The Ising Model on Diluted Graphs and Strong Amenability

Olle Häggström[*] Roberto H. Schonmann[†] Jeffrey E. Steif[‡]

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Abstract

Say that a graph has persistent transition if the Ising model on the graph can exhibit a phase transition (nonuniqueness of Gibbs measures) in the presence of a nonzero external field. We show that for nonamenable graphs, for Bernoulli percolation with p close to 1, all the infinite clusters have persistent transition. On the other hand, we show that for transitive amenable graphs, the infinite clusters for any stationary percolation do not have persistent transition. This extends a result of Georgii for the cubic lattice. A geometric consequence of this latter fact is that the infinite clusters are strongly amenable (i.e., their anchored Cheeger constant is 0). Finally we show that the critical temperature for the Ising model with no external field on the infinite clusters of Bernoulli percolation with parameter p, on an arbitrary bounded degree graph, is a continuous function of p.

1 Introduction

A great deal of interest has recently been dedicated to the study of statistical mechanics type processes on (infinite, locally finite, connected) graphs other than Euclidean lattices. Particularly important are the ways in which the geometry of the graph is reflected in the behavior of the process. Two of the most important models which are being studied in this context are

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percolation (see, e.g., [42], [43], [25], [44], [58], [45], [60], [7], [27], [40], [3], [4], [5], [26], [30], [51], [52], [31], [6], [47], [29], [15], [9]; see also [8] for an on-line, continuously updated, account of progress in this area, as well as for links to various related Web sites), and the Ising model (for the Ising model on homogeneous trees, see, e.g., [22, Chapter 12] and the references provided there, as well as the papers [11] and [33] that appeared afterwards; for the Ising model on more general graphs, see, e.g., [42], [54], [49], [1], [58], [59], [34], [53], [36], [29], [17]).

In this paper we are concerned with these two processes and also with their interrelations. Primarily we study the Ising model on a diluted graph, i.e., a graph from which sites have been randomly removed (in a quenched fashion, in the language of physics). As a byproduct, we obtain a new result about the geometry of infinite clusters; in this way this paper provides one more link between the theory of percolation and the Ising model.

There are various ways in which one can look at our investigation. On one hand, the study of Ising models on diluted graphs can be seen as a chapter in the study of statistical mechanics models in random environments, a very fundamental and active area of research. From this perspective we mention the papers [24], [18], [19], [20], [21], [14] and [2], where this problem was addressed primarily on Euclidean lattices and where one can find further references to this type of work, and the paper [42], where the problem was addressed on trees.

On the other hand, our investigation can be seen as part of the large project of understanding the properties of infinite clusters in percolation, and how these properties relate to the properties of the graph on which percolation is being performed. From this perspective it is natural to look at the phase diagram of statistical mechanics models on the infinite clusters and also to ask if the features of such phase diagrams unveil facts about the clusters' geometry. Such issues are akin to the study of random walks on graphs and Brownian motion on manifolds and the question of how their behaviors are related to the geometry of the underlying space.

2 Strong amenability and weak nonamenability

Throughout this paper, $G = (V, E)$ will be an infinite, locally finite, connected graph. If $W \subset V$ is finite, let $\partial W = \{u \in V \setminus W : \exists v \in W : v \sim u\}$ where $v \sim u$ means that v and u are neighbors. We write $\partial^G W$ if we need to make the graph G explicit. In the above, locally finite means that for each $x \in V$, $|\partial\{x\}| < \infty$. If for some finite D, for each $x \in V$, $|\partial\{x\}| \leq D$, then the graph is said to be of bounded degree.

Write $\mathrm{Aut}(G)$ for the group of graph automorphisms of the graph G.

Definition 2.1 *A graph $G = (V, E)$ is called* **transitive** *if for any $x, y \in V$ there exists a $\gamma \in Aut(G)$ which maps x to y. In other words G is transitive if $Aut(G)$ acts transitively on V, i.e., it produces a single orbit. A graph $G = (V, E)$ is called* **quasi-transitive** *if $Aut(G)$ acting on V produces a finite number of orbits.*

In general, qualitative features of transitive graphs are also shared by quasi-transitive graphs. For simplicity we will only consider transitive ones below, even when the statements could be extended to quasi-transitive graphs.

Definition 2.2 *Let $G = (V, E)$ be an infinite, locally finite, connected graph. The* **Cheeger constant** *$\kappa(G)$ for G, is defined by*

$$\kappa(G) := \inf\left\{\frac{|\partial W|}{|W|} \,:\, W \subseteq V,\, 0 < |W| < \infty\right\}.$$

If $\kappa(G) = 0$, G is said to be **amenable***, and if $\kappa(G) > 0$, G is said to be* **nonamenable***.*

It is a well established fact that that the (non-)amenability of a graph has important consequences for the behavior of various probabilistic processes associated with the graph. For instance, in the case of random walks and their spectral gaps this relation goes back to [38] and [39] (see, e.g., [4] for references to the subsequent developments on this relationship). In the current surge of interest in probabilistic processes on graphs, a driving force has been establishing many more relationships between the behavior of probabilistic processes and the amenability of the underlying graph (see e.g., [4], [36], [35]), while other relations are being conjectured (e.g., that for Bernoulli percolation on transitive graphs amenability is a necessary and sufficient condition for the absence of a phase with infinitely many infinite clusters – the sufficiency is known from the methods in [12]).

We should mention that the definition of the Cheeger constant varies somewhat from paper to paper, in that sometimes the vertex boundary used in the definition above is replaced with the edge boundary. This distinction is only important when the graph is not of bounded degree or when the Cheeger constant is used in numerical estimates (e.g., of a spectral gap, or a critical point), but is irrelevant in regard to the definition of amenability of graphs of bounded degree. Some other variations in the definition of amenability are also immaterial and are discussed in the Appendix. In contrast, the next definition produces an essentially distinct notion.

Definition 2.3 *Let $G = (V, E)$ be an infinite, locally finite, connected graph. Fix a vertex $0 \in V$. The* **anchored Cheeger constant** $\kappa^*(G)$ *for G, is defined by*

$$\kappa^*(G) := \lim_{n\to\infty} \inf \left\{ \frac{|\partial W|}{|W|} \,:\, 0 \in W \subseteq V,\, W \text{ connected },\, n \le |W| < \infty \right\}.$$

If $\kappa^*(G) = 0$, G is said to be **strongly amenable**, *and if $\kappa^*(G) > 0$, G is said to be* **weakly nonamenable**.

It is easy to see that the value of $\kappa^*(G)$, and hence also the notion of strong amenability, does not depend on the choice of 0 in V. It is also clear that $\kappa(G) \le \kappa^*(G)$ and hence strong amenability implies amenability, or equivalently nonamenability implies weak nonamenability. A graph that is amenable but is not strongly amenable can be obtained by attaching a sequence of paths of lengths 1,2,... at a very sparse sequence of vertices of a nonamenable graph. In contrast, it is elementary to see that a transitive amenable graph is strongly amenable (see the Appendix). As we will also mention in the appendix, replacing "$\lim_{n\to\infty}$" with "$\inf_n$" in the definition of $\kappa^*(G)$ will not affect its positivity (although there will then be a dependence on the vertex 0).

The anchored Cheeger constant was introduced in [6, Section 6], where one can find a detailed discussion of the motivation in introducing this object, references to earlier work which motivated this definition and conjectures on how it relates to random walks on the graph. It was further studied in [15] and [56]. In these two papers some fundamental questions from [6] were answered, including a proof in the former paper that non-degenerate Galton-Watson trees are weakly nonamenable and a proof in the latter paper that the lim inf speed of the symmetric random walk on any weakly nonamenable infinite connected graph of bounded degree is almost surely strictly positive.

