Amenability and phase transition in the Ising model

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Abstract

We consider the Ising model with external field h and coupling constant J on an infinite connected graph G with uniformly bounded degree. We prove that if G is nonamenable, then the Ising model exhibits phase transition for some $h \neq 0$ and some $J < \infty$. On the other hand, if G is amenable and quasi–transitive, then phase transition cannot occur for $h \neq 0$. In particular, a group is nonamenable if and only if the Ising model on one (all) of its Cayley graphs exhibits a phase transition for some $h \neq 0$ and some $J < \infty$.

1 Introduction

The first connection between probability theory and amenability of groups was obtained by H. Kesten (see [11] and [12]) where he proved that if one takes a finite symmetric generating set for a finitely generated group, then the group is nonamenable if and only if the return probabilities for simple random walk on the resulting Cayley graph decay exponentially (or equivalently the spectral radius for the resulting Markov operator on L_2 has spectral radius strictly less than one). This result was extended in [6] to any graph of bounded degree where it was shown that the return probabilities for simple random walk on the graph decay exponentially if and only if the graph is nonamenable (to be defined later).

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Recently, another connection between amenable groups and probability theory has been obtained. In [1], it is shown that a group is amenable if and only if for all $\alpha > 0$, there is a G–invariant site percolation on one (all) of its Cayley graphs such that the probability of a site being on is larger than α but for which there are no infinite components. (This result was motivated by an earlier result for regular trees in [8]). See [1] for details and where the above is stated in a more general setting. A conjecture concerning percolation on groups is that a group is nonamenable if and only if for one (all) of its Cayley graphs, there is a nontrivial interval of parameters p such that i.i.d. percolation with parameter p yields infinitely many infinite clusters. See [2] for details and a more general conjecture (Conjecture 6) as well as [9] for a related result. For related results concerning the Ising Model, see [19] and [20]. The latter paper proves a multiple phase transition for some hyperbolic graphs, a property conjectured to characterize nonamenability.

In this paper, we relate amenability to phase transition in the Ising model in the presence of an external field and we work in the more general context of graphs. We first define the Gibbs state for the Ising model on a finite graph.

Definition 1.1 *Let $G = (V, E)$ be a finite graph, $J \in [0, \infty)$ and $h \in (-\infty, \infty)$. The* **Gibbs state for the Ising model on G with parameters h and J** *is the probability measure $\nu_{h,J,G}$ on $\{-1, 1\}^V$ given by*

$$\nu_{h,J,G}(\omega) = \frac{1}{Z_{h,J,G}} e^{h \sum_{v \in V} \omega(v) + J \sum_{u,v \in V: u \sim v} \omega(u)\omega(v)}$$

where $u \sim v$ means that u is adjacent to v and $Z_{h,J,G}$ is a normalization constant.

The parameters h and J are known as the *external field* and the *coupling constant* respectively. When one extends this definition to infinite graphs such as $\mathbf{Z}^d$ interesting things happen. First of all one has to define what a Gibbs state on an infinite graph is. *Throughout this paper, all infinite graphs we consider will always be assumed to be connected and locally finite.*

Definition 1.2 *Let $G = (V, E)$ be an infinite graph. Let ν be a probability measure on $\{-1, 1\}^V$ and let X be a random element chosen according to ν. We say that ν is a* **Gibbs state for the Ising model on G with**

external field h **and coupling constant** J *if, for all finite sets* $W \subseteq V$, *all* $\omega' \in \{-1,1\}^W$ *and* ν*–a.a.* $\omega'' \in \{-1,1\}^{V \setminus W}$, *we have*

$$P(X(W) = \omega' | X(V \setminus W) = \omega'') = \frac{1}{Z^{\omega''}_{W,h,J}} e^{h a_W(\omega') + J b_W(\omega', \omega'')} \tag{1}$$

where $a_W(\omega') = \sum_{v \in W} \omega'(v)$, $b_W(\omega', \omega'') = \sum_{u,v \in W \cup \partial W : u \sim v} \omega(u)\omega(v)$, $Z^{\omega''}_{W,h,J}$ *is a normalization constant,* $\partial W = \{u \in V \setminus W : \exists v \in W : v \sim u\}$ *is the boundary of* W, *and* $\omega \in \{-1,1\}^V$ *is defined by letting* $\omega(v)$ *be* $\omega'(v)$ *for* $v \in W$ *and* $\omega''(v)$ *for* $v \in V \setminus W$.

