Thm: If $G \subseteq S_n$, 4-sharp trans, nontrivial, then $n = 11$.

We have ruled out $n = 10$ earlier.

Last time $n \leq 7$ rule out. → step 1.

Pf. $n \geq 8$.

STEP 2: Any 2 ~~in~~ involutions ~~or~~ in G are conjugate in G ($g^2 = 1$)

Pf Note an inv. just consists of transp. (2-cycles) and 1 cycles.

$$\underbrace{(\ )(\ )(\ )}_{\text{disjoint}} \quad ()()()$$
$$\ 2 \quad 2 \quad 2$$

If π inv., it has ≥ 2 transpos. Otherwise ($n \geq 8$) the # of fixed pts would be ≥ 6 $\Rightarrow \pi = 1$. Let π, σ be inv.

$\pi = (12)(34) \dots\dots$

$\sigma = (ab)(cd) \text{----}$

By 4 trans $\exists\, \tau \in G$ $1 \to a$, $2 \to b$, $3 \to c$, $4 \to d$

$\tau \pi \tau^{-1} = (ab)(cd) \dots \text{--}$

$\xRightarrow{\text{sharp } 4} \tau \pi \tau^{-1} = \sigma$ □

STEP 3: $G \supseteq$ "4-Klein group" $= \mathbb{Z}/2 \times \mathbb{Z}/2$ with very nice structure

Pf By 4 trans, $\exists\, \pi, \sigma \in G$ s.t.

$\pi = (1)(2)(34) \dots$

$\sigma = (12)(3)(4) \text{---}$

Claim π, σ inv.

Pf $\pi^2 = (1)(2)(3)(4) \dots \Rightarrow \pi^2 = 1$ □

claim: $\pi\sigma = \sigma\pi$

Pf $\pi\sigma = (12)(34) \dots$, $\sigma\pi = (12)(34) \dots$ $\xRightarrow{\text{sharp } 4} \pi\sigma = \sigma\pi$ □

$\underline{H} := \langle \pi, \sigma \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2$

π has (by 4 sharp) at most 1 other fixed pt

lets call this fixed pt "7" if exists.

(If no third fixed pt, argue similarly).

claim: If $g, g' \in S_n$ commute, then g permutes the fixed pts of g'.

Pf x be a fixed pt for g'. NTS gx is a fixed pt of g'. $g'gx = gg'x = gx$ □

$\Rightarrow \sigma(7) = 7$

Let $\tau = \pi\sigma$ ($H = \{id, \pi, \sigma, \tau\}$)

$\tau(7) = 7$.

STEP 2 $\Rightarrow \tau, \pi$ are conj. in G

$\Rightarrow \tau, \pi$ same cycle structure

$\Rightarrow \tau$ has ~~B~~ exactly 3 fixed pts

Besides 7, let 5, 6 denote the fixed pts of τ. Next, $\pi \tau$ commute $\Rightarrow$ π permutes 5, 6, 7

easy claim

$\Rightarrow$ ~~π~~ $\pi = (1)(2)(34)(56)(7)\cdots$

Similarly $\sigma =$ ~~...~~ $(12)(3)(4)(56)(7)$

$\pi \quad (1)(2)\ (34)\ (56)(7)\cdots$

$\sigma = (12)\ (3)(4)\cdot(56)\ (7)\cdots$

$\tau = (12)\ (34)(5)(6)(7)\cdots$ STEP 3 $\square$

STEP 4: $C_G(H) = H$

Pf. H abelian $\Rightarrow$ $H \subseteq C_G(H)$

Let $g \in C_G(H) \setminus 1$. nts $g \in H$.

$\because g$ commutes with $\pi, \sigma, \tau \Rightarrow$

g permutes $\{1,2,7\}$ $\{3,4,7\}$, $\{5,6,7\}$

$\Rightarrow g(7) = 7$. $\Rightarrow$

$g = (12)^* (34)^* (56)^* 7 \cdots$

$*$ ~~...~~ 0 or 1.

If only 1 or 0 real trans, then g has ≥ 5 f.p. ~~$\Downarrow$~~ $g \neq 1$.