It is usually equally natural to consider site or bond percolation on a graph. Except in Section 6, we will in this paper only consider site percolation. Results similar to the ones stated in this context hold also for bond percolation, either because the same proofs apply, or because bond percolation on a given graph is identical to site percolation on its cover (line) graph (and the cover graph of a quasi-transitive graph is also quasi-transitive). Recall that an *induced* subgraph is a graph obtained by taking a subset of the vertices and then all edges between these vertices which were present in the original graph.

In Bernoulli (that is, i.i.d.) site percolation with retention parameter $p \in [0,1]$ on $G = (V, E)$, each vertex (site) is independently assigned the value 1 (occupied, open) with probability p, and the value 0 (vacant, closed) with probability $1-p$. We write $\mathbf{P}_p^G$, or simply $\mathbf{P}_p$, for the resulting probability measure on $\{0,1\}^V$. *Clusters* are the connected components of the graph obtained by deleting from G all the vacant vertices and all the edges incident to these vertices. $C(x)$ will denote the cluster containing x (which might be empty). By Kolmogorov's zero-one law, the existence of *at least one* infinite cluster has probability 0 or 1, and one defines

$$p_c(G) = \inf\{p \in [0,1] : \mathbf{P}_p^G(\exists \text{ an infinite cluster}) = 1\}.$$

In this paper we will also consider more general *stationary* (or *invariant*) percolation processes on a transitive graph. Such a process can be seen as simply a probability measure on $\{0,1\}^V$ which is invariant under $\mathrm{Aut}(G)$. This binary random field is then interpreted in the same way as in the case of Bernoulli percolation, with clusters having the same definition as above. One says that the *finite energy condition* holds if on any finite set of vertices each configuration of occupied and vacant vertices has positive conditional probability given any configuration outside of this set. The arguments in [48] show that for any stationary finite energy percolation on a transitive graph, the number of infinite clusters is a.s. 0, 1 or ∞. The arguments of [12] show that if the graph is also amenable, then for any stationary finite energy percolation, the number of infinite clusters is a.s. 0 or 1.

In both the amenable and nonamenable cases, there has been recent interest in the structure and properties of the infinite cluster(s) that arise from percolation. For example, in [47] and [6], it is studied whether clusters are transient for simple random walk. For another example, in [30] and [51] it is shown that for Bernoulli percolation on a transitive graph G with retention parameter $p > p_c(G)$ every infinite cluster C has $p_c(C) = p_c(G)/p$. Other results concerning properties of infinite clusters can be found, e.g., in [27], [4], [31] and [47] (number of *topological ends* of the infinite clusters), in [5] (presence of infinitely many so-called *encounter points* in infinite clusters), and in [31] (number of infinite clusters in Bernoulli percolation performed on the infinite clusters). An interesting type of result obtained in various levels of generality in [30], [47] and [31], is that, under certain conditions, certain types of properties are shared by all the infinite clusters, in that with probability 1, either they all have the property or they all do not have the property. This is referred to as *indistinguishability.* We will return to this type of result in Section 4.

It is natural to ask whether the amenability or nonamenability of a graph is "inherited" by the infinite clusters on this graph, at least in the Bernoulli case. The answer is trivially "yes" when $p = 1$, but otherwise, and if the graph is of bounded degree, it is negative, since it is not hard to see that all the infinite clusters then a.s. contain arbitrarily large chains of vertices with degree 2 (see, e.g., [51, proof of Lemma 1]), and therefore are always amenable.

In [6, Question 6.5], it is asked whether for Bernoulli percolation on a nonamenable graph all the infinite clusters are always weakly nonamenable. In [15] this is proved to be the case if p is large. On the other hand we observe here that in general this cannot be the case for reasons that we explain next. First we recall that a weakly nonamenable graph has $p_c < 1$, as can be seen e.g. from [7, proof of Theorem 2]. Second, we recall that it is also known that there exist nonamenable graphs (of bounded degree) on which at the critical point there is a.s. an infinite cluster. (For this we can take G as a spherically symmetric tree, meaning that all vertices at the same distance from a given vertex, called the root, have the same degree, constructed as follows. Choose the degree of the vertices at distance n from the root recursively in n, either as 3, if the number of such vertices is more than $2n^2 2^n$, or as 4, otherwise. It is clear that for large n, the number of vertices at distance n from the root will be in the interval $(n^2 2^n, 4n^2 2^n)$. Then it is clear that G is nonamenable, and from [44, Theorem 2.1] we obtain that $p_c(G) = 1/2$ and that there is percolation on G at criticality.) Now, if such an infinite cluster at criticality were weakly nonamenable, by performing Bernoulli percolation on it with a large retention parameter, we would obtain infinite clusters. But then a standard coupling argument shows that our original nonamenable graph has infinite clusters at a retention parameter smaller than its critical point, and this is a contradiction. A minor modification of this example shows that one can also obtain strongly amenable infinite clusters for Bernoulli percolation on a nonamenable graph G with retention parameter $p > p_c(G)$. For this purpose it is enough to connect, by means of a single extra edge, two disjoint nonamenable graphs, one of which is the one of the previous argument, while the other has a strictly smaller critical point.

Combining the above mentioned result from [15] with one of our contributions in this paper, we state the following result.

Theorem 2.4 *Let G be an infinite, locally finite, connected graph.*

(i) ([15]) *If G is weakly nonamenable, then for p close to 1, a.s. all the infinite clusters for Bernoulli percolation on G are weakly nonamenable.*

(ii) *If G is transitive and amenable, then for any stationary percolation on G, a.s. all the infinite clusters (if any exist) are strongly amenable.*

The proof of (ii) will be explained after we state Theorem 3.4.

It is clear that the assumption of transitivity cannot be removed in (ii), since by connecting, by means of a single extra edge, two disjoint graphs, exactly one of which is amenable, one obtains an amenable graph which will, with positive probability, contain weakly nonamenable infinite clusters for large p (as follows by using (i) above).

To put part (ii) of Theorem 2.4 in the proper perspective, we note the following easy result. In this statement, the meaning of *subexponential growth* is, as usual, that the number of vertices within distance n from a given vertex is bounded above by functions of the form $C_1 \exp(C_2 n)$, with C_2 arbitrarily close to 0.

Proposition 2.5 *Let G be an infinite, locally finite, connected graph of subexponential growth. Then every infinite connected subgraph of G is strongly amenable.*

To see why this proposition is true, note that if a graph has subexponential growth, then all its connected subgraphs also have subexponential growth. On the other hand, if a graph is weakly nonamenable, then it is clear that the number of vertices at distance exactly $n+1$ from a given vertex 0 is larger than a fixed positive fraction of the number of vertices within distance n from 0. Hence the number of vertices within distance n from 0 grows exponentially fast with n.

Proposition 2.5 renders part (ii) of Theorem 2.4 interesting only in the case of transitive amenable graphs of exponential growth. A well known example of such a graph is the so-called lamplighter graph (see, e.g., [37] or [46]). This graph is known to have induced subgraphs which are Fibonacci trees, in which vertices have degree 2 or 3, with each vertex of degree 2 connected to at least one vertex of degree 3. This assures that the Cheeger constant (and hence that the anchored Cheeger constant) of such a tree is positive. Therefore part (ii) of Theorem 2.4 contains non-trivial information in this case.

In view of the remarks and results above, it is natural to replace [6, Question 6.5] with the following one.

Question 2.6 *If G is transitive and nonamenable, is it the case that all the infinite clusters for any Bernoulli percolation on G are weakly nonamenable?*

Note that a positive answer to this question would imply absence of percolation at criticality, by the reasoning explained above. Absence of percolation at criticality for transitive nonamenable graphs has been established in the case of graphs with unimodular automorphism groups, in [4] and [5], but the general transitive nonamenable case remains open.

In the case in which G is a tree, Question 2.6 was answered affirmatively in [15, Corollary 1.3].

