Game Theory, Lecture 4

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After this lecture, we are done with finite zero-sum games. At least as far as lecturing on it...

A few extra comments on finite zero-sum games

Domination

We have seen the use of removal of rows and columns by the use of dominant strategies. There also a thing which Jeff calls "mixed domination".

Example 1.

Look at:

$$\begin{bmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{bmatrix}$$

Here, we can not remove any column (or row) by domination. However, we have $c_2 \geq 1/2c_1 + 1/2c_3$ (where c_i is the i:th column) and so, if (q_1, q_2, q_3) is an optimal strategy for Player II, she would do just as well with $(q_1 + 1/2q_2, 0, q_3 + 1/2q_2)$, so this must also be be an optimal strategy. (In fact, we have strict mixed domination since $c_2 > \lambda c_1 + (1-\lambda)c_3$, by choosing $\lambda \in (1/3, 1/2)$, so no optimal strategy for II can give positive weight to column 2.)

By removing the second column, we get

$$\begin{bmatrix} 0 & 6 \\ 5 & 4 \\ 9 & 3 \end{bmatrix},$$

and here $r_2 \leq 1/3r_1 + 2/3r_3$ (where r_i is the i:th row), so by the analogous reasoning we it is enough to look for the value and optimal strategies in

$$\begin{bmatrix} 0 & 6 \\ 9 & 3 \end{bmatrix}.$$

Comment: When we don't have strict domination (mixed or ordinary), the order of the removal of columns and rows matters, in the sense that we might end up with different optimal strategies depending on the order. But, if we are only looking for **some** optimal strategy, this problem does not concern us.

In the example above, we actually have strict domination in both reductions, and therefore, all optimal strategies can be found from the resulting 2×2 matrix.

Diagonal Games

For a square and diagonal pay-off matrix

$$\begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{bmatrix},$$

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we get, by using ET, that (we assume that all $d_i > 0$, since if the elements have different signs we have a saddle point)

$$p_i = \frac{1}{d_i}\Big(\frac{1}{\sum_j 1/d_j}\Big).$$

Symmetric Games

Definition 1. *A finite game is* ***symmetric*** *if the pay-off matrix A is square and skew-symmetric:*

$$a_{ij} = -a_{ji}$$

for all i and j.

Example 2 (Paper-Scissors-Rock).

Denote the moves Paper, Scissors, and Rock by 1,2, and 3, respectively.

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Theorem 1. *A finite symmetric game has value 0 and an optimal strategy for I is an optimal strategy for II.*

The proof is an easy exercise.

Invariant Games

Example 3 (2-Generals Game, Revisited). Player I's moves are $(4,0)$, $(3,1)$, $(2,2)$, $(1,3)$, and $(0,4)$, where the two numbers denotes the number of regiments sent to the first and second post, respectively. For Player II, the moves are $(3,0)$, $(2,1)$, $(1,2)$, and $(0,3)$.

The pay-off matrix becomes:

$$\begin{bmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{bmatrix},$$

where the order of rows and columns correspond to the order above.

Note that we have no saddle point and no dominant rows or columns. However, it has a certain invariance.

In general: Let $A = (a_{ij})$ an $m \times n$ matrix and S_k the set of permutations of k elements.

Notation: Let $G_1 = \{\pi \in S_m : \exists \pi' \in S_n \text{ such that } a_{ij} = a_{\pi(i)\pi'(j)} \;\; \forall i,j\}$.

Note that if two rows are identical, then flipping the rows is in G_1, and the corresponding π' can be taken to be the identity permutation, denoted e.

For

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

we have $G_1 = \{e, (1,2)\}$ where corresponding π' is e and $(1,2)$, respectively.

For the columns, we define the corresponding G_2:
$G_2 = \{\pi \in S_n : \exists \pi' \in S_m \text{ such that } a_{ij} = a_{\pi'(i)\pi(j)} \ \ \forall i, j\}$.

We say that the rows i and k are **related** if $\exists\, g \in G_1 : g(i) = k$.

