Homework 1: Due April 12th

Page 3 and 4 just has some clarifications should it be needed.

You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, you should write up the final solutions yourselves. Solutions should always be correct, clear and concise.

The point of these 3 questions is just to get you to understand the concept of coupling which is a crucial tool in IPS.

1. (a) Let $\{X_1, \dots, X_n\}$ be independent random variables with $P(X_i = 1) = p_i$ and $P(X_i = 0) = 1 - p_i$ where $p_1, \dots, p_n \in [0, 1]$. Let $\{Y_1, \dots, Y_n\}$ be independent random variables with $P(Y_i = 1) = q_i$ and $P(Y_i = 0) = 1 - q_i$ where $q_1, \dots, q_n \in [0, 1]$. Find sufficient and necessary conditions on the p_i's and q_i's so that

$$\{X_1, \dots, X_n\} \leq_{stoch} \{Y_1, \dots, Y_n\}.$$

Prove that the conditions are sufficient and necessary.

Assume that $\{X_1, \dots, X_n\}$ are $0, 1$ random variables (not necessarily independent) and that $\{Y_1, \dots, Y_n\}$ are $0, 1$ random variables (not necessarily independent). Show by example, that $X_i \leq_{stoch} Y_i$ for each i is not sufficient for $\{X_1, \dots, X_n\} \leq_{stoch} \{Y_1, \dots, Y_n\}$. (It is easy to see however that it is necesasry).

2. (If you don't succeed with problem 2 after some time, ask me for a hint.)

(a) Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov chain on $\{0,1\}$ such that $q_{0,1}=1$ and $q_{1,0}=3$. Let $\{Y_t\}_{t\geq 0}$ be a continuous time Markov chain on $\{0,1\}$ such that $q_{0,1}=2$ and $q_{1,0}=3$. Show that $\{X_t\}_{t\geq 0} \leq_{stoch} \{Y_t\}_{t\geq 0}$.

(b) Let $\{X_t\}_{t\geq 0}$ be a continuous time Markov chain on $\{0,1,2\}$ such that $q_{0,1}=a$, $q_{1,0}=c$, $q_{1,2}=b$, $q_{2,1}=d$, and $q_{0,2}=q_{2,0}=0$. Let $\{Y_t\}_{t\geq 0}$ be a continuous time Markov chain on $\{0,1,2\}$ such that $q_{0,1}=a'$, $q_{1,0}=c'$, $q_{1,2}=b'$, $q_{2,1}=d'$, and $q_{0,2}=q_{2,0}=0$. Assume that $a\leq a', b\leq b', c'\leq c$, and $d'\leq d$. Show that $\{X_t\}_{t\geq 0} \leq_{stoch} \{Y_t\}_{t\geq 0}$.

As a consequence show that

$$P(X_5 = 1 \text{ or } 2, X_t = 2 \text{ for some t in [8,9] }, X_t = 1 \text{ or } 2 \text{ for all t in [11,12] }) \leq$$

$$P(Y_5 = 1 \text{ or } 2, Y_t = 2 \text{ for some t in [8,9] }, Y_t = 1 \text{ or } 2 \text{ for all t in [11,12] }).$$

The point is that the coupling method allows one to prove things like this last thing in a computationally free way.

3. Let X, Y and Z be such that $P(X=1)=p_1$ and $P(X=0)=1-p_1$, $P(Y=1)=p_2$ and $P(Y=0)=1-p_2$, and $P(Z=1)=p_3$ and $P(Z=0)=1-p_3$. Find sufficient and necessary conditions on p_1,p_2 and p_3 such that X,Y and Z can be coupled such that X and Y are independent, $X\leq Z$ and $Y\leq Z$.

Hint: Is it sufficent to be able to (1) separately couple X and Y so that they are independent, (2) separately couple X and Z so that $X\leq Z$ and (2) separately couple Y and Z so that $Y\leq Z$?

Some clarifications if needed.

I first make the following comment which you hopefully have understood but if not, it is very important. An equivalent definition of $X \leq_{stoch} Y$ (with X and Y being real-valued random variables) which was given in class, is that there exists a probability measure μ on the Borel sets of R^2 such the first and second marginals of μ are μ_X and μ_Y respectively and that $\mu\{(x,y) : x \leq y\} = 1$. [By the first marginal of a probability measure ν on R^2, I mean the probability measure ν_1 on R given by

$$\nu_1(B) := \nu(B \times R).$$

This is also called the projection of ν onto the first coordinate and is what a probabilist calls the marginal distribution.]

In question 2, when I write

$$\{X_t\}_{t\geq 0} \leq_{stoch} \{Y_t\}_{t\geq 0},$$

I mean precisely that there exist processes $\{\tilde{X}_t\}_{t\geq 0}$ and $\{\tilde{Y}_t\}_{t\geq 0}$ defined on the same probability space such that

$$\{X_t\}_{t\geq 0} =^{\mathcal{D}} \{\tilde{X}_t\}_{t\geq 0},$$

$$\{Y_t\}_{t\geq 0} =^{\mathcal{D}} \{\tilde{Y}_t\}_{t\geq 0}$$

and

$$P(\tilde{X}_t \leq \tilde{Y}_t \text{ for all } t) = 1.$$

Note that

$$\{X_t\}_{t\geq 0} =^{\mathcal{D}} \{\tilde{X}_t\}_{t\geq 0}$$

means that these two processes are equal in distribution. This does not only mean that for every t, X_t and $\tilde{X}_t$ have the same distribution; it means the much stronger fact that for all $t_1, \ldots, t_k$, $\{X_{t_i}\}_{i=1,\ldots,k}$ has the same joint distribution as $\{\tilde{X}_{t_i}\}_{i=1,\ldots,k}$.

In question 3, by the expression "such that X, Y and Z can be coupled such that X and Y are independent, $X \leq Z$ and $Y \leq Z$", what I mean precisely is that there exist random variables $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ defined on the same probability space such that $X =^{\mathcal{D}} \tilde{X}$, $Y =^{\mathcal{D}} \tilde{Y}$ and $Z =^{\mathcal{D}} \tilde{Z}$ such that (1) $\tilde{X}$ and $\tilde{Y}$ are independent, (2) $\tilde{X} \leq \tilde{Z}$ and (3) $\tilde{Y} \leq \tilde{Z}$.