Regarding stationary percolation on nonamenable transitive graphs, it is not hard to find examples in which there are strongly amenable infinite clusters. For instance [27, Examples 6.1–6.3] are all examples of invariant percolations on homogeneous trees in which all infinite clusters have 1 or 2 topological ends. Such infinite clusters are clearly strongly amenable.

3 Ising model on infinite clusters and persistent transition

We shall study some aspects of the phase diagram for the Ising model on the infinite clusters (viewed as graphs in themselves) of a percolation process. In this section we are mainly interested in whether the phase coexistence region contains some pair (h, J) with $h \neq 0$. Throughout this paper, we assume $J \geq 0$.

Definition 3.1 *Let $G = (V, E)$ be an infinite locally finite graph. Let ν be a probability measure on $\{-1, 1\}^V$ and let X be a random element chosen according to ν. We say that ν is a* **Gibbs measure for the Ising model on G with external field h and coupling constant J** *if, for all finite sets $W \subseteq V$, all $\omega' \in \{-1, 1\}^W$ and ν–a.a. $\omega'' \in \{-1, 1\}^{V \setminus W}$, we have*

$$P(X(W) = \omega' | X(V \setminus W) = \omega'') = \frac{1}{Z^{\omega''}_{W,h,J}} e^{h a_W(\omega') + J b_W(\omega', \omega'')}, \tag{1}$$

where $a_W(\omega') = \sum_{v \in W} \omega'(v)$, $b_W(\omega', \omega'') = \sum_{u,v \in W \cup \partial W : u \sim v} \omega(u)\omega(v)$, $Z^{\omega''}_{W,h,J}$ is a normalization constant, and $\omega \in \{-1, 1\}^V$ is defined by letting $\omega(v)$ be $\omega'(v)$ for $v \in W$ and $\omega''(v)$ for $v \in V \setminus W$.

It is well known (see, e.g., [22, p. 71]) that for any infinite graph G, and for any parameters h and J, there exists a least one Gibbs measure. The fundamental question of interest is whether there exists more than one Gibbs measure. When this is the case, we say that *phase transition* or *phase coexistence* occurs. Another well known result (see, e.g., [16, p. 152]) is that in the case of the d–dimensional integer lattice $G = \mathbf{Z}^d$, phase transition cannot occur if $h \neq 0$, whereas if G is the binary tree, then there can be phase transition for $h \neq 0$ (see [22, p. 250]). It is natural to ask (in greater generality) which graphs can have a phase transition for $h \neq 0$; this problem is studied in [36].

Definition 3.2 *We say that a graph G satisfies* **persistent transition** *if for some (h, J) with $h \neq 0$, the Ising model on G with parameters J and h exhibits phase transition.*

Theorem 3.3 ([36]) *Let G be an infinite, locally finite, connected graph.*

(i) *If G is weakly nonamenable, then G has persistent transition.*

(ii) *If G is transitive and amenable, then G does not have persistent transition.*

In fact, part (i) of this theorem is stated in [36] under the hypothesis that G is nonamenable, rather than just weakly nonamenable, but in its proof only weak nonamenability is used. Moreover in [36] the graph is assumed to be of bounded degree, but this hypothesis can be avoided by using [15, Lemma 2.1] in place of [36, Lemma 3.1] (in conjunction with the inequality $|\partial W| \leq |\partial_{\mathrm{E}} W|$, for an arbitrary set of vertices W, where $\partial_{\mathrm{E}} W$ is the edge boundary of W).

Here we extend parts (i) and (ii) of Theorem 3.3 by introducing dilution on the graph.

Theorem 3.4 *Let G be an infinite, locally finite, connected graph.*

(i) *If G is weakly nonamenable, then for p sufficiently close to 1, a.s. all the infinite clusters for Bernoulli percolation on G have persistent transition.*

(ii) *If G is transitive and amenable, then for any stationary percolation on G, a.s. all the infinite clusters (if any exist) do not have persistent transition.*

Part (i) of Theorem 3.4 is an immediate consequence of part (i) of Theorem 2.4 and part (i) of Theorem 3.3. In Section 2, we prove part (ii) of Theorem 3.4. Part (ii) of Theorem 2.4 is an immediate consequence of part (ii) of Theorem 3.4 and part (i) of Theorem 3.3.

In [19], part (ii) of Theorem 3.4 is proved for the special case $G = \mathbf{Z}^d$.

Note that in (ii) under the extra assumption of finite energy there is at most one infinite cluster to which the statement applies. But this is not necessarily the case without this extra assumption, and one can have exactly k infinite clusters for any $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$; see e.g. [13].

Note: When we first proved part (ii) of Theorem 2.4, it was surprising to us that this geometric statement was proved as a corollary to a statement concerning the Ising model on the infinite clusters. After receiving the first version of this paper, Oded Schramm (personal communication) found an alternative, more direct proof of it.

Question 3.5 *Is there an analogue of Proposition 2.5 regarding persistent transition? More precisely, is it the case that if G has subexponential growth, then every subgraph of G does not have persistent transition? Surprisingly, even for the graph $\mathbf{Z}^d$, with $d > 1$, it appears to be difficult to establish whether any of its subgraphs have persistent transition.*

4 The phase diagram of the Ising model on the infinite clusters

In this section we are concerned with some further properties of the phase diagram of the Ising model on the infinite clusters of Bernoulli percolation on a transitive graph H.

Given an infinite, locally finite, connected graph G (not necessarily transitive), define its phase coexistence region as

$$\mathbf{Coex}(G) = \{(h, J)\colon \text{the } (h, J)\text{-Ising model on } G \text{ has phase transition}\}.$$

Question 4.1 *Consider Bernoulli percolation on a transitive graph. Is it the case that, when infinite clusters are present, the phase coexistence region of the Ising model on each one of them is the same?*

We are only able to provide some partial answers to this question, as we explain next.

Theorem 4.2 *Let H be an infinite, locally finite, connected transitive graph. For Bernoulli percolation on H with retention parameter $p > p_c(H)$, and arbitrary fixed values of h and J, one of the following alternatives holds: Either*

$$\mathbf{P}_p^H(\textit{the } (h, J)\textit{-Ising model on each infinite cluster has phase transition}) = 1,$$

or else

$$\mathbf{P}_p^H(\textit{the } (h, J)\textit{-Ising model on each infinite cluster does not have phase transition}) = 1.$$

This result is a consequence of [31, Theorem 1.8] concerning indistinguishability of *robust* properties. To apply [31, Theorem 1.8] we recall that (in the context of site percolation) a property $\mathcal{P}$ of infinite connected subgraphs of a transitive graph H is said to be robust if for every infinite connected subgraph C of H and every vertex x of C, we have the equivalence: C satisfies $\mathcal{P}$ iff there is an infinite connected component of the subgraph of H obtained from C by deleting from it the vertex x and all edges incident to x that satisfies $\mathcal{P}$. The robustness of the property that the (h, J)-Ising model has phase transition on a connected subgraph of H is proved in [19, Theorem 3.1] (for a more general result, from which this robustness can also be easily derived, see [22, Theorem 7.33]). Theorem 4.2 is therefore established.

For any infinite locally finite connected graph H, and $p > p_c(H)$, define

$$\mathbf{Coex}(H,p) :=$$
$$\{(h,J)\colon \mathbf{P}_p^H(\text{the } (h,J)\text{-Ising model on some infinite cluster has phase transition}) > 0\}. \quad (2)$$

By the robustness property discussed in the previous paragraph and Kolmogorov's zero-one law, an equivalent definition would be to require that the $\mathbf{P}_p^H$-probability in (2) is 1. It is an easy consequence of robustness that (2) is also equivalent to requiring that for any fixed $x \in V$,

$$\mathbf{P}_p^H(\text{the } (h,J)\text{-Ising model on } C(x) \text{ has phase transition}) > 0. \quad (3)$$

In the case where H is transitive, the set in (2) is (by Theorem 4.2) the same as

$$\{(h,J)\colon \mathbf{P}_p^H(\text{the } (h,J)\text{-Ising model on each infinite cluster has phase transition}) = 1\}.$$

For H transitive, it is tempting to conjecture that

$$\mathbf{P}_p^H(\text{each infinite cluster } C \text{ has } \mathbf{Coex}(C) = \mathbf{Coex}(H,p)) = 1, \quad (4)$$

but we have no proof of this.

