x Def: A subgroup N of G is normal, $N \trianglelefteq G$ if $gNg^{-1} = N$ $\forall g \in G$.
(N is inv. under conjugation).

Def: G is simple if G has no nontrivial ~~sgs~~ normal sg's i.e. different from G and 1.
Equiv. to G has no nontrivial factors; i.e. if $f: G \to H$ onto then either $|H| = 1$ or f is an iso.

$\ker f \trianglelefteq G$	$G \to G/N$

Simple groups are the building blocks of groups

For primes $\subseteq \mathbb{N}$ x

✓ (1) every $n \in \mathbb{N}$ is a prod. of primes

✓ (2) (1) occurs in a unique way

X (3) Given a finite # of primes they give rise to only 1 integer when multiplied.

x

Def: a (subnormal) series for G is a sequence of sg's

$$1 = G_0 \underset{\neq}{\trianglelefteq} G_1 \underset{\neq}{\trianglelefteq} G_2 \cdots \underset{\neq}{\trianglelefteq} G_{n-1} \underset{\neq}{\trianglelefteq} G_n = G.$$

Remark. $A \trianglelefteq B$ not transitive relation

$A \trianglelefteq B \trianglelefteq C \not\Rightarrow A \trianglelefteq C$ ($C = A_4$)

In particular, G_i's need not be normal in G.

Def. A composition series is a sub normal series s.t. G_{i+1}/G_i simple $\forall i$

Def: The factors of a comp. series are the groups $G_1, G_2/G_1, G_3/G_2 \cdots G_n/G_{n-1}$. (all simple)

x

Thm: Every finite group has a comp series.

Pf: key step (correspondence thm) $H \trianglelefteq G$, G/H simple iff $\nexists K$ s.t. $H \underset{\neq}{\trianglelefteq} K \underset{\neq}{\trianglelefteq} G$
i.e. H maximal normal s.g.

$1 \trianglelefteq G$ not simple if not a c.s.

$1 \underset{\neq}{\trianglelefteq} H \underset{\neq}{\trianglelefteq} G$

$1 \underset{\neq}{\trianglelefteq} H \underset{\neq}{\trianglelefteq} K \underset{\neq}{\trianglelefteq} G.$

process stops since $|G_{i+1}| \geq 2|G_i|$. If length n, $|G| \geq 2^n$ $\square$

Given G, we can construct a C.S. and look at the factors of the CS yielding a finite # of simple groups

(1) ✓ (2) ?

Although G does not determine a CS, the factors are determined

Thm. (Jordan Hölder) If G has 2 C.S.

$1 = G_0 \lhd G_1 \ldots \lhd G_n = G$

$1 = H_0 \lhd H_1 \ldots \lhd H_m = G$

then $n = m$ and the factors, including repetitions, are the same (not in order)

"pf" Induction on $|G|$ and 2nd isomorphism theorem ie. $N \lhd G$, $H \leq G$ not necess normal, then $\frac{NH}{N} \cong H/H \cap N$. (Hint) □

Factors do not determine G.

Example $n = 4$ $\mathbb{Z}/4 \not\cong \underbrace{\mathbb{Z}/2 \times \mathbb{Z}/2}_{\text{Klein 4-group}}$

Ex. factors are $\mathbb{Z}/2$, $\mathbb{Z}/2$

$n = 6$ $\mathbb{Z}/6$ S_3

comp. factors are $\mathbb{Z}/2$, $\mathbb{Z}/3$.

Remark: G solvable iff the simple factors are $\mathbb{Z}/p$ p prime

2 steps to understand all groups.

(1) Determine all simple groups. 10,000, "class. of finite simple groups"

(2) Given a finite # of simple groups (rep. allowed) find all groups which have these as their factors

consider intractible

Extension problem Given N and H, find all G s.t $\exists \tilde{N} \lhd G$ s.t $N \cong \tilde{N}$ and $G/\tilde{N} \cong H$. simple

complements, semi-direct, split extension

Asked late 1880's which groups simple

Rutgers 78-82.

Examples of simple groups

1. abelian s. groups $\mathbb{Z}/p$ p prime
2. Alternating groups $A_n \leq S_n$ $n \geq 5$, $n \neq 4$
3. Lots of infinite families of s. groups (17?) (Lie ~~group~~ type) one such family are the "projective special Linear groups"
4. Simple sporadic groups 26 of them. 1st 5 discovered by Mathieu in late 19th century.

6th S. sporadic 1965

$n \geq 2$ $q = p^k$ ~~p r r~~ p prime

F_q = finite field with q elements.

$GL(n,q)$ general linear group

$= \{ n \times n$ invertible matrices with elements in $F_q \}$

$SL(n,q)$ special linear group

$= \{$ matrices in $GL(n,q)$ with determinant $1 \}$

$SL(n,q) \trianglelefteq GL(n,q)$

Def $PSL(n,q) = SL(n,q) / \text{center}$

ex. center $= SL(n,q) \cap$ scalar matrices $= \{ cI : c^n = 1 \}$

Thm: $PSL(n,q)$ simple except

$(n,q) = (2,2)$ S_3

$(n,q) = (2,3)$ A_4

$PGL(n,q) = GL(n,q)/\text{center} = GL(n,q)/\text{scalar matrices}$

$PGL(2,q)$, $PSL(3,4)$ } cardinal.

Def: $k \geq 1$ a subgroup $G \leq S_n$ [G perm. group] is (sharp) k-transitive if

$\forall$ distinct $(a_1, a_2, \dots a_k) \subseteq \{1 \dots n\}$

and distinct $(b_1 \dots b_k) \subseteq \{1 \dots n\}$,

$\exists (!)\ g \in G$ s.t. $g a_i = b_i$ $i = 1 \dots k$.

Thm Jordan 1872.

Let $G \leq S_n$ be sharp k-trans and non trivial.

(1) If $k = 4$, then $n = 11$

(2) If $k = 5$, then $n = 12$

(3) If $k \geq 6$, cannot happen.

Remarks.

(1) Cannot talk about a group being transitive.

$G_1 \leq S_6$ $G_1 = \langle (123456) \rangle$

$G_2 \leq S_6$ $G_2 = \langle (123), (45) \rangle$

G_1 1-trans. G_2 not 1-trans.

but $G_1 \cong G_2 \cong \mathbb{Z}/6$ not 1-trans

(2) 1-trans $\Longleftrightarrow$ trans.

(3) $k+1$ trans $\Longrightarrow$ k-trans
almost always false if "sharp"

(4) $S_n \leq S_n$ sharp n-trans

(5) $S_n \leq S_n$ sharp $(n-1)$-trans

(6) $A_n \leq S_n$ sharp $(n-2)$ trans

(7) $G = A_n$ or $S_n \leq S_n$ trivial