Homework 1: Due October 30

You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, you should write up the final solutions yourselves and tell me who you have discussed the problems with. Solutions should always be clear and concise (and hopefully of course correct).

1. Let f be a function from $\{0,1\}^n$ to R (with the domain have uniform measure) and let $\mathcal{S}$ be the spectral sample associated to f. Show that if A is a subset of $1,\ldots,n$, then $P(\mathcal{S} \subseteq A) = E[E[f|\mathcal{F}_A]^2]$ where $\mathcal{F}_A$ is the σ-algebra generated by the variables in A. (Observe what this says in the two special cases when $A = \emptyset$ or $A = \{1,\ldots,n\}$.)

2. Recall given a Boolean function, $\mathcal{P}$ is the set of pivotal bits for f. Determine the distribution of the pivotals and the spectral sample for (a) the function which is 1 if $X_1 = 1$ and -1 otherwise and (b) the function which is 1 if $X_1 = 1$ or $X_2 = 1$ and -1 otherwise.

3. (a). Find an example of a Boolean function where the events $\{1 \in \mathcal{P}\}$ and $\{2 \in \mathcal{P}\}$ are independent.

(b). Find an example of a Boolean function where the events $\{1 \in \mathcal{P}\}$ and $\{2 \in \mathcal{P}\}$ are strictly positively correlated.

(c). Find an example of a Boolean function where the events $\{1 \in \mathcal{P}\}$ and

$\{2 \in \mathcal{P}\}$ are strictly negatively correlated.

4. We have defined an operator T_ϵ on the space H of functions on $\{0,1\}^n$ to R. We now define another operator $\tilde{T}_\epsilon$ on the space H as follows. Consider the discrete time Markov chain on $\{0,1\}^n$ where the n coordinates change independently where each one changes its value with probability $(1-\epsilon)/2$. Let $\tilde{T}_\epsilon$ be the corresponding Markov operator. (This means that $\tilde{T}_\epsilon(g)(x)$ is the expected value of $g(X_1)$ when the Markov chain is started at x and X_1 is the state of the Markov chain at time 1.)

Show that $\tilde{T}_\epsilon = T_\epsilon$.

Hint: This Markov chain is the same as the Markov chain where each coordinate independently 'updates' its state with probability $(1-\epsilon)$ where 'updates' means that in then independently of everything else chooses to be in state 0 or 1 with probability 1/2 each.

5. Recall a Boolean function is a function from $\{0,1\}^n$ to ± 1 or to $\{0,1\}$. In this exercise, all functions are Boolean functions.

Note that if f is constant or a dictator (which means that there is some i such that $f(x)$ is 1 if and only if x_i is 1) or an anti-dictator (which means that there is some i such that $f(x)$ is 1 if and only if x_i is 0), then all the Fourier coefficients for sets of size 2 or larger are 0. Show, conversely, that if a Boolean function has all the Fourier coefficients for sets of size 2 or larger being 0, then f is constant, a dictator or an anti-dictator.

6(a). For the tribes example presented in class, show that there is a constant $c > 0$ so that for all n, the probability of the event occuring is in $[c, 1-c]$. (This shows that these events are a nontrivial example (nontrivial in the sense that the variances stay uniformly bounded away from 0) where the maximum influences is of the order $\log n/n$.)

(b). Show that the variant of the tribes presented (where the intervals have length $\log_2(n)$ rather than $\log_2(n) - \log_2(\log_2(n))$), there is no c as above in part (a) (and so these events are not nontrivial in the above sense). It follows from the KKL theorem that this must be the case (since the maximum influence is of size $1/n$) but the point is to do this directly.

7. Show that the set of functions $\{\chi_S\}_{S\subseteq\{1,\ldots,n\}}$ is an orthonormal basis.

8. A function from $\{0,1\}^n$ to ± 1 is called monotone if $x \leq y$ (meaning $x_i \leq y_i$ for each i) implies that $f(x) \leq f(y)$. Show that for a monotone function taking values in ± 1, it is the case that $\hat{f}(\{i\}) = Inf_f(i)$. Is this true without the monotone assumption?

9. If f is a function with $E(f^2) < 1$ (for example if f is the indicator function of a nontrivial event), then we still have the spectral measure but this becomes a subprobability measure (of total weight $E(f^2)$) rather than a probability measure. For this reason, sometimes when dealing with events A, it can be convenient to deal with the function $I_A - I_{A^c}$ (which is the function which is 1 on A and -1 on A^c) since this function is always ± 1 and hence its spectrum is a probability measure. Describe the exact relationship between the spectral (sub)probability measure corresponding to I_A and the spectral probability measure corresponding to $I_A - I_{A^c}$.