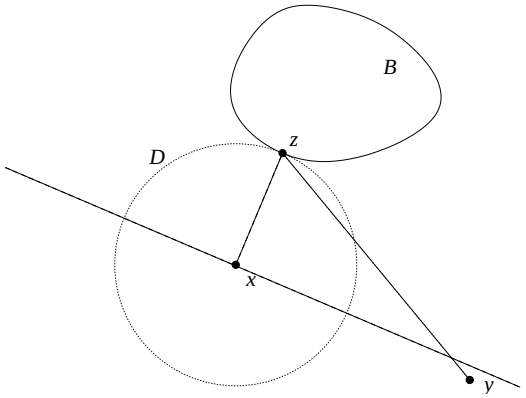
proof:

let $z \in B$ be the closest point to x (a closest point exists because B is closed and is unique because B is convex);

let P be the hyperplane passing through x orthogonal to $\overrightarrow{xz}$

this is the hyperplane we are after. To see that, suppose y is a "bad" point (*i. e.* $y \in B$, but y is either on P or on the wrong side of P)

let D be the disk of center x of radius xz. No point of B can lie strictly in D, but since B is convex, all point of the segment $[y, z]$ are in B.



But (2 dimensional geometry) there must be a point of this segment (close to z) which lies in D!
Contradiction... **QED**

Another lemma:

> **Lemma:** (theorem of the alternative for matrices)
> Let $A = (a_{i,j})$ be an $m \times n$ matrix. Then, either *(i)* or *(ii)* holds:
>
> *(i)* : in $\mathbb{R}^m$, the convex hull C of the n columns of A (as vectors in $\mathbb{R}^m$) and the m unit vectors $(e_1, \dots e_m)$ contains 0;
>
> *(ii)* : there exist numbers $p_1, \dots p_m$ s.t.:
> - $p_i > 0$ for all $i = 1, \dots m$,
> - $\sum_{i=1}^m p_i = 1$,
> - and $\sum_{i=1}^m p_i a_{i,j} > 0$ for all $j = 1, \dots n$
>
> ◁ **Remark:** in terms of game matrices, point *(ii)* just asserts the existence of a completely mixed strategy (all moves are chosen with strictly positive probability) giving strictly positive gain to I.

proof:

suppose *(i)* is false, *i. e.* $0 \notin C$. By the preceding lemma, take a hyperplane separating 0 and C:

$\exists a_1, \dots a_m$ (since it goes through 0, $a_{m+1} = 0$)
s.t. $\sum_{i=1}^m a_i y_i > 0$ for all $y \in C$.

In particular:

- for $y = e_i$, it gives $a_i > 0$,
- for y the jth column of A, it gives $\sum_{i=1}^m a_{i,j} a_i > 0$ for all $j = 1, \dots n$.

Finally, put $p_k = a_k / \sum_{i=1}^m a_{i,j}$. (p_i) satisfies the conditions of the lemma. **QED**

One last lemma:

> **Lemma:** suppose A is a game matrix, and that clause *(i)* from lemma above holds.
> Then, $V_{\text{II}} \leq 0$.

proof:

since $0 \in C$, 0 can be written as a convex combination of the A_i (columns of A) and the e_i:

$$\overrightarrow{0} = \sum_{j=1}^{n} s_j A_j + \sum_{j=n+1}^{n+m} s_j e_{j-n} \qquad \text{with } s_j \geq 0 \text{ and } \sum s_j = 1.$$

i. e. , for any $i = 1, \ldots m$, $0 = \sum_{j=1}^{n} s_j a_{i,j} + s_{n+i}$. $\qquad (*)$

The $s_1, \ldots s_n$ can't be all 0 since the e_i are linearly independent.

Let $q_i = s_i / \sum_{j=1}^{n} s_j$ for $i = 1, \ldots n$. (q_i) is a mixed strategy for II.

By $(*)$ we have that $\sum_{j=1}^{n} q_j a_{i,j} = -s_{n+i} / \sum_{j=1}^{n} s_j \leq 0$ for all $i = 1, \ldots n$.

This implies that V_{II} is negative. **QED**

proof (of the minimax theorem):

since either *(i)* or *(ii)* holds, we have that either $V_{\mathrm{II}} \leq 0$ (case *(i)*, see above) or $V_{\mathrm{I}} > 0$ (case *(ii)*) holds.

So, it is impossible to get $V_{\mathrm{I}} \leq 0 < V_{\mathrm{II}}$.

Suppose $V_{\mathrm{I}} < V_{\mathrm{II}}$, and choose $\alpha \in \mathbb{R}$ s.t. $V_{\mathrm{I}} + \alpha \leq 0 < V_{\mathrm{II}} + \alpha$.

Consider the new game matrix $\widetilde{A} = (a_{i,j} + \alpha)$. It is easy to see that $\widetilde{V_{\mathrm{I}}} = V_{\mathrm{I}} + \alpha$ and $\widetilde{V_{\mathrm{II}}} = V_{\mathrm{II}} + \alpha$.

This gives a game $\widetilde{A}$ s.t. $\widetilde{V_{\mathrm{I}}} \leq 0 < \widetilde{V_{\mathrm{II}}}$.

Contradiction, this is impossible. **QED**

We can now define the value of a game:

- **Definition:** the value of a game V is its upper (lower) value.

The equilibrium theorem

- **Proposition:** fix some $k \in X$ (choice of a strategy for I); then the following are equivalent:
  (1) there is an optimal strategy (p_i) for I giving strictly positive weight to k;
  (2) k is a best response against *all* optimal strategies for II (*i. e.* for any optimal (q_j), for any i, $\sum_j q_j a_{i,j} = V$).

proof: we will only show (1)⇒(2).

Suppose the implication is not true: *i. e.* there is an optimal strategy for I (p_i) s.t. $p_k > 0$ and there is an optimal strategy for II (q_j) s.t. $\sum_j a_{k,j} q_j < V$.

Since $\sum_i p_i \sum_j a_{i,j} q_j = V$ and the kth term $(\sum a_{k,j} q_j)$ which is strictly less than V is given strictly positive weight, some other term (with strictly positive weight) needs to be strictly greater than V!

This is impossible since q is optimal for II. **QED**