It is well known (see [5], p.71) that for any infinite graph G, and for any parameters h and J, there exists a corresponding Gibbs state. The fundamental question of interest is whether there exists more than one Gibbs state. When this is the case, we say that *phase transition* occurs. It is well known, (see [4], p.152), that in the case of the d–dimensional integer lattice $G = \mathbf{Z}^d$, phase transition cannot occur if $h \neq 0$, whereas if G is the binary tree, then there can be phase transition for $h \neq 0$ (see [5], p.250). It is natural to ask the general question of for which graphs phase transition can occur for nonzero h and the point of this paper is to give a partial answer to this question.

Definition 1.3 *Let* $G = (V, E)$ *be an infinite graph. The* **Cheeger constant** *for* G, $\kappa(G)$, *is defined by*

$$\kappa(G) = \inf_{W \subseteq V : |W| < \infty} \frac{|\partial W|}{|W|}.$$

If $\kappa(G) = 0$, G *is said to be* **amenable** *and in case* $\kappa(G) > 0$, G *is said to be* **nonamenable**.

Definition 1.4 *An infinite graph* $G = (V, E)$ *is* **quasi–transitive** *if there exists a finite number of vertices* $x_1, \ldots, x_k$ *in* V *such that for any* $x \in V$, *there is an automorphism of* G *taking* x *to some* x_i *(i.e., the automorphism group of* G *acting on* V *has a finite number of orbits). If* k *can be taken to be 1, the graph is* **transitive**.

Theorem 1.5 *Let G be an infinite graph with maximum degree $d < \infty$.*

(a) If G is nonamenable, $h > 0$ and $J > (2\kappa(G))^{-1}(2h+1+\log(3(d+1)))$, then a phase transition occurs. In particular, there exist $h > 0$ and $J \in [0, \infty)$ such that G exhibits a phase transition.

(b) If G is amenable and quasi–transitive, then the Ising model on G does not exhibit a phase transition for any $h > 0$ and $J \in [0, \infty)$.

Part (a) will be obtained from a Peierls type argument. For (b), we extend the classical convexity and differentiability of pressure argument for $\mathbf{Z}^d$.

Remarks.

- Part (b) is already suggested by the comment in [18] that the uniqueness of the Gibbs distribution for the Ising model for $h \neq 0$ on $\mathbf{Z}^d$ depends only on the fact that the graph is transitive and that the number of sites at distance N from a fixed site grows slower than the number of sites at distances smaller than N as $N \to \infty$. The latter property however does not characterize graphs for which there is uniqueness for $h \neq 0$ since there are transitive graphs (in fact, Cayley graphs of groups), which have exponential growth (and hence the latter property fails) but which are amenable. One such example is the group G_1 in [10], also called the lamplighter group.

- Part (b) of Theorem 1.5 is false for general amenable graphs. (Therefore the analogue in our case of the extension in [6] of Kesten's result is not true). Consider for instance the lattice $\mathbf{Z}^2$ and a binary tree where an edge is placed between the root of the tree and the origin of the lattice. The resulting graph is clearly amenable but one can show that there is a phase transition for some $h > 0$.

- Griffiths' inequality (see [15], p.186) implies that when $h = 0$, if there is phase transition at J, then there is also phase transition at any larger J. This inequality also implies that if there is no phase transition at J with $h = 0$ on a graph G, then the same is true for any subgraph of G. Interestingly, when $h > 0$, both of the latter statements are false; see [18].

To relate the above result to groups, we first define amenability of a group.

Definition 1.6 *A group G is* **amenable** *if given $\epsilon > 0$ and a finite set $B \subseteq G$, there exists $F \subseteq G$ such that*

$$|BF \triangle F| \leq \epsilon |F|.$$

Definition 1.7 *Given a finitely generated group G' (amenable or not) and a finite symmetric generating set S, the* **Cayley graph** *associated to G' and S is the graph whose vertices are the elements of G' and an edge exists between x and y if and only if $sx = y$ for some $s \in S$ (or equivalently $sy = x$ for some $s \in S$ since S is symmetric).*

It is a easy to see that a group is amenable if and only if the Cayley graph associated with some (all) generating set S is amenable. Since Cayley graphs are transitive, Theorem 1.5 yields a phase transition characterization of amenable groups.

