Homework 2: Due April 26th

You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, you should write up the final solutions yourselves. Solutions should always be correct, clear and concise.

Some parts of 2 might be somewhat challenging. Don't get discouraged if you can't do them all. But I do think they are fun and very educational and so I encourage you to try your hand at them.

1. Consider an interacting particle system on $X = \{0, 1\}^Z$ where $c(x, \eta)$ is 1 for all $\eta \in X$ and $x \in Z$. (So each location independent of other locations waits an exponential time with parameter 1 and changes its value.) Show
(1) there is a unique stationary distribution μ for this system,
(2) every initial state converges to μ
(3) if we start out with all 1's, then for all $s \neq t$, the distribution at time s and the distribution at time t are mutually singular.

2. (a) Consider the voter model in 1 dimension. Given an example of an initial state such that

$$P(\text{ for every } n, \exists t > n \text{ such that } \eta_t(0) \neq \eta_t(1)) = 1$$

(b). Find an initial configuration η_0 so that $\eta_t^{\eta_0}(0)$ does not converge in probability but so that $\eta_t^{\eta_0}$ converges in distribution.

(c). Find an initial configuration η_0 so that $\eta_t^{\eta_0}$ does not converge in distribution.

(d). If we start with finitely many 0's, does the system converge a.s. to the all 1 configuration?

(e). Does there exist a start configuration with infinitely many 0's such that the system converges a.s. to the all 1 configuration?

(f). ("challenge problem".) Can you find a start configuration so that $\eta_t^{\eta_0}(0)$ converges in probability but not a.s.?

3. Let μ_p be product measure on $X = \{0,1\}^Z$ "with density p". So

$$\mu_p = \prod_Z [p\delta_1 + (1-p)\delta_0].$$

Show that $\lim_{p\to 1/2} \mu_p = \mu_{1/2}$. Find a set A so that

$$\lim_{p\to 1/2} \mu_p(A) \neq \mu_{1/2}(A).$$

Why is this not a contradiction?