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Recall:

Thm. $G \leq S_n$, sharp k-trans, $G \not\leq S_n$, $\not\leq A_n$.

1. $k = 4 \Rightarrow n = 11$
2. $k \geq 5 \Rightarrow n = 1$
3. $k = 6 \Rightarrow n = n$

1st step

Prop: 1. $\nexists\, G \leq S_{10}$ which is sharp 4-

2. $\nexists\, G \leq S_{13}$ ... — sharp 5.

Lemma: Let $G \leq S_X$ k-trans.

Let $Y = \{a_1 \ldots a_k\} \subseteq X$.

Let $H = G_Y$ = point stab of $Y = \{g : g a_i = a_i \ \forall i = 1 \ldots k\}$

Let P be a Sylow p-sg. of H

Let F_P = fixed pts of $P = \{x : gx = x \ \forall g \in P\} \supseteq Y$

Let $N_G(P)$ = normalizer of P in G

$= \{g \in G : g P g^{-1} = P\}$

Then $N_G(P)$ leaves F_P inv. and is k-trans. of F_P.

$$P \subseteq \underset{H}{N_G(P)} \subseteq G$$

1. $N_G(P)$ leaves F_P inv.

i.e. $x \in F_P$, $\pi \in N_G(P) \Rightarrow \pi x \in F_P$

i.e. $g \pi x = \pi x \ \forall g \in P$

i.e. $\pi^{-1} g \pi x = x \ \forall g \in P$

$\in \pi^{-1} P \pi = P \Rightarrow$ above $=$ $\underset{\in P}{g'} x = x$

2. (Frattini arg.)

Now $Y = \{a_1 \ldots a_k\} \in F_P$. We NTS if $(b_1, \ldots, b_k) \subseteq F_P$ $\exists\, \pi \in N_G(P)$ s.t. $\pi a_i = b_i \ \forall i$.

G k-trans $\Rightarrow \exists\, \sigma \in G$ s.t. $\sigma a_i = b_i \ \forall i = 1 \ldots k$. Since P fixes $b_1 \ldots b_k$, $\sigma^{-1} P \sigma$ fixes $a_1 \ldots a_k$.

$\Rightarrow \sigma^{-1} P \sigma \leq H$.

$P, \sigma^{-1} P \sigma \in H$. $\underset{\text{Sylow}}{\Rightarrow} \exists\, \tau \in H$

s.t. $\tau^{-1} P \tau = \sigma^{-1} P \sigma \Leftrightarrow$

$\sigma \tau^{-1} P (\sigma \tau^{-1})^{-1} = P \Rightarrow \sigma \tau^{-1} \in N_G(P)$

$\sigma \tau^{-1} a_i = b_i$ $\square$

Prop: $\nexists\, G \leq S_{10}$ sharp 4-trans

Pf STEP 1: If G exists, $|G| = 10 \cdot 9 \cdot 8 \cdot 7$

Choose $P \subseteq G$ 7-Sylow s.g. $|P| = 7$.

$P = \langle \pi \rangle$, π - 7 cycle. $\underset{\text{WLOG}}{\Rightarrow} \pi = (1 2 3 \ldots 7)$

Apply lemma with $k = 3$, $Y = \{8, 9, 10\}$

$H = \{g : g(8) = 8, g(9) = 9, g(10) = 10\}$.

$P \in H$. Note $F_P = Y$. Prev Lemma $\Rightarrow N_G(P)$ is 3 trans. of $\{8, 9, 10\}$.

$(\Leftrightarrow) N_G(P) \xrightarrow[\text{onto}]{\phi} S_{\{8,9,10\}}$ (ϕ restriction)

STEP 2:

$C_G(P)$ = cent. of P in G

$= \{g \in \underset{N_G(P)}{G} : g g' = g' g \ \forall g' \in P\}$

fix P p.w under conj	$N_G(P)$ - fixes P setwise under conj

$\ast\, C_G(P) \supseteq N_G(P)$

pf. Let $N_G(P)$ act on P by conj

$(g g') := g g' g^{-1} \in P$. $\Rightarrow$

$N_G(P) \to \mathrm{Aut}(P)$.