This breaks up the rows into equivalence classes, i.e. different sets where in each set, everyone is related, and if i and k are in different classes, they are not related.

Definition 2. *A mixed strategy p for I is* ***invariant****, if we have that, whenever k and l are related (i.e. $\exists g \in G_1 : g(k) = l$), then $p_k = p_l$.*

Theorem 2. *For a finite game, an invariant optimal strategy always exists.*

This is of course useful when we want to calculate optimal strategies.

Example 4.

- For $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $G_1 = \{e, (1, 2)\}$, so there is only one equivalent class, and hence, by Theorem 2, p_1 and p_2 must be equal, so $p_1 = p_2 = 1/2$.
- For
$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$
we have $G_1 = \{e, (1, 2, 3), (1, 3, 2)\}$ and so, once again, only one equivalence class (note however that $(1, 2) \notin G_1$) $\Rightarrow$ only one invariant strategy $\Rightarrow p = (1/3, 1/3, 1/3)$ an optional strategy for I.

Example 3 (cont.).

In the 2-Generals Game, there exists an invariant optimal strategy (since the two posts are identical, so interchanging their roles doesn't effect the game), with G_1 having 2 elements; $G_1 = \{e, (5, 4, 3, 2, 1)\}$ (with corresponding $\pi' = e$ and $(4, 3, 2, 1)$, respectively).

We have two equivalence classes each for the two players:

- For Player I:
  1. $\{(4, 0), (0, 4)\}$
  2. $\{(3, 1), (1, 3)\}$
  3. $\{(2, 2)\}$
- For Player II:
  1. $\{(3, 0), (0, 3)\}$
  2. $\{(2, 1), (1, 2)\}$

Knowing this, Theorem 2 tells us that

- $\exists$ an o.s. $p = (p_1, \ldots, p_5)$ for I: $p_1 = p_5$, $p_2 = p_4$
- $\exists$ an o.s. $q = (q_1, \ldots, q_4)$ for II: $q_1 = q_4$, $q_2 = q_3$

Think of Player I's moves as $(4,0)^*, (3,1)^*$, and $(2,2)$, where $(4,0)^*$ means that you choose $(4,0)$ or $(0,4)$ each with probability $1/2$.

Similarly, for Player I; $(3,0)^*$ and $(2,1)^*$.

This gives us a 3×2 game instead:

$$\begin{bmatrix} 2 & 3/2 \\ 0 & 3/2 \\ -2 & 2 \end{bmatrix} \tag{1}$$

where the rows are the moves of Player I given in the order as above. Likewise for the columns.

Since $r_2 \leq r_1$, in suffices to look for the value and optimal strategies in

$$\begin{bmatrix} 2 & 3/2 \\ -2 & 2 \end{bmatrix},$$

resulting in $v = 14/9$, and optimal strategies $(8/9, 1/9)$ for I, and $(1/9, 8/9)$ for II. This corresponds to

- $p = \frac{1}{9}(4,0,1,0,4)$ and
- $q = \frac{1}{18}(1,8,8,1)$

for the "original" moves.

Remarks:

1. Player I can really guarantee (not just the expected value) himself 0 by a $(4,0)$ split.
2. If $(3,1)$ is played against the optimal strategy $\frac{1}{18}(1,8,8,1)$ for II, the expected pay-off is $12/9 < v$. Hence, the Equilibrium Theorem tells us that **no** optimal strategy for I gives positive weight for $(3,1)$, so our removal of the second row in (1) did in fact not matter.

Infinite Games

Infinite games are games which are played once, but where we have have infinite number of choices. We have two types:

1. moves are countable
2. moves are not countable

The Minimax Theorem can fail for the case 1 above:

Example 5. Each player simultaneously calls out a positive number, and the player with the smallest number gives a dollar to the other player.

Here $X = Y = \{1, 2, \ldots\}$ with pay-off function a as

$$a(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x = y \\ -1 & \text{if } x < y \end{cases}$$

It turns out that the value for this game doesn't exists.

Exercise: Show that $V_I = 1$ and $V_{II} = -1$.