# Homework 3: Due April 1th

You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, you should write up the final solutions yourselves. Solutions should always be correct, clear and concise.

1. Let $\eta_t^{\overline{\nu}}$ be the contact process on $Z^d$ started with stationary distribution $\overline{\nu}$. Assume that $\lambda$ is such that $\overline{\nu} \neq \delta_{\overline{0}}$. Fix $x \in Z^d$.

(a) Does $\eta_t^{\overline{\nu}}(x)$ converge in distribution?

(b) Does $\eta_t^{\overline{\nu}}(x)$ converge a.s.?

(c) Does $\eta_t^{\overline{\nu}}(x)$ converge on a set of positive measure?

(d) Does $\eta_t^{\overline{\nu}}(x)$ converge in probability?

2. Consider the contact process now with $\lambda$ such that $\overline{\nu} = \delta_{\overline{0}}$.

(a) What is the probability that there exists $T < \infty$ such that $\eta_T^{Z^d} = \overline{0}$?

(b) Given $x \in Z^d$, does $\eta_t^{Z^d}(x)$ converge in probability?

(c) Given $x \in Z^d$, does $\eta_t^{Z^d}(x)$ converge a.s.?

3. For each $n$, let $c_n(x, \eta)$ be spin rates for a nearest neighbor spin system (nearest neighbor means that the rate only depends on the states of the point itself and its neighbors). Assume we also have rates $c_\infty(x, \eta)$. Assume that the spin rates $c_n(x, \eta)$ converge to $c_\infty(x, \eta)$ in the sense that

for all $x$,

$$\lim_{n\to\infty} c_n(x,\eta) = c_\infty(x,\eta)$$

uniformly in $\eta$.

Show that if, for each $n$, $\mu_n$ is a stationary distribution for the spin rates $c_n(x,\eta)$ and $\lim_{n\to\infty}\mu_n = \mu_\infty$, then $\mu_\infty$ is a stationary distribution for the spin rates $c_\infty(x,\eta)$.

Hint: You may use the fact (without proving it) that $c_n$ approaching $c_\infty$ in the above sense implies that $\forall t > 0$, for all $f \in C(X)$,

$$\lim_{n\to\infty} T^n(t)f(\eta) = T^\infty(t)f(\eta)$$

uniformly in $\eta$ where $T^n(t)$ denotes our semigroup of Markov operators for the rates $c_n$ (and $T^\infty(t)$ corresponds to the rates $c_\infty$).

4. You will want to use problem 3 above for this problem. If you can't do 3, you should of course just assume it for this problem.

(a). Show that $\overline{\nu}_\lambda(\eta(x) = 1)$ is a right continuous function of $\lambda$ on $[0,\infty)$.

(b) Show that if $\overline{\nu}_\lambda \neq \delta_{\overline{0}}$, $\overline{\nu}_\lambda(\overline{0}) = 0$.

(c) Use part (b) and the following nontrivial theorem to that that $\overline{\nu}_\lambda(\eta(x) = 1)$ is a left continuous function of $\lambda$ on $(\lambda_c,\infty)$.

Nontrivial theorem: If $\overline{\nu}_\lambda \neq \delta_{\overline{0}}$, then all stationary distributions are of the form $p\overline{\nu}_\lambda + (1-p)\delta_{\overline{0}}$.