Game theory : Lecture 7

To finish the proof that Brouwer's Fixed Point Theorem implies existence of a NE for a k-person zero-sum game, it remains to prove

Lemma *$(\tilde{\pi_1}, ..., \tilde{\pi_k})$ is a NE if and only if it is a fixed point of g.*

Proof : If $(\tilde{\pi_1}, ..., \tilde{\pi_k})$ is a NE then no player has any incentive to unilaterally change strategy $\Rightarrow \psi_{ij} = 0 \; \forall \; i, j \Rightarrow g(\tilde{\pi_1}, ..., \tilde{\pi_k}) = (\tilde{\pi_1}, ..., \tilde{\pi_k})$.

In the other direction, simply observe that if some $\Psi_{ij} > 0$ then $(g(\tilde{\pi}))_i \neq \tilde{\pi}_i$ since $\Psi_{ij'} = 0$ for some $j' \neq j$.

Proof of Brouwer F.P.T.

We will prove the theorem for a triangle in $\mathbf{R}^2$. The proof employs a result called Sperner's Lemma, for which we will present two different proofs. All the ideas can then be extended to the case of a general compact, convex subset of $\mathbf{R}^n$.

Before embarking on our journey, we remark that for subsets of $\mathbf{R}^1$, there is a particularly simple proof of the FPT using the Intermediate Value Theorem (IVT), which unfortunately does not generalise.

A compact, convex subset of $\mathbf{R}^1$ is just a finite closed interval. So WLOG let $X = [0, 1]$ and let $g : X \to X$ be a continuous function. Define $h : X \to X$ by $h(x) := g(x) - x$. Then clearly $h(0) \geq 0$ and $h(1) \leq 0$, and since h is continuous, IVT implies that $h(x_0) = x_0$ for some $x_0 \in (0, 1)$. Then x_0 is a fixed point of g.

Now let T be a triangle in the plane with vertices A, B, C. Sperner's lemma reads as follows :

Sperner's lemma *Label the vertices A, B, C with $0, 1, 2$ respectively. Consider any triangulation of T and extend the labelling to all the vertices in the triangulation in a 'nice' manner, i.e.: such that*

- *all the vertices on the edge AB get either 0 or 1,*
- *all the vertices on the edge AC get either 0 or 2,*

- *all the vertices on the edge BC get either 1 or 2.*

Then there exists a (0,1,2)-triangle in the triangulation, that is, a triangle whose three vertices get one each of the labels 0,1 and 2.

We first show how the FPT follows from Sperner's Lemma. The proof is by contradiction. So let $g : T \to T$ be a continuous mapping without fixed points. Take any sequence T_n of triangulations whose meshes go to zero. We describe a 'nice' labelling of each T_n :

For each $v \in T_n$ other than A, B, C, choose a side s of T such that $g(v)$ is strictly closer to s than v. Note that there could be two such sides for a given vertex v, but in that case we may choose one arbitrarily. We then assign to the vertex v the label of that of A, B, C which is opposite the chosen side s.

It is obvious that this labelling is 'nice'. Hence by Sperner's lemma, there exists a $(0,1,2)$-triangle for each T_n. Choose one such triangle for each n and call it t_n. Then choose any point $x_n \in t_n$ (including interior).

By compactness there exists a subsequence x_{n_k} which converges in T, to a point x_∞, say. We claim that x_∞ is a fixed point of g. For suppose the contrary. Then $g(x_\infty)$ is further away than x_∞ from some side s of T. WLOG, suppose A is the vertex of T opposite s. By continuity of g, there thus exists a neighbourhood N of x_∞ such that, for every point $y \in N$, $g(y)$ is further away from s than y. Choose k sufficiently large such that $t_{n_k} \subset N$. This containment certainly implies that no vertex of t_{n_k} can get the same label as A. But this contradicts the fact that t_{n_k} is a $(0,1,2)$-triangle.

This completes the proof of the FPT.

