Game Theory, Lecture 3

Course given by Jeff Steif,
notes made by Mats Kvarnström[1]

First, recall:

Theorem 1 (Equilibrium Theorem). *Fix $k \in X$. The following is equivalent:*

1. *$\exists$ an optimal strategy (o.s.), p, for I such that $p_k > 0$*
2. *k is a best response against all o.s. for II, i.e. $\forall$ o.s. q*

$$\sum_j a_{kj} q_j = V \tag{1}$$

where V is the value of the game.

Remark. *Last time we proved $1 \Rightarrow 2$. Note also that the book by Ferguson only states this theorem as the implication downwards.*

The "all" in 2 is critical:

Example 1.

Consider the pay-off matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Obviously, the value is 1 and the only optimal strategy for I is $x = \{2\}$.

It turns out, that every strategy for II is optimal, since whatever II does, he is guaranteed a loss at most the value of the game.

Check with the ET for $k = 1$: 1) is false, but still k is a best response to some optimal strategy for II, namely $y = \{1\}$.

Example 2 (Odds-Even).

Each player simultaneously yells out one of the numbers 0,1, or 2. If the sum of the numbers is odd (even), player I wins (loses) this sum. The pay-off matrix becomes:

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix}$$

Questions: What's the value and what are some optimal strategy for I and II?

First, notice that there are no dominant strategies (i.e. we can't rule out any rows or columns).

Now, **assume** that each player has an o.s. which gives positive weight to each move. Then ET says that for all optimal strategies q for Player II, we have

$$\sum_j a_{kj} q_j = V$$

[1] email:matskv@math.chalmers.se

for all $k = 1, 2, 3$. Also, for all optimal strategies p for I,

$$\sum_i p_i a_{ik} = V$$

for all $k = 1, 2, 3$.

For p we get a set of linear equations

$$\begin{cases} p_2 - 2p_3 = V \\ p_1 - 2p_2 + 3p_3 = V \\ -2p_1 + 3p_2 - 4p_3 = V \end{cases} \tag{2}$$

Together with the requirement $p_1 + p_2 + p_3 = 1$, we get $V = 0$ and $p = (p_1, p_2, p_3) = (1/4, 1/2, 1/4)$ by elementary calculations. Notice that this doesn't yet tell us that the value is 0 because of the assumption we made. However, we have that this strategy p guarantees I a profit of at least 0.

By symmetry of the matrix (or by repeating everything above for q), we see that II is guaranteed a loss of at most 0 by using the mixed strategy $(q_1, q_2, q_3) = (1/4, 1/2, 1/4)$.

By the definition of the value of the game, these two results says that the value of the game actually is 0 and then that $p = q = (1/4, 1/2, 1/4)$ are optimal strategies for I and II. Observe, now we know that the assumption holds for some optimal strategy, and hence any optimal strategy must satisfy the linear equations above (by the ET). Since these equations have a unique solution we have found the unique optimal strategy.

The Minimax Theorem revisited, with an alternative proof

Assume that the pay-off matrix A is an 2 x n matrix. ("Of course, there is an analog of this in higher dimensions")

Theorem 2 (Minimax Theorem). *Let H be the convex hull of the columns of the matrix A and $v = \sup\{w : (-\infty, w] \times (-\infty, w] \cap H = \emptyset\}$.*

Then v is the value of the game.

Remark. *We don't assume $V_I = V_{II}$ here; it will come out of the analysis.*

Proof. Write $H = \{Aq : q \in R^n, \sum q_i = 1\}$ and denote $K = (-\infty, v] \times (-\infty, v]$ with v defined as in the statement of the theorem.

Let furthermore $\tilde{q}$ be a probability vector such that $A\tilde{q} \in K$ (so $A\tilde{q} \in H \cap K$) and let l be a line separating K and H.

Let $\tilde{p}$ be the probability vector perpendicular to l.

Note that $l = \{x \in R^2 : \tilde{p}^T x = v\}$ and that $\tilde{p}^T h \geq v$, $\forall h \in H$, or, by the definition of H:

$$\tilde{p}^T Aq \geq v \quad \forall q \tag{3}$$

so that the value is at least v (since whatever II does, $\tilde{p}$ guarantees I a profit of at least v).

Now for the opposite inequality. For all probability vectors p and for all $x \in K$

$$p^T x \leq v$$

since K is "on the left" of any line touching (v, v) of negative slope (a line touching (v,v) with p as normal vector has negative slope since p has non-negative components). In particular, we have, since $A\tilde{q} \in K$,

$$p^T A\tilde{q} \leq v \quad \forall p \tag{4}$$

and so the value can be at most v.

Now, the two inequalities (3) and (4) together implies that the value is v, and that $\tilde{p}$ and $\tilde{q}$ are optimal strategies. $\square$

Observation (or exercise): The optimal strategy for I is unique if and only if the line separating K and H is unique.

Example 3 (The 2-Generals Game). General I has 4 regiments and general II has 3 regiments.

The generals will send their regiments to two posts. The rules are: regiments can not be split into smaller entities than 1, and you have to send them all.

For each post, whoever sent the most regiments captures this post and the regiments from the other team. No one wins in case of a draw.

Example: if I sends 2 regiments to each post and II sends all his 3 to the second post, the pay-off becomes 1-(1+2)=-2.

Our first observation is that the value should be ≥ 0 since I can send all of his four regiments to one post, thereby guaranteeing himself a profit of at least zero.

Note that this simple argument doesn't work if there are three posts...

The pay-off matrix is pretty large (5×4) and we want a simple way to calculate the value an the optimal strategies by hand. There is a concept called invariance which is useful for this situation and that we will look at next time.

Example 4. Look at the pay-off matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 1 & 2 & 5 & 3 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

First: we have no saddle points.

Since the fourth column is (strictly) dominated by the second (the definition of domination used here is that a column j dominates the k:th column if $a_{ij} \leq a_{ik}$ for all $i \in X$; if we have strict inequality in every component, we have **strict** domination), we reduce the game to

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$

since if (q_1, q_2, q_3, q_4) were an optimal strategy with $q_4 > 0$, then $(q_1, q_2 + q_4, q_3, 0)$ does better against all moves (of player I), and hence must also be an optimal strategy.

Generally: if the j:th column **strictly** dominates the k:th column, then no optimal strategy charges k.

But, if the j:th column **weakly** dominates the k:th, there exists an optimal strategy that doesn't charge it, and so we can remove the k:th column **if** we only are looking for **some** optimal strategy. Analogously for rows.

In this example, (strict) domination reduces the problem to

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

which is easy to solve.