Def: $G \subseteq S_n$ is (sharp) k-transitive if $\forall (a_1, \dots, a_k) \subseteq \{1, \dots, n\}$ and $(b_1, \dots, b_k) \subseteq \{1, \dots, n\}$, $\exists$ (unique) $g \in G$ s.t. $g(a_i) = b_i$ $\forall i = 1 \dots k$.

Thm: Let $G \subseteq S_n$ be sharp k-trans and nontrivial ($G \neq S_n, A_n$).

(1) If $k = 4$, then $n = 11$

(2) If $k = 5$, then $n = 12$

(3) $k \geq 6$ cannot happen

Thm:

(1) $\exists\, M_{11} \subseteq S_{11}$ which is sharp-4-trans. and simple

(2) $\exists\, M_{12} \subseteq S_{12}$ which is sharp-5-transitive and simple.

(M_{11}, M_{12} first 2 simple sporadic groups)

Cor. (i) $|M_{11}| = 11 \times 10 \times 9 \times 8 = 7920$

(ii) $|M_{12}| = 12 \times 11 \times 10 \times 9 \times 8 = 95{,}040$.

Pf (i). Let $\mathcal{O}_4 = \{ (b_1, b_2, b_3, b_4) \subseteq \{1 \dots 11\}$ distinct $\}$

$|\mathcal{O}_4| = 11 \times 10 \times 9 \times 8$.

$$G \longrightarrow \mathcal{O}_4$$
$$g \longrightarrow (g(1), g(2), g(3), g(4))$$

sharp 4 $\Leftrightarrow$ bij. $\square$

Def: G finite group. X finite set. a (Left) G-action of G on X is a map
$$G \times X \to X$$
$$(g, x) \longrightarrow gx \quad \text{s.t.}$$

(1) $1x = x$

(2) $(g_1 g_2)x = g_1(g_2 x)$ $\forall x, g_1, g_2$

__Exer__ For fixed g, the map $x \to gx$ is a bij. ϕ_g on X.

(2) $\Leftrightarrow$ $\phi_{g_1 g_2} = \phi_{g_1} \phi_{g_2}$

Ie. get a homom. from G to S_X
$$g \longrightarrow \phi_g$$

Def: K = kernel of this homo.

$K = \{ g : \phi_g = \text{identity} \} = \{ g : gx = x \ \forall x \in X \}$

Def: Given $x \in X$, $\{ gx : g \in G \}$ is the __orbit__ of x.

$x \sim y$ if $\exists g$, $gx = y$
orbits are the equiv. classes

Usually, 1 orbit — transitive

Def: $x \in X$, the stab. of x, $G_x = \{ g : gx = x \}$.

Exercise $K = \bigcap_{x \in G} G_x$

Def: a group action is faithful if $K = \text{Kernel} = 1$

This yields an injection of G into S_X

$\Rightarrow$ think of G as a s.g. of S_X.

(More generally G/K is a s.g. of S_X

- 1st iso. thm.

Thm orbit-stabilizer thm

$\forall x \in X$, $|G/G_x| = |\mathcal{O}(x)|$

Left cosets of G_x in G.

"Pf" $G/G_x \longrightarrow \mathcal{O}(x)$

$gG_x \longmapsto gx$ ✓ welldefd bij. $\square$

Alt. proof: $|\{(g,y) : gx = y\}| \subseteq G \times X$

Double count $\Rightarrow$

$|G| = |G_x|\,|\mathcal{O}x|$

Example 1: Let $H < G$.

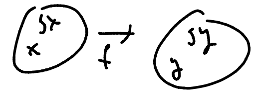
G acts on G/H by "translations"

$g(g'H) = gg'H$

Exercise: (1) GA. (2) Trans (3) If $H = 1$, what happens?

Thm: Every ~~transitive~~ action "looks like this".

Def: If G acts on X and acts on Y, the actions are equiv if $\exists$ bij: $X \to Y$ s.t. $f(gx) = g f(x)$ $\forall x$ $\forall g$.



renaming of the elements.

"pf" exercise.

$$G/G_x \xrightarrow{f} O(x) = X$$

✓ f works $\square$

Thm If $x \sim y$, G_x and G_y are conjugate s.g. i.e.

$\exists\, g \in G$ $g G_x g^{-1} = G_y$

pf. choose $g \in G$ s.t. $gx = y$ $\square$

Return to our universal trans. example

G acts on G/H $\quad H < G$

$g(g'H) = gg'H$ $\quad$ Kernel?

