Homework 2: Due March 4th

You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, you should write up the final solutions yourselves. Solutions should always be correct, clear and concise.

Problems 3(b) 4, and 5 are more "challenge problems" and so are not "required" but I encourage you to try your hand at them.

1. Consider an interacting particle system on $X = \{0,1\}^Z$ where $c(x,\eta)$ is 1 for all $\eta \in X$ and $x \in Z$. (So each location independent of other locations waits an exponential time with parameter 1 and changes its value.) Show
(1) there is a unique stationary distribution μ for this system,
(2) every initial state converges to μ
(3) if we start out with all 1's, then for all $s \neq t$, the distribution at time s and the distribution at time t are mutually singular.

2. Show that if $\{X_t^x\}$ is a continuous time simple random walk in Z^d starting at x and $\{Y_t^y\}$ is a continuous time simple random walk in Z^d starting at y, then they can be coupled so that they eventually agree. This is made precise as I described in the lectures (on Feb 19th). To do this, you need to modify the proof that I presented in class for the discrete time case.

3. (a) Consider the voter model in 1 dimension. Given an example of an initial state such that

$$P(\text{ for every } n, \exists t > n \text{ such that } \eta_t(0) \neq \eta_t(1)) = 1$$

(b) (This is harder and so perhaps should be viewed more as a challenge problem (and due to our own Peter Hegarty.)) One proof I know uses ergodicity. But perhaps you can find an easier proof. If the initial state is

$$\ldots 0, 1, 0, 1, 0, 1 \ldots,$$

show that

$$P(\text{ for every } n, \exists t > n \text{ such that } \eta_t(0) \neq \eta_t(1)) = 1$$

Comments related to this exercise and to the exercise at the top of page 27 in my notes and definitely worth understanding. I wrote these comments up very fast and so might contain mistakes; tell me if something does not seem clear.

Here are a few comments explaining the exercise at the top of page 27: this concerns exactly the point Nisse brought up in the lecture. The subtle point comes down to the distinction between convergence in probability and convergence a.s. If we let

$$f_t = I_{\eta_t(0) \neq \eta_t(1)},$$

what we proved in class together with what we showed here is that f_t approaches 0 in probability but not a.s. It is a similar (but more complicated) phenomena to the following much simpler situation. If U_n are independent events with $P(U_n) = 1/n$, then I_{U_n} approaches 0 in probability but not a.s.

In the voter model case, the reason we don't get the a.s. convergence is exactly because if $t < t'$,

$$X_t^{x,t} = X_t^{y,t}$$

does NOT imply that

$$X_{t'}^{x,t'} = X_{t'}^{y,t'}.$$

Draw some realizations to convince yourself of this. **End of Comments**

4. (This is also a little harder; i.e., a challenge problem.) Find an initial configuration η such that $\delta_\eta T(t)$ (which is just the distribution at time t if we start with configuration η) does not converge.

5. (Another "challenge problem".) Let μ_p be product measure on $X = \{0,1\}^Z$ "with density p". So

$$\mu_p = \prod_Z [p\delta_1 + (1-p)\delta_0].$$

Convince yourselve that $\lim_{p\to 1/2} \mu_p = \mu_{1/2}$. Find a set A so that

$$\lim_{p\to 1/2} \mu_p(A) \neq \mu_{1/2}(A).$$