Next, set for arbitrary H:

$$\mathbf{Coex}_{(h,\cdot)}(H) = \{J\colon (h,J) \in \mathbf{Coex}(H)\},$$
$$\mathbf{Coex}_{(h,\cdot)}(H,p) = \{J\colon (h,J) \in \mathbf{Coex}(H,p)\},$$
$$\mathbf{Coex}_{(\cdot,J)}(H) = \{h\colon (h,J) \in \mathbf{Coex}(H)\},$$
$$\mathbf{Coex}_{(\cdot,J)}(H,p) = \{h\colon (h,J) \in \mathbf{Coex}(H,p)\}.$$

For arbitrary G, it is well known that $\mathbf{Coex}_{(0,\cdot)}(G)$ is an infinite interval although M. Salzano has an example which shows that $\mathbf{Coex}_{(h,\cdot)}(G)$ is not necessarily an interval if $h \neq 0$. From [53] it is also known that, for each J, $\mathbf{Coex}_{(\cdot,J)}(G)$ is an interval. Therefore the following is an easy consequence of Theorem 4.2.

Corollary 4.3 *Let H be an infinite, locally finite, connected transitive graph. For Bernoulli percolation on H with retention parameter $p > p_c(H)$,*

$$\mathbf{P}_p^H(\text{each infinite cluster } C \text{ has } \mathbf{Coex}_{(0,\cdot)}(C) = \mathbf{Coex}_{(0,\cdot)}(H,p)) = 1,$$

and, for each value of J,

$$\mathbf{P}_p^H(\text{each infinite cluster } C \text{ has } \mathbf{Coex}_{(\cdot,J)}(C) = \mathbf{Coex}_{(\cdot,J)}(H,p)) = 1.$$

Given a graph H, it is natural to define

$$J_c(H,p) = \inf \mathbf{Coex}_{(0,\cdot)}(H,p),$$

and

$$h_c(H,p,J) = \sup \mathbf{Coex}_{(\cdot,J)}(H,p),$$

and to ask how these critical values behave. Also set $J_c(H) = J_c(H,1)$, i.e. $J_c(H)$ is the critical value for the Ising model with zero external field on H. From [2] and comparison results between site and bond percolation (see [26] or [29]) we know also that if H has bounded degree, then for $p > p_c$ we have $J_c(H,p) < \infty$, and $\lim_{p \searrow p_c(H)} J_c(H,p) = \infty$. It is well known that $J_c(G)$ is non-increasing in G, and therefore $J_c(H,p)$ is non-increasing in p. Furthermore, it is also known (see, e.g., [14, p.403]) that from the arguments in [18] one can obtain the inequality $J_c(H,p) \geq J_c(H)/p$. Applying this inequality to the infinite clusters at level p_2, one obtains, for $0 < p_1 < p_2 \leq 1$, $J_c(H,p_1) \geq J_c(H,p_2)p_2/p_1$, which in particular implies that $J_c(H,p)$ is strictly decreasing in p in the interval $(p_c, 1]$. We will prove, in Section 6,

Theorem 4.4 *Let H be any infinite, connected graph of bounded degree. Then $J_c(H,p)$ is a continuous function of p on the interval $(p_c, 1]$.*

This result is different in spirit from our other results, in that it has little to do with the geometry of H. We do not know whether the bounded degree assumption is essential (with bond percolation instead of site percolation, it is not; see Theorem 6.1).

Regarding the behavior of $h_c(H,p,J)$, one can ask similar questions.

Question 4.5 *Let H be any infinite, locally finite, connected graph. For fixed J, is $h_c(H,p,J)$ a continuous function of p? Is it a (strictly) decreasing function of p on $[p_c(H),1]$? If this is false, then perhaps it is true for graphs of bounded degree, or at least for transitive graphs.*

Note that it is not even clear whether $h_c(H,p,J)$ is decreasing in p. Indeed, in [53] it was shown that it is possible for a subgraph to have an Ising model phase coexistence region larger than the full graph.

5 No persistent transition in the amenable case

In this section we prove part (ii) of Theorem 3.4. There is no loss of generality in assuming that our percolation process is ergodic, i.e., that the σ-algebra of automorphism invariant events is trivial under the law of the percolation process. (This is because stationary processes are mixtures of the ergodic ones. See, e.g., [22, Section 14.1].) So in this section we add the assumption of ergodicity.

Given an arbitrary graph G (infinite, locally finite and connected as above), let $\nu_G^{+,J,h}$ ($\nu_G^{-,J,h}$) be the plus (minus) measure for the Ising model with parameters J and h on G. These are the measures obtained by taking plus (minus) boundary conditions on an increasing sequence of sets and taking the limit. Having phase transition is equivalent to having $\nu_G^{-,J,h} = \nu_G^{+,J,h}$ (see, e.g., [16], [36] or [53] for details).

We need to create an automorphism invariant measure which represents our first performing percolation and then coupling monotonically the plus and minus measures for each of the infinite clusters.

To do this, we first need to construct a *canonical* coupling of $\nu_G^{+,J,h}$ and $\nu_G^{-,J,h}$ for an arbitrary graph $G = (V, E)$ (and for any values for the parameters J and h). We do this by considering a Markov chain which has the two measures above as stationary distributions and by using ideas based on the Propp–Wilson *coupling-from-the-past* algorithm (see [50] and [10]).

Let $\{A_{i,t}, U_{i,t}\}_{i\in V, t\in \mathbf{Z}, t<0}$ be independent random variables with $P(A_{i,t} = 1) = 1 - P(A_{i,t} = 0) = 1/2$ and $U_{i,t}$ uniform on $[0, 1]$ for each i and t . The coupling of $\nu_G^{+,J,h}$ and $\nu_G^{-,J,h}$ that we will obtain will be measurable with respect to the above random variables. For $t < 0$, let f_t be the (random) map from $\{\pm 1\}^V$ to itself given by

$$(f_t(\eta))_i = \begin{cases} \eta_i & \text{if } A_{i,t} = 0 \text{ or } A_{j,t} = 1 \text{ for some } j \sim i \qquad (*) \\ +1 & \text{if } (*) \text{ fails and } U_{i,t} < a(\eta, i) \\ -1 & \text{if } (*) \text{ fails and } U_{i,t} \geq a(\eta, i) \end{cases}$$

where $a(\eta, i)$ is the conditional probability, with respect to any Gibbs measure with parameters J and h, of having a 1 at i given its neighbors agree with η. It is straightforward (using arguments in [10]) to show that

$$\lim_{n\to\infty} (f_{-1} \circ f_{-2} \circ \dots f_{-n}(+), f_{-1} \circ f_{-2} \circ \dots f_{-n}(-))$$

exists *always*, where $+$ $(-)$ denote the configuration of all 1's (all -1's), and that the distribution of this limit is a coupling of $\nu_G^{+,J,h}$ and $\nu_G^{-,J,h}$ with the former coupled stochastically above the latter.

We denote the coupling of $\nu_G^{+,J,h}$ and $\nu_G^{-,J,h}$ that we have just constructed by $\nu_G^{+-,J,h}$.

Remark: The above immediately yields, what we feel is the simplest proof that the plus and minus states for the Ising model at any temperature for an amenable group are Bernoulli shifts (in the sense of ergodic theory), a result originally due to D. Ornstein and B. Weiss for $\mathbf{Z}^d$ and later extended by S. Adams to amenable groups.