WLOG

$g = (12)(34)\cdots$

$\Rightarrow g = \tau \in H$. $\square$

4 sharp

STEP 5. H-orbits and $N_G(H)$

Note $\{1, \ldots 7\}$ is a union of H-orbits; ie $\{1, \ldots 7\}$ is invariant under H. $n \geq 8$.

Consider an $\boxed{H\text{-orbit } \mathcal{O}} \subseteq \{8, 9, \ldots n\}$

Let $S := \{g \in G : g\mathcal{O} = \mathcal{O}\}$

Claim: $S = N_G(H)$. (true $\forall \mathcal{O} \subseteq \{8, \ldots n\}$)

No $g \in H \setminus 1$ has a fixed pt in $\{8, 9 \ldots n\}$

$\Rightarrow x \in \mathcal{O}$, the stabilizer H_x trivial

$\Rightarrow |\mathcal{O}| = [H : H_x] = |H| = 4$

orbit stab

So $|\mathcal{O}| = 4$.

---

wts $|N_G(H)| \leq 24$. Pf let $N_G(H)$ act on H by conj. $\Rightarrow N_G(H)/C_G(H) \hookrightarrow \mathrm{Aut}(H) = S_3$.

$$[N_G(H) : \underset{4}{H}] \overset{\text{STEP 3,4}}{=} [N_G(H) : C_G(H)] \leq 6$$

$\Rightarrow |N_G(H)| \leq 24$ $\square$

Note $H \in S$

$$S \cong S_{\mathcal{O}} \ (\cong S_4)$$

$g \longrightarrow g|_{\mathcal{O}}$ 4 trans $\Rightarrow$ onto, sharp 4 $\Rightarrow$ inj

$\Rightarrow |S| = 24$. $\mathcal{O} = x_1 \ldots x_4$

$S_4 \supseteq 4$ ~~Klein~~ Klein 4 group

$\{e,$ ~~...~~ $(12), (34), (12)(34)\}$ $\times 3$

$\{e, (12)(34), (13)(24), (14)(23)\}$ $\times 1$ $\triangleleft S_4$

Identify H with $H|_{\mathcal{O}}$

$g \longleftrightarrow g|_{\mathcal{O}}$

$H|_{\mathcal{O}} = \{1, (x_1 x_2)(x_3 x_4), \ldots\}$

$H|_{\mathcal{O}} \triangleleft S_{\mathcal{O}} \Rightarrow H \triangleleft S$

$\Rightarrow S \subseteq N_G(H)$.

If we show $|N_G(H)| \leq 24$, then since $|S| = 24 \Rightarrow S = N_G(H)$

STEP 6 ~~(final)~~ At most 1 H-orbit in $\{8, 9, \ldots n\}$

If assume $\exists$ another H-orbit $\sigma' \neq \sigma$.

$\sigma = \{x_1\, x_2\, x_3\, x_4\}$ $\quad \sigma' = \{y_1, y_2, y_3, y_4\}$

$S' = \{g \in G : g\sigma' = \sigma'\}$. $S = N_G(H) = S'$

$S = S|_\sigma$, $S = S|_{\sigma'}$. $(S \cong S_\sigma)$

Let $g \in S$ $\quad (x_1)(x_2)\ (x_3 x_4)$. $\Rightarrow$ g inv.

Note $g \in N_G(H) \setminus H$.

$H|_{\sigma'} = \{1, (y_1 y_2)(y_3 y_4), \ldots\ldots\}$

$g \notin H \Rightarrow g|_{\sigma'}$ not of above form.

$g|_{\sigma'}$ be of the form ~~(...)~~ $(y_1 y_3)(y_2)(y_4)$

Since g inv.

$\Rightarrow g$ has 4 fixed pt $x_1, x_2, y_2, y_4 \Rightarrow g = 1$ ↯

$\square$

Final step exactly

$n \geq 8$, ~~at least~~ 1 H orbit outside $\{1, \ldots 7\}$ $\Rightarrow$

$n = 7 + 4 = 11$.

Recall assume π had a 3rd fixed pt

If π did not have a 3rd f.p.

Then $n = 10$.

But $n = 10$ ruled out earlier.

$\square$