We mention here that much of classical ergodic theory has been extended to the case where the group acting is an amenable group. This is because it is precisely these groups which contain "good averaging sets", so-called Fölner sequences.

§2 contains some preliminaries and §3 contains the proof of Theorem 1.5.

2 Monotonicity preliminaries

Given a finite set $W \subseteq V$ and a $\delta \in \{-1,1\}^{V \setminus W}$, we consider the probability measure $\nu^{\delta}_{W,h,J}$ on $\{-1,1\}^V$ obtained by placing the configuration δ on $V \setminus W$, and on W using (1) with $\omega'' = \delta$. We refer to δ as a *boundary condition* and call $\nu^{\delta}_{W,h,J}$ the *finite volume Gibbs state on W with parameters h and J and with boundary condition δ.*

One way to construct Gibbs states is to fix a sequence $\{W_n\}$ of subsets of V such that $W_n \uparrow V$ (meaning that $W_1 \subseteq W_2 \subseteq W_3 \ldots$ and $\cup_i W_i = V$) and a sequence $\delta_n \in \{-1,1\}^{V \setminus W_n}$, and consider weak subsequential limits of the sequence $\{\nu^{\delta_n}_{W_n,h,J}\}$ as $n \to \infty$. Any such weak subsequential limit is a Gibbs state (see [5] p.67). When the sequence $\{W_n\}$ is understood from context, we will write $\nu^{\delta_n}_{n,h,J}$ for $\nu^{\delta_n}_{W_n,h,J}$.

The fact that $J > 0$ results in a good deal of monotonicity in the Ising model. For two probability measures ν and μ on $\{0,1\}^S$, we write $\nu \leq_d \mu$ to indicate that ν is stochastically smaller than μ, which means that $\int f d\nu \leq \int f d\mu$ for all increasing functions f on $\{0,1\}^S$. Now, with $\{W_n\}$ still fixed, consider the two particular sequences, $\{\nu^+_{n,h,J}\}$ and $\{\nu^-_{n,h,J}\}$, of measures corresponding to $\delta_n \equiv 1$ and $\delta_n \equiv -1$ respectively. Standard monotonicity arguments based on Holley's Theorem (see e.g. [15], p.188) implies that $\nu^-_{n,h,J} \leq_d \nu^\delta_{n,h,J} \leq_d \nu^+_{n,h,J}$ for any $\delta \in \{-1,1\}^{V \setminus W_n}$ and that $\{\nu^+_{n,h,J}\}$ and $\{\nu^-_{n,h,J}\}$ are stochastically decreasing and increasing respectively. Such arguments also imply that the weak limits $\nu^+_{h,J} = \lim_{n\to\infty} \nu^+_{n,h,J}$ and $\nu^-_{h,J} = \lim_{n\to\infty} \nu^-_{n,h,J}$ exist and are independent of the sequence of sets $\{W_n\}$. In addition, one has that $\nu^-_{h,J} \leq_d \nu \leq_d \nu^+_{h,J}$ for any Gibbs state on G with the same parameters. The latter implies that phase transition for the parameters h and J occurs if and only if $\nu^+_{h,J} \neq \nu^-_{h,J}$. The following lemma which is a trivial extension of Corollary 2.8 in ([15], p.75) provides a simple tool to check whether this is the case.

Lemma 2.1 *$\nu^+_{h,J} = \nu^-_{h,J}$ if and only if $\nu^+_{h,J}(X(v) = 1) = \nu^-_{h,J}(X(v) = 1)$ for every $v \in V$.*

It is straightforward to show that $\nu^+_{h,J}$ and $\nu^-_{h,J}$ are invariant under all graph automorphisms of G. Therefore, if G is quasi–transitive, then $\nu^+_{h,J}(X(v) = 1)$ and $\nu^-_{h,J}(X(v) = 1)$ each take on at most finitely many values.