Lemma 5.1 *Fix a transitive graph $G = (V, E)$, a stationary ergodic percolation μ, $J \geq 0$, $h \geq 0$ and let $S := \{0, ++, +-, --\}$. Then there exists an automorphism invariant measure $\mu_G^{J,h}$ on S^V such that*

(1) $\{x : \eta(x) \in S\backslash\{0\}\}$ *is equal in distribution to the union of the infinite components of a* μ*–percolation on* G*, and*

(2) *Given that the infinite components of* $\{x : \eta(x) \in S\backslash\{0\}\}$ *are* $\{C_i\}_{i\in I}$*, the distribution of* η *on the* $\{C_i\}$*'s are conditionally independent with the conditional distribution of* η *on* C_i *being such that* $\{x \in C_i : \eta(x) \in \{++, +-\}\}$ *is the same as the distribution of 1's in* $\nu_{C_i}^{+,J,h}$ *and the conditional distribution of* $\{x \in C_i : \eta(x) \in \{++\}\}$ *is the same as the distribution of 1's in* $\nu_{C_i}^{-,J,h}$*.*

Proof: If $\{C_i\}_{i\in I}$ are nonadjacent infinite (induced) subgraphs of G, let $\nu_{\{C_i\}}^{+-,J,h}$ be the measure on S^V which is 0 on $V\backslash\bigcup_{i\in I} C_i$ and is $\nu_{C_i}^{+-,J,h}$ on each C_i, independent for different C_i's. Finally let $\mu_G^{J,h} := \int \nu_{\{C_i\}}^{+-,J,h} dP(\{C_i\})$ where P is the distribution of the union of the infinite components of μ–percolation on G. It is clear that $\mu_G^{J,h}$ is automorphism invariant and satisfies (1) and (2).

□

Definition 5.2 *Given a (possibly disconnected) graph* $G = (V, E)$*, we call a sequence* $\{F_n\}_{n\geq 1}$ *of subsets of* V *a* **Følner sequence** *for* G *if*

$$\lim_{n\to\infty} \frac{|\partial F_n|}{|F_n|} = 0.$$

We write $F_n \nearrow V$ if $F_1 \subset F_2 \subset ...$ and $\bigcup_{n=1}^{\infty} F_n = V$.

Lemma 5.3 *Let* $G = (V, E)$ *be a transitive and amenable graph,* μ *a stationary ergodic percolation on* G *and* $\mathcal{C}$ *be the union of the infinite clusters for* μ*. Suppose that* μ *is such that a.s.* $\mathcal{C}$ *is not empty. If* $\{F_n\}_{n\geq 1}$ *is a Følner sequence for* G*, such that* $F_n \nearrow V$*, then there exists* $n_1, n_2, \ldots \to \infty$ *such that a.s.*

$$\{F_{n_i} \cap \mathcal{C}\}_{i\geq 1}$$

is a Følner sequence for $\mathcal{C}$*, and* $F_n \cap \mathcal{C} \nearrow V \cap \mathcal{C}$*.*

Proof: Let $0 \in V$ be an arbitrary vertex. We have that $P(0 \in \mathcal{C}) > 0$. Theorem 7.5 in the Appendix implies that

$$\lim_{n\to\infty} \frac{|F_n \cap \mathcal{C}|}{|F_n|} = P(0 \in \mathcal{C}),$$

in probability. Hence there exist $n_1, n_2, \ldots \to \infty$ such that

$$\lim_{i\to\infty} \frac{|F_{n_i} \cap \mathcal{C}|}{|F_{n_i}|} = P(0 \in \mathcal{C})$$

a.s.. It is easy to see that

$$\partial^{\mathcal{C}}(F_n \cap \mathcal{C}) \subseteq \partial^G(F_n).$$

Hence

$$\frac{|\partial^{\mathcal{C}}(F_{n_i} \cap \mathcal{C})|}{|F_{n_i} \cap \mathcal{C}|} \le \frac{|\partial^G(F_{n_i})|}{|F_{n_i} \cap \mathcal{C}|} = \frac{|\partial^G(F_{n_i})|}{|F_{n_i}|} \frac{|F_{n_i}|}{|F_{n_i} \cap \mathcal{C}|}.$$

Since the first factor approaches 0 and the second factor remains bounded a.s. the first statement is proved. The second statement that $F_n \cap \mathcal{C} \nearrow V \cap \mathcal{C}$ is trivial. $\square$

Proof of Theorem 3.4 (ii): Let μ denote the distribution of the stationary percolation process and denote by P the measure $\mu_G^{J,h}$ on S^V introduced in Lemma 5.1. We also let $\mathbf{E}$ denote expectation with respect to P. Let $x \in V$ be arbitrary; we are done if we can show that $P(\eta(x) = +-) = 0$. For the sake of deriving a contradiction, we assume that $P(\eta(x) = +-) > 0$.

Let $U_x = I_{\{\eta(x)=+-\}}$. From Proposition 7.1 and Lemma 5.3, there exists a Følner sequence $\{F_n\}$ for G, with the properties that $F_n \nearrow V$ and $\{F_n \cap \mathcal{C}\}_{n\ge 1}$ is a Følner sequence for $\mathcal{C}$ μ a.s. where $\mathcal{C}$ is the union of the infinite clusters for μ. By Theorem 7.5, we have that

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{v\in F_n} U_v = \mathbf{E}[U_0|\mathcal{I}],$$

in probability, where $\mathbf{E}[U_0|\mathcal{I}]$ is the projection of U_0 onto the automorphism invariant functions. Hence we can choose $n_i \to \infty$ such that

$$\lim_{i\to\infty} \frac{1}{|F_{n_i}|} \sum_{v\in F_{n_i}} U_v = \mathbf{E}[U_0|\mathcal{I}]$$

a.s.. Since $U_v = 1$ implies that $v \in \mathcal{C}$ and $\mathbf{E}[U_0|\mathcal{I}]$ is not a.s. 0 (as $P(U_0 = 1) > 0$), we get

$$\liminf_{i\to\infty} \frac{1}{|F_{n_i} \cap \mathcal{C}|} \sum_{v\in F_{n_i}\cap\mathcal{C}} U_v > 0, \tag{5}$$

on a set of positive probability. It follows that there exists an integer N and $\gamma > 0$ such that

$$P(A) \ge \gamma, \tag{6}$$

where

$$A := \left\{ \frac{1}{|F_{n_i} \cap \mathcal{C}|} \sum_{v \in F_{n_i} \cap \mathcal{C}} U_v \geq \gamma \ \ \forall i \geq N \right\}.$$

On the other hand, if we condition on $\mathcal{C}$, then the fact that $\{F_{n_i} \cap \mathcal{C}\}_{i \geq 1}$ is a Følner sequence for $\mathcal{C}$ a.s. allows us to use [36, Proposition 3.2] (applied to $\mathcal{C}$ and the Følner sequence $\{F_{n_i} \cap \mathcal{C}\}_{i \geq 1}$, where one also needs to observe that this proposition holds even if the graph is not connected) to conclude that there exists a random subsequence $\{n_i'\} = \{n_i'(\omega)\}$ of $\{n_i\}$ with

$$\lim_{i \to \infty} \mathbf{E} \left[\frac{1}{|F_{n_i'} \cap \mathcal{C}|} \sum_{v \in F_{n_i'} \cap \mathcal{C}} U_v \middle| \mathcal{C} \right] = 0$$

a.s. and hence, by the Bounded Convergence Theorem, we get

$$\lim_{i \to \infty} \mathbf{E} \left[\frac{1}{|F_{n_i'} \cap \mathcal{C}|} \sum_{v \in F_{n_i'} \cap \mathcal{C}} U_v \right] = 0.$$