Kernel $= C_G(P) \Rightarrow$

$\Rightarrow$ 1st iso. $\dfrac{N_G(P)}{C_G(P)} \hookrightarrow \mathrm{Aut}\, P$ $= \mathbb{Z}/6$ = abelian.

$\Longrightarrow C_G(P) \geq N_G'(P)$

STEP 3: $C_G(P)$ has an element of order 3.

Pf

Exercise: homo and taking commss's commute ie

$f: G_1 \to G_2 \quad f(G_1') = (f(G_1))'$

$\phi(C_G(P)) \underset{S.2}{\supseteq} \phi(N_G'(P)) = (\phi(N_G(P)))'$

$\underset{S.1}{\cong} \; S_{\{8,9,10\}}' \cong S_3' = \mathbb{Z}/3$

$\Rightarrow C_G(P) \ni \sigma$ of order 3.

Choose g s.t. $o(\phi(g)) \mid o(g)$.

$\phi(g)$ has order 3. $\Rightarrow 3 \mid o(g)$

$\underset{\text{Cauchy's thm}}{\Longrightarrow} \langle g \rangle \geq$ element of order 3. D

STEP 4

$\pi = (1234567) \quad \sigma \in C_G(P) \quad o(\sigma) = 3$.

$\sigma\pi$ has order 21.

$\sigma\pi$ must be a 7-cycle & a 3-cycle

$(\sigma\pi)^7$ = 3 cycle.

$(\sigma\pi)^7$ non trivial but fixed 7 elements. $\not\downarrow$ 4 sharp trans

Prop: $\nexists$ $G \leq S_{13}$ 6-trans. $\to$ sharp

Pf S.1. If $\exists$ $|G| = 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$

Let P be a 5-Sylow SS. $|P| = 5$.

$P = \langle \pi \rangle$. π is either 5-cycle or 2 5 cycles.

π not 1 5 cycle because fix 8 pts $\not\downarrow$ 6 trans

$\Rightarrow \pi = (12345)(678910)$.

Let $Y = \{11, 12, 13\}$

$H = \{g : g \text{ fixes } Y \text{ p.w.}\}$

$P \subseteq H$. . . ✓

$C_G(P)$ contain σ of order 3.

$\pi = (12345)(678910) \quad o(\sigma) = 3$

$\Rightarrow \sigma\pi$ order 15

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$(\sigma\pi)^5$ = 1 or 2 3-cycles.

$\Rightarrow (\sigma\pi)^5$ fixes 7 elements

$\not\downarrow$ 6 sharp trans

$\square$

Jordan's Thm.

First prove: $G \le S_n$, 4-trans (sharp), non trivial $\Rightarrow n = 11$.

Step 1: Rule out $n \le 7$.

Pf. $n=4$ $|G| = 4 \cdot 3 \cdot 2 \cdot 1 = 4! \Rightarrow G = S_4$.

$n=5$ $|G| = 5 \cdot 4 \cdot 3 \cdot 2 = 5! \Rightarrow G = S_5$.

$n=6$ $|G| = 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6!}{2} \Rightarrow G = A_6$

$n=7$ $|G| = 7 \cdot 6 \cdot 5 \cdot 4 = \frac{7!}{6}$

$[S_7 : G] = 6$

Let S_7 action S_7/G (left cosets of G in S_7) by translation. $g, g' \in S_7$

$g(g'G)$

$= (gg'G)$

$K =$ Kernel $=$ core$(G) = \bigcap_{g \in S_n} gGg^{-1} \le G$

$S_7 \to S_6$ = group action

$S_7/K \hookrightarrow S_6$

$K \trianglelefteq S_7 \Rightarrow K = 1, A_7, S_7$

$|K| \le |G| = \frac{7!}{6} \Rightarrow K \ne A_7, S_7$

$\Rightarrow K = 1$. Then $S_7 \hookrightarrow S_6$.

$\square$