3 Proofs

3.1 The nonamenable case

We shall need the following lemma due to Kesten (see [13]).

Lemma 3.1 *Let G be an infinite graph with maximum degree d and let $\mathcal{C}_m$ be the set of connected sets with m vertices containing a fixed vertex v. Then $|\mathcal{C}_m| \leq (e(d+1))^m$.*

Proof of Theorem 1.5 (a).
In order to prove phase transition, note the well-known fact that the measures $\nu^+_{h,J}$ and $\nu^-_{h,J}$ are stochastically increasing in h. Therefore, for $h > 0$, we have $\nu^+_{h,J}(X(v) = 1) \geq \nu^+_{0,J}(X(v) = 1) \geq 1/2$ (the second inequality following from

symmetry) for all vertices v. Therefore if we can show that $\nu^-_{h,J}(X(v)=1) < 1/2$ for some v, then phase transition follows.

Let $v \in V$ be arbitrary and $\mathcal{C}_m$ be as in the above lemma. For $C \in \bigcup_{m=1}^{\infty} \mathcal{C}_m$, let $A_C = \{X(C) \equiv 1, X(\partial C) \equiv -1\}$, i.e. the event that the cluster of 1's containing v is exactly C. Now for any $n \in \{1,2,\ldots\}$, we have

$$\nu^-_{n,h,J}(X(v)=1) = \nu^-_{n,h,J}\Big(\bigcup_{m=1}^{\infty} \bigcup_{C\in\mathcal{C}_m} A_C\Big) = \sum_{m=1}^{\infty} \sum_{C\in\mathcal{C}_m} \nu^-_{n,h,J}(A_C). \qquad (2)$$

If we show that for each n and for each $C \in \cup_m \mathcal{C}_m$,

$$\nu^-_{n,h,J}(A_C) \le e^{(2h-2J\kappa(G))|C|}, \qquad (3)$$

then by inserting this into (2) and using Lemma 3.1, we get

$$\nu^-_{n,h,J}(X(v)=1) \le \sum_{m=1}^{\infty} (e(d+1)e^{2h-2J\kappa(G)})^m$$

which is less than 1/2 uniformly in n if $J > (2\kappa(G))^{-1}(2h+1+\log(3(d+1)))$.

To prove (3), fix n, fix $C \in \cup_m \mathcal{C}_m$, identify A_C with a subset of $\{-1,1\}^{W_n}$ and let T_C be the injective map from A_C into $\{-1,1\}^{W_n}$, which changes all the 1's in C to -1's and leaves the other vertices alone. For $\omega \in \{-1,1\}^{W_n}$, let $H_n(\omega) = ha_{W_n}(\omega) + Jb_{W_n}(\omega,\omega'')$ where these terms come from (1) and with ω'' being all -1's. Letting $\partial_E(C)$ be the *edge boundary* of C which is the set of all edges connecting C to ∂C, we have since $|\partial_E(C)| \ge |\partial C|$ that

$$H_n(T_C(\omega)) = H_n(\omega) + 2J|\partial_E(C)| - 2h|C| \ge H_n(\omega) + (2J\kappa(G) - 2h)|C|$$

and hence

$$\nu^-_{n,h,J}(A_C) = \frac{\sum_{\omega\in A_C} e^{H_n(\omega)}}{Z} \le e^{-(2J\kappa(G)-2h)|C|} \frac{\sum_{\omega\in A_C} e^{H_n(T_C(\omega))}}{Z}$$
$$\le e^{-(2J\kappa(G)-2h)|C|}$$

the last inequality following from the fact that T_C is injective. □

3.2 The amenable case

The first proofs that for $\mathbf{Z}^d$ there is no phase transition for $h > 0$ can be found in [14] and [17] and relied on the Lee-Yang Circle Theorem. Afterwards, Preston ([16], see also Chapter V in [4]) was able to obtain the same result, using the GHS inequality (see [7]) instead of the Lee-Yang Circle Theorem. This latter method will be the one followed in the proof below. This method was also exploited to study phase transition in the hard core model (see [3]).

The key step here is to establish the following proposition where quasi–transitivity is not required.