Hence there exists $r \geq N$ such that

$$P(B) < \gamma, \tag{7}$$

where

$$B := \left\{ \frac{1}{|F_{n_r'} \cap \mathcal{C}|} \sum_{v \in F_{n_r'} \cap \mathcal{C}} U_v \geq \gamma \right\}.$$

However $r \geq N$ implies that $n_r' \geq n_r \geq n_N$ and hence $A \subseteq B$ contradicting equations (6) and (7). □

6 Continuity of $J_c(H, p)$

In this section, we prove Theorem 4.4. We will first prove an analogous result (without the bounded degree assumption) with bond percolation instead of site percolation; this is Theorem 6.1 below. For $p \in [0, 1]$ and $H = (V, E)$, write $\mathbf{P}^H_{p,bond}$ for the probability measure on $\{0,1\}^E$ where each edge independently takes value 1 (occupied) or 0 (vacant) with respective probabilities p and $1-p$. Let $p_{c,bond} = p_{c,bond}(H)$ be the critical value for bond percolation on H. Define $\mathbf{Coex}(H,p,bond)$ as $\mathbf{Coex}(H,p)$ with $\mathbf{P}^H_{p,bond}$ in place of $\mathbf{P}^H_p$. Also set $J_c(H,p,bond) = \inf \mathbf{Coex}_{(0,\cdot)}(H,p,bond)$. As $J_c(H,p)$, the function $J_c(H,p,bond)$ is non-increasing in p. We shall prove

Theorem 6.1 *Let H be any infinite, locally finite connected graph. Then $J_c(H,p,bond)$ is a continuous function of p on the interval $(p_{c,bond},1]$.*

An important tool in analyzing the Ising model with zero external field is the (Fortuin–Kasteleyn) random-cluster model; see e.g. [28] for an introduction to this technique.

Definition 6.2 *The* **random-cluster measure** $\phi^{r,q}_G$ *with parameters $r \in [0,1]$ and $q > 0$ for a finite graph $G = (V,E)$, is the probability measure on $\{0,1\}^E$ which to each $\eta \in \{0,1\}^E$ assigns probability*

$$\phi^{r,q}_G(\eta) = \frac{r^{n_1(\eta)}(1-r)^{n_0(\eta)}q^{k(\eta)}}{Z^{r,q}_G}$$

where $n_0(\eta)$ (resp. $n_1(\eta)$) is the number of edges taking value 0 (resp. 1) in η, $k(\eta)$ is the number of connected components in η, and $Z^{r,q}_G$ is a normalizing constant.

The key to drawing conclusions about the Ising model from the random-cluster model is the following well-known result. (Note that the Ising model on a finite graph of course has a unique Gibbs measure.)

Proposition 6.3 *Let $G = (V,E)$ be a finite graph, fix $r \in (0,1)$ and consider the following way of picking a random spin configuration $X \in \{-1,1\}^V$. First pick a random edge configuration $Y \in \{0,1\}^E$ according to the random-cluster measure $\phi^{r,2}_G$. Then toss an independent fair coin for each connected component C in Y to decide whether all vertices $v \in C$ should take value $X(v) = -1$, or whether they all should take value $+1$. Then X is distributed according to the Gibbs measure for the Ising model on G with $J = -\frac{1}{2}\log(1-r)$ and $h = 0$.*

Next, we consider another model living on the edge set of G, which we can view as an edge-diluted random-cluster model. Fix $p, r \in [0,1]$, and let $\psi_G^{r,2,p}$ be the probability measure on $\{-1,0,1\}^E$ corresponding to first assigning value -1 independently to each edge with probability $1-p$, and then assigning values 0 and 1 to the remaining edges according to the random-cluster measure with parameters r and $q=2$ on the subgraph of G obtained by deleting all edges with value -1.

Also let $\tilde{\psi}_G^{r,2,p}$ be the probability measure on $\{0,1\}^E$ corresponding to first picking $Y \in \{-1,0,1\}^E$ according to $\psi_G^{r,2,p}$, and then obtaining $\tilde{Y} \in \{0,1\}^E$ by letting $\tilde{Y}(e) = \max\{0, Y(e)\}$ for each $e \in E$ (i.e. by changing all -1's to 0's in Y).

Suppose now that we pick $\tilde{Y} \in \{0,1\}^E$ according to $\tilde{\psi}_G^{r,2,p}$, and then pick $X \in \{0,1\}^V$ by assigning independent random spins (-1 or 1 with probability $\frac{1}{2}$ each) to the connected components of $\tilde{Y}$. It follows from the construction and Proposition 6.3 that X has the same distribution as the Ising model with parameters $J = -\frac{1}{2}\log(1-r)$ and $h=0$ on the percolation clusters of G under $\mathbf{P}_{p,bond}^G$. (This fact will be implicitly used in (14) below.)

Next, we need to discuss stochastic domination in some detail. For two configurations $\eta, \eta' \in \{0,1\}^E$, we write $\eta \preceq \eta'$ if $\eta(e) \leq \eta'(e)$ for all $e \in E$. For two probability measures P and P' on $\{0,1\}^E$, we say that P is stochastically dominated by P', writing $P \overset{\mathcal{D}}{\preceq} P'$, if there exists a pair (Y, Y') of $\{0,1\}^E$-valued random variables such that (i) Y has distribution P, (ii) Y' has distribution P', and (iii) $Y \preceq Y'$ with probability 1.

The following stochastic domination result is a major step towards proving Theorem 6.1.

Proposition 6.4 *Let $G = (V,E)$ be a finite graph, and fix $r, r' \in (0,1)$ such that $r < r'$. We then have*

$$\phi_G^{r,2} \overset{\mathcal{D}}{\preceq} \tilde{\psi}^{r',2,p}$$

whenever

$$p \geq \frac{r}{r'}. \tag{8}$$

The proof of this result is based on Holley's Lemma [32], a close variant of which we state (and will use) next; see also [28] for a formulation (and a proof) which includes the variant stated here.

Lemma 6.5 ([32]) *Let P and P' be two probability measures on $\{0,1\}^E$, where E is any finite set, and assume that P and P' both assign positive probability to every element of $\{0,1\}^E$. Let Y and Y' be $\{0,1\}^E$-valued random objects with distributions P and P'. Suppose that for every $e \in E$ and all $\eta, \eta' \in \{0,1\}^{E\setminus\{e\}}$ such that $\eta \preceq \eta'$ we have*

$$P(Y(e)=1 \,|\, Y(E\setminus\{e\})=\eta) \leq P'(Y'(e)=1 \,|\, Y'(E\setminus\{e\})=\eta').$$

Then $P \overset{\mathcal{D}}{\preceq} P'$.

