Some background on ergodic theory

These notes have been rather hastily written up. As such, please keep a list of all mistakes, inaccuracies or things which are not clear and let me know about them. I will perhaps we updating this file. This file was last updated on March 4.

I have decided to write a few pages giving some background concerning stationary processes and ergodicity.

Let A be a finite set (often it will be $\{0,1\}$ when dealing with particle systems or $\{(0,0),(0,1),(1,0),(1,1)\}$ when dealing with couplings of particle systems).

As always, we completely identify a probability measure on A^{Z^d} with a family of random variables $\{X_i\}_{i\in Z^d}$ taking values in A and indexed by Z^d. (It is crucial that you understand that these are essentially the same thing and can go back and forth between them).

Defintion: The process $\{X_i\}_{i\in Z^d}$ is *stationary* if for all finite sets S and any $v \in Z^d$ $\{X_i\}_{i\in S}$ has the same joint distribution as $\{X_i\}_{i\in S+v}$.

Exercise: For $j = 1, \ldots, d$, let T^j be the transformation on A^{Z^d} which shifts a configuration 1 unit in the jth direction. Then $\{X_i\}_{i\in Z^d}$ is stationary if and only if the corresponding probability measure μ on A^{Z^d} (its distribution) is invariant under each of the T^j's meaning that for any j and for any Borel subset B of A^{Z^d}, $\mu(B) = \mu(T^j B)$. This is easy; it is just unravelling definitions more or less; at least for B which are cylinder sets.

Let $\mathcal{S}$ be the collection of all probability measures on of A^{Z^d} which are invariant under each of the T^j's (so these just correspond to the collection of all stationary processes taking values in A and indexed by Z^d).

Exercise: $\mathcal{S}$ is a closed (in the weak* topology), convex set.

As such, it has extreme points; elements which cannot be written as a convex combination of other elements in this set.

Theorem: $\mu \in \mathcal{S}$ is an extreme element if and only if all sets $B \subset A^{Z^d}$ which are T^j invariant for each j (meaning $T^j(B) = B$ for each j) have μ measure 0 or 1.

Exercise: Think of sets which are are T^j invariant for each j. Show the "only if" direction. This is easier than the "if" direction.

Definition: Measures which satisfy the equivalent definitions of the previous theorem are called *ergodic*.

Theorem: An i.i.d. process (i.e., a uniform product measure) is ergodic.

Proof: See my notes.

Exercise: Let $\{X_i\}_{i\in Z}$ and $\{Y_i\}_{i\in Z}$ each (individually be) i.i.d. processes taking the values 1 and 0 each with probability 1/2. A coupling of these two processes is of course just a probability measure on $\{(0,0),(0,1),(1,0),(1,1)\}^Z$.
(a) Find a coupling of these processes which is not stationary.
(b) Find a coupling of these processes which is stationary but not ergodic. (Observe that the independent coupling of these processes is ergodic.)

For the statement of the important ergodic theorem, see Theorem 2 on page 2 of my notes. There it is stated just for i.i.d. processes but the ergodic theorem stated there holds for any *stationary* process. The a.s. limit however might not be a constant. (Example??). However, if the process is also ergodic, then the limit is a constant.