FIRST PROOF OF SPERNER'S LEMMA : We will in fact prove that the number of $(0,1,2)$-triangles is odd. The following simple observation will prove useful :

Observation : Divide the interval $[0,1]$ into a finite number of subintervals. Give every vertex of the subdivision either 0 or 1 as a label, with the restriction that the point 0 gets the label 0 and the point 1 gets the label 1. Then there are an odd number of $(0,1)$-subintervals.

We consider now a fixed triangulation of T (which we also denote by T)

together with a fixed 'nice' labelling. Let $\mathcal{C}$ denote the collection of triangles with both 0 and 1 and $\mathcal{C}' \subseteq \mathcal{C}$ the collection of $(0,1,2)$-triangles. Further, let $\mathcal{D}$ denote the collection of $(0,1)$-edges in T and $\mathcal{D}' \subseteq \mathcal{D}$ those $(0,1)$-edges which lie on the side AB.

We claim that

$$|\mathcal{C}'| + 2 \cdot |\mathcal{C} \backslash \mathcal{C}'| = |\mathcal{D}'| + 2 \cdot |\mathcal{D} \backslash \mathcal{D}'|. \tag{1}$$

If true then this implies that $|\mathcal{C}'|$ is odd, as required, since from the observation above we can immediately state that $|\mathcal{D}'|$ is odd.

To prove the claim is quite straightforward. One need only verify that both sides of (1) equal the cardinality of the set of pairs

$$(D, C) \in \mathcal{D} \times \mathcal{C} \;\text{ such that }\; D \subset C.$$

Second proof of Sperner's lemma : Here's a rather more ingenious proof. Again we consider a fixed triangulation and a fixed 'nice' labelling.

Consider each triangle of the triangulation as a 'room' and each edge as either a 'door' or a 'wall'. More precsiely, the $(1,2)$-edges are doors and all other edges walls. By the observation above, there are an odd number of doors along the edge BC. Enter the triangle via one such door and walk around inside in such a way that you never use any door more than once. Continue walking until either

(i) you get stuck in some room with no way out

(ii) you exit through another door along the edge BC.

If (ii) happens then, since there are an odd number of entrances, you can reenter the triangle and continue walking. It follows that at some point you must in fact get stuck in some room. But then it's clear that any such room is a $(0,1,2)$-triangle. Q.E.D.

Question *Is BFPT still true in infinite dimensions ?*

The answer is 'yes' in the following sense :

Theorem *Let $\mathcal{B}$ be a Banach space and X a compact, convex subset of $\mathcal{B}$. If $g : X \to X$ is a continuous function, then g has a fixed point.*

However, this theorem fails if we take X to be the unit ball in an infinite-dimensional Hilbert space H. We illustrate by means of an example :

EXAMPLE : Let $H = l_2 = l_2(\mathbf{R})$ be the space of square-summable doubly-infinite sequences of real numbers, i.e.:

$$l_2 = \{x = (x_n)_{-\infty}^{\infty} : \sum_{-\infty}^{\infty} |x_n|^2 < \infty\}.$$

Let $U : l_2 \to l_2$ be the right-shift operator, i.e.:

$$(Ux)_n = x_{n-1} \qquad \forall\, x \in l_2.$$

Let X be the unit ball in l_2 and let $g : X \to X$ be the function

$$g(x) = U(x) + (1 - ||x||)z,$$

where $z \in l_2$ is the sequence such that $z_0 = 1$ and $z_n = 0\ \forall\, n \neq 0$.

That $g(X) \subseteq X$ follows from the triangle inequality. If $x = (x_n)$ is a fixed point of g, then it's easy to see that

$$\begin{aligned} x_0 = x_1 = x_2 = \cdots \\ \text{and } x_{-1} = x_{-2} = x_{-3} = \cdots \end{aligned}$$

But since $||x|| \leq 1$ it follows that every component of x must equal zero, i.e.: $x \equiv 0$. But $g(0) = z \neq 0$, a contradiction.