Proposition 3.2 *Let $G = (V, E)$ be an amenable graph with uniformly bounded degree and let $h_0, J > 0$. Then for any sequence, $\{W_n\}$, of subsets of V such that $W_n \uparrow V$ and $|\partial W_n|/|W_n| \to 0$ it is the case that for some subsequence $\{n_i\}$,*

$$\lim_{i\to\infty} |W_{n_i}|^{-1}(\mathbf{E}^{+}_{n_i,h_0,J} a_{W_{n_i}}(X(W_{n_i})) - \mathbf{E}^{-}_{n_i,h_0,J} a_{W_{n_i}}(X(W_{n_i}))) = 0$$

where the function $a_{W_{n_i}}$ is as in (1); the sum of the values of the variables over the set W_{n_i} and $\mathbf{E}^{\delta_n}_{n,h,J}$ refers to expectation with respect to the measure $\nu^{\delta_n}_{n,h,J}$.

Proof: Let d be the maximum degree of a vertex. Fix any sequence, $\{W_n\}$, of subsets of V such that $W_n \uparrow V$ and $|\partial W_n|/|W_n| \to 0$. For $h \in [0, 2h_0]$ and a boundary condition δ_n on W_n, define $f^{\delta_n}(n, h, J)$ as $|W_n|^{-1} \log Z^{\delta_n}_{n,h,J}$, where $Z^{\delta_n}_{n,h,J} = \sum_{\omega\in\{-1,1\}^{W_n}} e^{h a_{W_n}(\omega) + J b_{W_n}(\omega,\delta_n)}$ is the normalization constant in (1) with $\omega'' = \delta_n$ and $W = W_n$. It follows from inspection of this expression that there is $K = K(J, h_0)$ such that $f^{\delta_n}(n, h, J) \in [0, K]$ for every n, δ_n and $h \in [0, 2h_0]$. Now fix a sequence of boundary conditions $\{\delta_n\}$. By compactness, there exists a sequence $\{n_i\}$ so that the subsequence $\lim_{i\to\infty} f^{\delta_{n_i}}(n_i, h, J)$ exists for all rational h in $[0, 2h_0]$.

Now, for a fixed n, δ and h, we have that

$$\frac{\partial}{\partial h} f^{\delta}(n, h, J) = |W_n|^{-1}\mathbf{E}^{\delta}_{n,h,J} a_{W_n}(X(W_n)) \tag{4}$$

and

$$\frac{\partial^2}{\partial h^2} f^{\delta}(n, h, J) = |W_n|^{-1}(\mathbf{E}^{\delta}_{n,h,J}(a_{W_n}(X(W_n))^2) - (\mathbf{E}^{\delta}_{n,h,J} a_{W_n}(X(W_n)))^2 \geq 0.$$

Thus $f^{\delta_n}(n,h,J)$ is convex in h for each n, and it follows from Theorem V1.3.3(a) in [4] that $\lim_{i\to\infty} f^{\delta_{n_i}}(n_i,h,J)$ exists for all h in $[0,2h_0]$ and is convex in h. Denote this limit by $f(h,J)$ which is defined for h in $[0,2h_0]$ and which (possibly) depends on $\{W_n\}$, $\{\delta_n\}$ and the sequence $\{n_i\}$. Next, for any $\omega \in \{-1,1\}^{W_n}$ and any boundary conditions δ'_n we have that $|b_{W_n}(\omega,\delta_n) - b_{W_n}(\omega,\delta'_n)| \leq 2d|\partial W_n|$ so that $|f^{\delta_n}(n,h,J) - f^{\delta'_n}(n,h,J)| \leq 2dJ|\partial W_n|/|W_n| \to 0$. Hence for any boundary conditions $\{\delta'_n\}$, and any h in $[0,2h_0]$, we have

$$\lim_{i\to\infty} f^{\delta'_{n_i}}(n_i,h,J) = f(h,J).$$

Let h be a point where $f(h,J)$ is differentiable. By Lemma IV.6.3 in ([4], p.114), convexity and the above we have that

$$\frac{\partial}{\partial h} f^{\delta'_{n_i}}(n_i,h,J) \to \frac{\partial}{\partial h} f(h,J)$$

for any boundary conditions $\{\delta'_n\}$. Applying the latter statement to the two boundary conditions of all pluses and all minuses together with (4) yields the conclusion of the proposition provided $f(h,J)$ is differentiable at h_0.

