Game theory, lecture 5

Infinite games

First, recall the example at the end of the last lecture. The problem with this example was the lack of compactness.

A probability measure μ on $[0,1]$ is an optimal strategy if

$$\inf_{y\in[0,1]} \int A(x,y)d\mu(x) = \sup_{\nu} \inf_{y\in[0,1]} \int A(x,y)d\nu(x) := V_I$$

We now state and prove a theorem that we always have an optimal strategy.

Theorem Let X and Y be $[0,1]$. Let $A : X \times Y \to \mathbb{R}$ be a continuous function. Then there exists an optimal strategy for each of the players I and II, and $V_I = V_{II}$.

Proof By continuity, for any measure ν,

$$\inf_{y\in[0,1]} \int A(x,y)d\nu(x)$$

is achieved at some $y \in Y$. First, for all n there exists a probability measure μ_n such that

$$\inf_{y} \int A(x,y)d\mu_n(x) \geq V_I - \frac{1}{n} .$$

(This convergence is called either vague convergence, weak convergence or weak* convergence depending on who you are.)

There exists a subsequence of the μ_n, μ_{n_k} such that $\mu_{n_k} \to \mu_\infty$. We leave it as an exercise to prove that μ_∞ is an optimal strategy.

To prove $V_I = V_{II}$, we approximate A by finite games. It is trivial that $V_I \leq V_{II}$.

Let $\varepsilon > 0$. We show $V_I \geq V_{II} - \varepsilon$. Given ε there exists n such that for any u, $v \in [0,1]$, whenever $|u-v| < \frac{1}{n}$, we have $|A(u) - A(v)| < \varepsilon$. Let A_n denote the game with $X = Y = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$, $A_n(x,y) = A(x,y)$.

Let V_I^n, V_{II}^n be the values for A_n (we know that $V_I^n = V_{II}^n$ by the minimax theorem) and let $p^{(n)}$ and $q^{(n)}$ be optimal strategies for players I and

II. Now

$$V_I \geq \inf_y \sum_{i=0}^{n} A(\frac{i}{n}, y) p_i^{(n)} \geq \min_j \sum_{i=0}^{n} A(\frac{i}{n}, \frac{j}{n}) p_i^n - \varepsilon = V_I^n - \varepsilon$$

So we have that $V_I \geq V_I^n - \varepsilon$.

The same analysis gives $V_{II} \leq V_{II}^n + \varepsilon$, so we get

$$V_I \geq V_I^n - \varepsilon = V_{II}^n - \varepsilon \geq V_{II} - 2\varepsilon$$

$\square$

The duel problem

Let $X, Y = [0, 1]$. At time 0 the two players are 1 mile apart, and they begin to walk towards each other at constant speed such that they meet at time 1. Each player has a "silencer", which they can use once. If they shoot at time x, the probability of hitting is x. This is a symmetric game; player I wins 1$ if I shoots and hits II, and player II wins 1$ if II shoots and hits I. If they shoot at the same time and both hit or don't hit, or if no one shoots, the payoff to both is 0.

Does this game have a value? We can't use the theorem since the expected payoff is not continuous (e.g. $A(0.9, 0.9 + \varepsilon)$ and $A(0.9, 0.9 - \varepsilon)$ are very different). But if a value exists, then $V = 0$ by symmetry.

What are optimal strategies for the players? What is $A(x, y)$?

If $x < y$ then $A(x, y) = 1 \cdot x + (-1)(1 - x)y = x - y + xy$,

and if $x > y$ then $A(x, y) = -1 \cdot y + 1 \cdot (1 - y)x = x - y - xy$.

Let $K(x, y)$ be the payoff if $x < y$, and let $L(x, y)$ be the payoff if $x > y$. We have that $K(x, y) = -L(y, x)$, i.e. this is a symmetric game.

Let us assume that the distribution function for an (the?) optimal strategy is given by F. We will derive a number of properties F must satisfy. We can then solve for F, and verify afterwards that it is an optimal strategy. Let

$$E[F, y] := \int_0^1 A(x, y) F'(x) dx \ .$$

Assume $F' > 0$ on some interval (a, b) and $F' = 0$ outside of (a, b) ($[a, b]$ is the support of the measure $dF = F'(x)dx$). By the equilibrium theorem for finite games and assuming the value exists and is 0 we should have that $E[F, y] = 0$ for all $y \in [a, b]$. This implies that $\frac{d}{dy}E[F, y] = 0$ on (a, b), i.e.

$$\frac{d}{dy}[\int_a^y K(x,y)F'(x)dx + \int_y^b L(x,y)F'(x)dx] = 0$$

$$\Rightarrow \int_a^y K_y(x,y)F'(x)dx + \lim_{x \nearrow y} K(x,y)F'(y) + \int_y^b L_y(x,y)F'(x)dx - \lim_{x \searrow y} L(x,y)F'(y) =$$

$$= \int_a^y (x-1)F'(x)dx + y^2F'(y) + \int_y^b (-1-x)F'(x)dx + y^2F'(y) = 0$$

$$\Leftrightarrow -2y^2F'(y) = \int_a^y (x-1)F'(x)dx + \int_y^b (-x-1)F'(x)dx$$

Differentiate with respect to y:

$$-4yF'(y) - 2y^2F''(y) = (y-1)F'(y) + (y+1)F'(y) = 2yF'(y)$$

$$\Leftrightarrow -2y^2F''(y) = 6yF'(y)$$

$$\Leftrightarrow -2yF''(y) = 6F'(y)$$

$$\Leftrightarrow F'(y) = \frac{k}{y^3}$$

(The latter being derived from elementary differential equations.)

We need to find k, a and b.

Claim $b = 1$.

We know that $E[F, y] = 0 \ \forall y \in (a, b)$. This implies by continuity that $E[F, b]$ should also be 0, and so

$$\int_a^b K(x,b)F'(x)dx = 0$$

$$\Rightarrow \int_a^b (x - b + xb)F'(x)dx = 0$$

If $b < 1$, increase b to 1 in the integrand (not in the limit). The integrand decreases strictly since clearly $\frac{\partial}{\partial b}[x - b(1-x)] < 0$. This implies that

$$\int_a^b K(x,1)F'(x)dx < 0 \ ,$$

which contradicts the fact that F is an optimal strategy. Hence $b = 1$, as claimed.

We now compute a. We know that $E[F, 1] = 0$, so

$$\int_a^1 (2x - 1)\frac{k}{x^3}dx = 0$$

From this equation we compute a. It turns out that $a = \frac{2 \pm \sqrt{4-3}}{3} = \{1, \frac{1}{3}\}$. It is unreasonable that $a = 1$, so $a = \frac{1}{3}$. Given a and b we can now compute $k = \frac{1}{4}$. So the density function is

$$F'(x) = \begin{cases} 0 & \text{if } x < \frac{1}{3} \\ \frac{1}{4}\frac{1}{x^3} & \text{if } x \geq \frac{1}{3} \end{cases}$$

The last step is to verify that F works. One easily computes

$$E[F, y] = \begin{cases} 0 & \text{if } y \geq \frac{1}{3} \\ > 0 & \text{if } y < \frac{1}{3} \end{cases}$$

Hence $V_I \geq 0$. Since $V_{II} \leq 0$ by an identical argument (or symmetry), it follows that $V_I = V_{II}$ and F is an optimal strategy. Is this the only optimal strategy or are there other ones?