~~centralizer~~ G_H = stab. of $H \overset{ex}{=} H$.

Ex. $K = \bigcap_{g \in G} G_{gH} = \bigcap_g g G_H g^{-1} = \bigcap_g gHg^{-1}$

core(H) = largest normal s.g. $\subseteq H$.

Example 2: $H < G$

Let G act on the conjugates of H, $\{gHg^{-1} : g \in G\}$ by conjugation.

$$g(g'Hg'^{-1}) = g g' H g'^{-1} g^{-1}$$

Exercise

(1) GA

(2) G_H = stab. of $H = \{g : gHg^{-1} = H\}$ $\triangleq N_G(H)$ normalizer of H in G.

(3) orbit

orbit stabilizer thm $\Rightarrow$

$[G : N_G(H)]$ = # of conjugate of H

both sides $=1 \iff H \trianglelefteq G$

Primitive action (between trans. and 2-trans.). assume trans.

Def: G acts on X. $B \subseteq X$ is a block if $\forall g \in G$ either $gB = B$ or $gB \cap B = \emptyset$.

2 trivial blocks.

(1) X is a block.

(2) each $x \in X$ is a block.

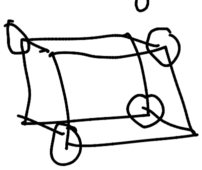
Def: G is primitive if no other blocks.

Ex: $\mathbb{Z}/6$ acting on $\{1-6\}$ by trans. not PRIM since $\{1, 3, 5\}$ block.

Ex 2. $\mathbb{Z}/7$ act on $\{1..7\}$ by trans is primitive.

Ex 3: ~~Automorphism~~ symmetries of the vertices of a cube in 3-space

$\subseteq S_8$ not prim



4 elements block

not primitivity $\iff$

$\exists$ a non trivial partition of X

s.t each $g \in G$ preserves.

Remark: Prim means there are no "factor actions".

"primitivity" $\iff$ "simplicity"

Ex: 2 trans $\Rightarrow$ prim.

Thm: Let G act trans. on X. Then G is primitive iff G_x is a max s.g. $\forall x$ (for some x) won't prove

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Thm: Let G act trans. on X. Then if G is 2-trans, then G_x is a max s.g. prove

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Commutator s.g. and abelianization

G group

Def The commutator s.g. G' of G is the s.g. generated by

$$\{\underbrace{aba^{-1}b^{-1}}_{\text{commutator}} : a,b \in G\}$$

Remark: G abelian iff $G' = 1$ (all comm. triv.)

Prop: $G' \trianglelefteq G$.

"pf" Key step: conj. of a comm is a comm.

$$g\,aba^{-1}b^{-1}g^{-1} = \boxed{gag^{-1}}\boxed{gbg^{-1}}\boxed{ga^{-1}g^{-1}}gb^{-1}g^{-1}$$

$\square$

Since G' represents the nonabelianness of G, one might hope if we mod out by G' we get an abelian group

Thm: $H \trianglelefteq G$. Then G/H abelian iff $G' \subseteq H$.

G/G' "abelianization of G".

Pf. G/H abelian $\overset{\text{remark}}{\Longleftrightarrow}$ all comm. in G/H are triv.

$\Leftrightarrow aHbH(aH)^{-1}(bH)^{-1} = H \quad \forall a,b \in G$

$\Leftrightarrow (aba^{-1}b^{-1})H = H \quad \forall a,b$

$\Leftrightarrow aba^{-1}b^{-1} \in H \quad \forall a,b \Leftrightarrow G' \subseteq H$ $\square$

Remarks:

(1) G abelian $\Leftrightarrow G' = 1 \Leftrightarrow C(G) = G$ ($C(G)$: center of G.)

(2) nonabelianness corresponds to large G' and/or small $C(G)$.

(3) If G is simple, nonabelian then $\underline{G' = G}$, $\underline{C(G) = 1}$

(4) In general, no relationship between $G' = G$ and $C(G) = 1$ $\Rightarrow$ different notions of nonab.

Ex $G = S_5$ $\quad C(G) = 1$

$G' = A_5$.

Next time:
First step for Jordan's thm.

Thm: (1) no sharp 4-trans. s.g. of S_{10}.
(2) no sharp 5-trans. s.g. of S_{13}