Proof of Proposition 6.4: Fix G, r, r' and p as in the proposition. Fix an edge $e \in E$ connecting two vertices $x, y \in V$. For $\eta \in \{0,1\}^{E\setminus\{e\}}$, write $x \overset{\eta}{\leftrightarrow} y$ (resp. $x \overset{\eta}{\not\leftrightarrow} y$) to indicate that there exists (resp. does not exist) a path of open edges in η from x to y. It is immediate from Definition 6.2 that

$$\phi_G^{r,2}(Y(e)=1 \,|\, Y(E\setminus\{e\})=\eta) = \begin{cases} r & \text{if } x \overset{\eta}{\leftrightarrow} y \\ \frac{r}{2-r} & \text{if } x \overset{\eta}{\not\leftrightarrow} y. \end{cases} \tag{9}$$

Since the event $x \overset{\eta}{\leftrightarrow} y$ is increasing in η, we are done (using Lemma 6.5) if we can show that

$$\tilde{\psi}_G^{r',2,p}(\tilde{Y}(e)=1 \,|\, \tilde{Y}(E\setminus\{e\})=\eta) \geq r \quad \text{on the event } x \overset{\eta}{\leftrightarrow} y \tag{10}$$

and

$$\tilde{\psi}_G^{r',2,p}(\tilde{Y}(e)=1 \,|\, \tilde{Y}(E\setminus\{e\})=\eta) \geq \frac{r}{2-r} \quad \text{on the event } x \overset{\eta}{\not\leftrightarrow} y. \tag{11}$$

To prove (10), it suffices to show that

$$\psi^{r',2,p}(Y(e)=1 \,|\, Y(E\setminus\{e\})=\eta) \geq r$$

for all $\eta \in \{-1,0,1\}^{E\setminus\{e\}}$ such that $x \overset{\eta}{\leftrightarrow} y$ (where an edge e' counts as open in η if and only if $\eta(e')=1$). Fix such an η. From the construction of $\psi^{r',2,p}$, it is immediate that

$$\frac{\psi^{r',2,p}(Y(e)=1 \,|\, Y(E\setminus\{e\})=\eta)}{\psi^{r',2,p}(Y(e)=0 \,|\, Y(E\setminus\{e\})=\eta)} = \frac{r'}{1-r'}\,. \tag{12}$$

Next, let $\tilde{\tilde{Y}}$ denote the $\{-1,0\}^E$-valued random element obtained by picking $Y \in \{-1,0,1\}^E$ according to $\psi^{r',2,p}$ and then letting $\tilde{\tilde{Y}}(e') = \min(Y(e'),0)$ for each $e' \in E$ (so that $\tilde{\tilde{Y}}$ indicates only which edges are removed in the dilution step). We obtain $\tilde{\tilde{\eta}}$ from η similarly. Note that for any finite graph H, the random-cluster measure with given parameter values on H conditioned on a given edge e' being absent, is the same as the (unconditioned) random-cluster measure with the same parameter values on the graph obtained from H by removing e'. Hence,

$$\frac{\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, Y(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})}{\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = -1, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})} = 1.$$

Note also that

$$\psi^{r',2,p}(Y(e) = 0 \,|\, \tilde{\tilde{Y}}(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}}) \geq 1 - r'$$

using (9). Combining these observations, we get

$$\begin{aligned}
&\frac{\psi^{r',2,p}(Y(e) = 0, Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})}{\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = -1, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})} \\
&= \frac{\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, Y(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})\psi^{r',2,p}(Y(e) = 0 \,|\, \tilde{\tilde{Y}}(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})}{\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = -1, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})} \\
&\geq 1 - r'.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\psi^{r',2,p}(Y(e) = 0 \,|\, Y(E\setminus\{e\}) = \eta)}{\psi^{r',2,p}(Y(e) = -1 \,|\, Y(E\setminus\{e\}) = \eta)} &= \frac{\psi^{r',2,p}(Y(e) = 0, Y(E\setminus\{e\}) = \eta))}{\psi^{r',2,p}(Y(e) = -1, Y(E\setminus\{e\}) = \eta))} \\
&= \frac{\psi^{r',2,p}(\tilde{\tilde{Y}}(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})\psi^{r',2,p}(Y(e) = 0, Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = 0, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})}{\psi^{r',2,p}(\tilde{\tilde{Y}}(e) = -1, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})\psi^{r',2,p}(Y(E\setminus\{e\}) = \eta \,|\, \tilde{\tilde{Y}}(e) = -1, \tilde{\tilde{Y}}(E\setminus\{e\}) = \tilde{\tilde{\eta}})} \\
&\geq \frac{p(1-r')}{1-p}.
\end{aligned}$$

Combining this with (12), we get

$$\psi^{r',2,p}(Y(e) = 1 \,|\, Y(E\setminus\{e\}) = \eta) \geq \frac{1}{1 + \frac{1-r'}{r'}(1 + \frac{1-p}{p(1-r')})} = pr' \geq r,$$

where we used (8). So (10) is established. A similar calculation shows that (11) holds whenever (8) holds, so that the proposition follows. $\square$

Now let $H = (V, E)$ be an infinite locally finite connected graph as in Theorem 6.1. Let $F_1, F_2, \dots$ be a sequence of finite subsets of V such that $F_1 \subset F_2 \subset \cdots$ and $\bigcup_{i=1}^{\infty} F_i = V$. For each i, let Λ_i be the (finite) graph whose vertex set V_{Λ_i} is $F_i \cup \{\Delta\}$, where Δ is an auxiliary vertex, and whose edge set E_{Λ_i} consists of

(i) the edges in E that connect two vertices in F_i, and

(ii) for each edge in E that connects some $x \in F_i$ to some $y \in V \setminus F_i$, we include an edge $\langle x, \Delta \rangle$ in E_{Λ_i}.

Fix $x \in F_1$. It is well known (see e.g. [28]) and easy to show using Proposition 6.3, that the Ising model on H with coupling constant J and zero external field, has phase transition if and only if

$$\lim_{i \to \infty} \phi_{\Lambda_i}^{r,2}(x \leftrightarrow \Delta) > 0 \tag{13}$$

where $r = 1 - e^{-2J}$. By similar arguments, it is easy to see that the Ising model with coupling constant J' and zero external field on the union of the clusters of H obtained with Bernoulli percolation with parameter p, has phase transition if and only if

$$\lim_{i \to \infty} \tilde{\psi}_{\Lambda_i}^{r',2,p}(x \leftrightarrow \Delta) > 0 \tag{14}$$

where $r' = 1 - e^{-2J'}$. (That the limits in (13) and (14) exist follows from standard stochastic monotonicity arguments based on Holley's Lemma.)

Suppose now that

$$J' \geq -\frac{1}{2} \log \left(\frac{e^{-2J} + p - 1}{p} \right)$$

which is equivalent to $p \geq \frac{r}{r'}$. By Proposition 6.4, we then have that

$$\phi_{\Lambda_i}^{r,2}(x \leftrightarrow \Delta) \leq \tilde{\psi}_{\Lambda_i}^{r',2,p}(x \leftrightarrow \Delta).$$

By letting $i \to \infty$ and using (13) and (14), we arrive at the conclusion that if the $(0, J)$-Ising model on H has phase transition, then so does the $(0, J')$-Ising model on the union of the clusters of H obtained from Bernoulli (p) bond percolation. This immediately implies the following result.

Lemma 6.6 *Let $H = (V, E)$ be an infinite, locally finite connected graph satisfying $J_c(H) < \infty$. For any $p > 1 - e^{-2J_c(H)}$, we have*

$$J_c(H, p, bond) \leq -\frac{1}{2} \log \left(\frac{e^{-2J_c(H)} + p - 1}{p} \right). \tag{15}$$

In particular,

$$\lim_{p \nearrow 1} J_c(H, p, bond) = J_c(H).$$

Proof of Theorem 6.1: Continuity of $J_c(H, p, bond)$ at $p = 1$ was established in Lemma 6.6; what remains is to prove continuity for $p \in (p_c, 1)$. Fix such a p. Let $\varepsilon > 0$ be arbitrary and let $J = J_c(H, p, bond) + \varepsilon$. Now fix an arbitrary $x \in V$ and define the event

$$E_x := \{J_c(C(x)) \leq J\}.$$

Then, by (3), we have $\mathbf{P}^H_{p,bond}(E_x) > 0$. Note that for $p' < p$, Bernoulli (p') bond percolation on H can be obtained by first performing Bernoulli (p) percolation on H, and then doing Bernoulli $(\frac{p'}{p})$ percolation on the clusters of the first percolation. If $\rho > 1 - e^{-2J}$, then $\rho > 1 - e^{-2J_c(C(x))}$ for $\omega \in E_x$ and hence by (15), we have that for $\omega \in E_x$,

$$J_c(C(x), \rho, bond) \leq -\frac{1}{2} \log \left(\frac{e^{-2J_c(C(x))} + \rho - 1}{\rho} \right) \leq -\frac{1}{2} \log \left(\frac{e^{-2J} + \rho - 1}{\rho} \right).$$

Hence by (3), for $\omega \in E_x$,

$$\mathbf{P}^{C(x)}_{\rho,bond} \left(J_c(\tilde{C}(x)) \leq -\frac{1}{2} \log \left(\frac{e^{-2J} + \rho - 1}{\rho} \right) + \varepsilon \right) > 0$$

where $\tilde{C}(x)$ is the cluster of x when performing ρ percolation on $C(x)$ and so

$$\mathbf{P}^{H}_{p\rho,bond} \left(J_c(C(x)) \leq -\frac{1}{2} \log \left(\frac{e^{-2J} + \rho - 1}{\rho} \right) + \varepsilon \right) > 0.$$

This gives that

$$J_c(H, p\rho, bond) \leq -\frac{1}{2} \log \left(\frac{e^{-2J} + \rho - 1}{\rho} \right) + \varepsilon,$$

which establishes the left continuity, as $\varepsilon > 0$ is arbitrary.