Being convex in h on $[0,2h_0]$, $f(h,J)$ is differentiable for all but at most countably many values of h on $[0,2h_0]$ but we need to show this for all h, which would complete the proof. To do this we apply the standard argument for $\mathbf{Z}^d$ (see [4], p.151) based on concavity of $\mathbf{E}^f_{n,h,J} a_{W_n}(X(W_n))$ for $h > 0$, where the superscript f corresponds to the *free* boundary condition, i.e. $\nu^f_{n,h,J}$ is defined according to Definition 1.1 on the finite graph spanned by W_n. Clearly $\lim_{i\to\infty} f^f(n_i,h,J) = f(h,J)$ where $f^f(n,h,J)$ is defined in the obvious way. Concavity of $\mathbf{E}^f_{n,h,J} a_{W_n}(X(W_n))$ follows directly from the GHS inequality which in [7] is stated for subsets of $\mathbf{Z}^d$ but which is clearly valid on any graph as it allows general interactions. Thus the desired concavity follows and the rest of the argument in ([4], p.151) goes through unchanged to conclude that $f(h,J)$ is differentiable for all $h > 0$, completing our proof. $\square$

Remark. The last paragraph of the above proof can be avoided by appealing to Proposition 2 of [18].

Proof of Theorem 1.5 (b).
Fix $h_0 > 0$ and $J \geq 0$. If $\nu^+_{h_0,J} \neq \nu^-_{h_0,J}$, then by Lemma 2.1, there must exist

v such that $\mathbf{E}^{+}_{h_0,J}X(v) > \mathbf{E}^{-}_{h_0,J}X(v)$. It follows (since G is connected) that this strict inequality holds for all v and hence by quasi-transitivity,

$$\inf_{v \in V}(\mathbf{E}^{+}_{h_0,J}X(v) - \mathbf{E}^{-}_{h_0,J}X(v)) := \epsilon_0 > 0.$$

Since G is amenable, one can show that it is possible to pick a sequence $\{W_n\}$ of subsets of V (not necessarily connected) so that $W_n \uparrow V$ and $\lim_{n\to\infty} |\partial W_n|/|W_n| = 0$. By Proposition 3.2, choose a sequence $n_i \to \infty$ so that the conclusion of Proposition 3.2 holds. Next by stochastic monotonicity, we have that for any fixed n

$$|W_n|^{-1}\mathbf{E}^{+}_{n,h_0,J}a_{W_n}(X(W_n)) \geq |W_n|^{-1}\mathbf{E}^{+}_{h_0,J}a_{W_n}(X(W_n)) \geq$$
$$|W_n|^{-1}\mathbf{E}^{-}_{h_0,J}a_{W_n}(X(W_n)) \geq |W_n|^{-1}\mathbf{E}^{-}_{n,h_0,J}a_{W_n}(X(W_n)). \qquad (5)$$

If there were a phase transition, then the difference of the two middle terms would be at least ϵ_0. However the difference of the expressions on the extreme left and right converge to 0 as $n \to \infty$ along the sequence n_i and hence phase transition cannot occur for $h = h_0$. $\Box$

Remark. The reason that the above proof breaks down for general amenable graphs is that the strict inequality

$$\inf_{v \in V}(\mathbf{E}^{+}_{h_0,J}X(v) - \mathbf{E}^{-}_{h_0,J}X(v)) > 0,$$

used above does not hold as the expected spins might take on infinitely many different values. It is still true however that Proposition 3.2 holds, a fact which is interesting in its own right as it tells us that for a suitable choice of $\{W_n\}$, the average magnetism over W_n will ultimately be approximately the same for $\nu^{+}_{h,J}$ and $\nu^{-}_{h,J}$. In the example of the tree and the lattice attached by an edge, the W_n's whose boundary/volume ratio is going to 0 will consist of "large" parts of the lattice and "small" parts of the tree and so most of the contribution in the terms above are coming from the lattice part (where there is no phase transition for $h > 0$). This explains why the expression in Proposition 3.2 can be going to 0 even in the presence of a phase transition.

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