Game theory : Lecture 8

For finite, zero-sum games, the concepts of 'Nash equilibrium' and 'optimal strategy' coincide. More precisely, we have

Theorem *Consider a 2-person, finite, zero-sum game. Let $\mathcal{E}$ denote the set of Nash equilibria. Let $\mathcal{O}_i$, $i = 1, 2$, denote the set of optimal strategies for player i. Then $\mathcal{E} = \mathcal{O}_1 \times \mathcal{O}_2$.*

PROOF : Let A denote the matrix of the game and v its' value. First suppose $(p, q) \in \mathcal{O}_1 \times \mathcal{O}_2$. Then $pAq = v$. That q is optimal for player 2 implies that $pAq \geq \tilde{p}Aq$ for any $\tilde{p} \neq p$. Hence, player 1 has no incentive to unilaterally change strategy. Similarly for player 2 ; hence, $(p, q) \in \mathcal{E}$.

Now suppose $(p, q) \notin \mathcal{O}_1 \times \mathcal{O}_2$. WLOG, say $p \notin \mathcal{O}_1$. We consider two cases :

Case 1 : $pAq \geq v$. Since p is not optimal for player 1, there must exist (by the minimax theorem) SOME strategy $\tilde{q}$ for player 2 such that $pA\tilde{q} < v$. But then $pAq > pA\tilde{q}$, so player 2 has an incentive to unilaterally change and $(p, q) \notin \mathcal{E}$.

Case II : $pAq < v$. We know there exists SOME optimal strategy $\tilde{p}$ for player 1, and since $\tilde{p}Aq = v > pAq$, player 1 has the incentive to unilaterally change to this strategy. Hence, $(p, q) \notin \mathcal{E}$ in this case either, which completes the proof of the theorem.

Now let's go back to the setting of general sum games.

EXAMPLE 1 :

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix}.$$

This is a rather uninteresting game. There are two PNE : $(1, 1)$ or $(2, 2)$. If the players could communicate, they would quickly agree to play $(2, 2)$ and there'd be no need to enforce the agreement with a contract.

EXAMPLE 2 (PRISONER'S DILEMMA) :

$$\begin{pmatrix} (3,3) & (0,4) \\ (4,0) & (1,1) \end{pmatrix}.$$

The only NE is $(2,2)$. If the players could communicate, they would agree to play $(1,1)$, but only if a contract were signed.

REMARK : Consider the game which consists of N rounds of Prisoner's Dilemma, where N is a fixed positive integer, known to both players in advance. Then, by induction, it is easy to see that the same analysis applies as in the $N=1$ case. Things get a lot more interesting if N is a randomly chosen number, unknown to the participants. We will return to analyse this situation in a later lecture.

EXAMPLE 3 (COURNOT MODEL FOR DUOPOLY) : This example should appeal to anyone interested in economics.

Suppose you have two firms producing the same good. Let C be the unit production cost, and assume there are no other costs. Let q_i be the number of units produced by firm i. Suppose the unit market price P depends on the total output Q according to the formula

$$P(Q) = \max\{0, a-Q\} := (a-Q)^+,$$

where a is some fixed positive real number. Let u_i denote the profit made by firm i. We easily compute

$$u_1(q_1,q_2) = q_1\,[a-(q_1+q_2)]^+ - cq_1, \tag{1}$$
$$u_2(q_1,q_2) = q_2\,[a-(q_1+q_2)]^+ - cq_2. \tag{2}$$

In this situation, we have a 2-player game with $X = Y = [0,a]$. Assume $a > c$, as otherwise neither firm has any incentive to produce anything. We are interested in the NE for this game.

First, though, let us quickly analyse the monopoly situation ($q_2 = 0$). Then

$$u(q_1) = q_1(a-c) - q_1^2.$$

Obviously this is the quantity the firm will want to maximise and we easily compute that the maximum is achieved at $q_1 = (a-c)/2$ and equals

$(a-c)^2/4$. The unit price is then $(a+c)/2$.

Now let's return to the duopoly model. At a PNE we have

$$\frac{\partial u_1}{\partial q_1} = \frac{\partial u_2}{\partial q_2} = 0.$$

Substituting into (1),(2) and solving yields the unique solution

$$q_1 = q_2 = \frac{a-c}{3}.$$

Hence the total production is now $2(a-c)/3$. The corresponding unit price is $(a+c)/3$ and the profit to each firm is $(a-c)^2/9$, thus giving a total profit of $2(a-c)^2/9$.

The conclusion we can draw from this is that it would be better for both firms to set up a cartel, whereas it is better for 'society' that they don't do so.

Correlated equilibrium

This notion generalises that of NE. We will define and discuss it formally in the next lecture and content ourselves now with a motivational example.

EXAMPLE : Consider the game

$$\begin{pmatrix} (5,1) & (0,0) \\ (4,4) & (1,5) \end{pmatrix}.$$

Both $(1,1)$ and $(2,2)$ are PNE. A third NE is where both players play with probability vectors $(1/2, 1/2)$. The payoffs in each of the three situations are respectively $(5,1)$, $(1,5)$ and $(5/2, 5/2)$.

Now if the two players could meet and sign a contract, they would obviously be willing to agree on playing $(2,1)$. But suppose this is not possible. Is there anything we can do to make sure both get a higher payoff than $5/2$? The answer is yes, if there is a trusted third party who can correlate the strategies of the 2 players.

First consider that it may be beneficial to use a convex combination of NE. In general, suppose there are k NE with payoff vectors (a_i, b_i), $i = 1, ..., k$. If both players play a convex combination $(p_1, ..., p_k)$ of these strategies then the resulting payoff vector will be

$$\sum_{i=1}^{k} p_i(a_i, b_i).$$

We can ensure that both players adopt this strategy by having the trusted 3rd party flip a k-sided 'coin' with probability distribution $(p_1, ..., p_k)$. He announces the result of the toss to each player. Neither player has any incentive to disobey the 3rd party's recommendation as long as he believes his opponent will obey.

In our example, we have $k = 3$. It's easy to check that the total payoff is maximised when $p_1 = p_2 = 1/2$, and the payoff vector is then $(3, 3)$.

If we make use of the 3rd party in a more clever way, however, we can do even better than this. The idea is that the 3rd party should only partly reveal the result of each coin toss to each player. For example : let the three NE be called A,B,C. Suppose the 3rd party tosses a $(1/3, 1/3, 1/3)$-coin and then reveals the following information :

- player I is only told whether the result of the toss were A or not,
- player II is only told if the result of the toss were C or not.
- player I is told to play move 1 if A occurs and move 2 if either B or C occurs
- player II is told to play move 1 if either A or B occurs and move 2 if C occurs.

Observe that, if the players do what they're told, then each of the moves $(1, 1)$, $(2, 1)$ and $(2, 2)$ is equally likely, and so the expected payoff is $(10/3, 10/3)$, i.e.: better than before ! But notice also that neither player has an incentive to unilaterally disobey.