As for the right continuity, assume for contradiction that for some $p > p_c$, we have that

$$\delta := J_c(H, p, bond) - \lim_{\varepsilon \searrow 0} J_c(H, p+\varepsilon, bond) > 0.$$

For any $\varepsilon > 0$, we then have by (3) that

$$\mathbf{P}^H_{p+\varepsilon, bond}\left(J_c(C(x)) \le J_c(H, p, bond) - \frac{3\delta}{4}\right) > 0.$$

Choose $\varepsilon > 0$ such that

$$\frac{p}{p+\varepsilon} > 1 - e^{-2[J_c(H,p,bond) - \frac{3\delta}{4}]}$$

and

$$-\frac{1}{2}\log\left(\frac{e^{-2[J_c(H,p,bond)-\frac{3\delta}{4}]} + \frac{p}{p+\varepsilon} - 1}{\frac{p}{p+\varepsilon}}\right) + \varepsilon < J_c(H, p, bond) - \frac{\delta}{2}.$$

Let E be the event that under $(p+\varepsilon)$ percolation, $J_c(C(x)) \le J_c(H, p, bond) - 3\delta/4$. On E (which has positive probability), we have that

$$\frac{p}{p+\varepsilon} > 1 - e^{-2J_c(C(x))}.$$

Hence, on the event E, (15) (together with (3)) implies that when we perform Bernoulli $(\frac{p}{p+\varepsilon})$ percolation on $C(x)$, the (new, smaller) cluster of x has a J_c value smaller than

$$-\frac{1}{2}\log\left(\frac{e^{-2J_c(C(x))} + \frac{p}{p+\varepsilon} - 1}{\frac{p}{p+\varepsilon}}\right) + \varepsilon$$

with positive probability. On E, the latter is at most

$$-\frac{1}{2}\log\left(\frac{e^{-2[J_c(H,p,bond)-\frac{3\delta}{4}]} + \frac{p}{p+\varepsilon} - 1}{\frac{p}{p+\varepsilon}}\right) + \varepsilon$$

which in turn is less than $J_c(H, p, bond) - \delta/2$. Hence, under $\mathbf{P}^H_{p,bond}$, $J_c(C(x))$ is less than $J_c(H, p, bond) - \delta/2$ with positive probability, contradicting the definition of $J_c(H, p, bond)$. □

We finally want to bring Lemma 6.6 and Theorem 6.1 over to a site percolation setting. To this end, we want to be able to stochastically compare site percolation on H to bond percolation on H. It is not immediately clear what such a stochastic domination should mean, since the measures $\mathbf{P}_p^H$ and $\mathbf{P}_{p,bond}^H$ are defined on different spaces ($\{0,1\}^V$ and $\{0,1\}^E$, respectively). However, a site percolation process is naturally identified with a (usually dependent) bond percolation process in which a bond $e = \langle x, y\rangle$ is considered to be open if and only if the two vertices x and y are open. We denote by $\tilde{\mathbf{P}}_{p'}^H$ the probability measure for a bond model which arises from an i.i.d. site model with parameter p' in this way. Note that the components in a site model are always induced subgraphs while for bond models, this is not true. However, a bond model constructed from a site model as above has all of its components being induced subgraphs and moreover, the infinite components in this bond model are precisely the same as the infinite components in the original site model. With this identification in mind, we have the following stochastic domination.

Lemma 6.7 *Let $H = (V, E)$ be a bounded degree graph in which each vertex has at most D neighbors. If p and p' satisfy*

$$p' = 1 - (1 - \sqrt{p})^D, \tag{16}$$

then

$$\mathbf{P}_{p,bond}^H \overset{\mathcal{D}}{\preceq} \tilde{\mathbf{P}}_{p'}^H. \tag{17}$$

Proof: See [47, Remark 6.2], which was motivated by the results in [41]. □

Lemma 6.8 *Let $H = (V, E)$ be an infinite, locally finite graph with degrees bounded by D and $J_c(H) < \infty$. For any p' satisfying*

$$\left(1 - (1 - p')^{\frac{1}{D}}\right)^2 > 1 - e^{-2J_c(H)},$$

we have

$$J_c(H, p') \leq -\frac{1}{2}\log\left(\frac{e^{-2J_c(H)} + \left(1 - (1 - p')^{\frac{1}{D}}\right)^2 - 1}{\left(1 - (1 - p')^{\frac{1}{D}}\right)^2}\right). \tag{18}$$

In particular,

$$\lim_{p' \nearrow 1} J_c(H, p') = J_c(H).$$

Proof: Choose p so that (16) holds. A simple computation shows that $p = \left(1-(1-p')^{\frac{1}{D}}\right)^2$ and so Lemma 6.6 gives

$$J_c(H, p, bond) \leq -\frac{1}{2}\log\left(\frac{e^{-2J_c(H)}+p-1}{p}\right)$$

which equals

$$-\frac{1}{2}\log\left(\frac{e^{-2J_c(H)}+\left(1-(1-p')^{\frac{1}{D}}\right)^2-1}{\left(1-(1-p')^{\frac{1}{D}}\right)^2}\right).$$

By Lemma 6.7 (and the comments above it), we have that $J_c(H, p') \leq J_c(H, p, bond)$, completing the proof. □

Proof of Theorem 4.4: Follows by applying Lemma 6.8 in the same way that Lemma 6.6 was used to prove Theorem 6.1. □

7 Appendix: Remarks on the definition of amenability and strong amenability, and a mean ergodic theorem

It is clear that according to Definition 2.2 an infinite, locally finite, connected graph $G = (V, E)$ is amenable iff there exists a Følner sequence for it, i.e., a sequence $(W_n)_{n\geq 1}$ of finite subsets of V such that

$$\lim_{n\to\infty}\frac{|\partial W_n|}{|W_n|} = 0. \tag{19}$$

Note that clearly we must have $\lim_{n\to\infty}|W_n| = \infty$, for (19) to hold. The propositions below address the issue of whether additional conditions can be imposed on the sequence $(W_n)_{n\geq 1}$. As before, we write $W_n \nearrow V$ if $W_1 \subset W_2 \subset \ldots$ and $\cup_{n=1}^{\infty} W_n = V$.

Proposition 7.1 *Suppose that $G = (V, E)$ is an infinite, locally finite, connected graph. Then G is amenable iff there exists a sequence $(W_n)_{n\geq 1}$ of finite subsets of V such that $W_n \nearrow V$ and* (19) *holds.*

Proof: We know that there exists a sequence of finite subsets of V, $(W'_n)_{n\geq 1}$, such that

$$\lim_{n\to\infty} \frac{|\partial W'_n|}{|W'_n|} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} |W'_n| = \infty. \tag{20}$$

Fix an arbitrarily $0 \in V$ and let $B(r)$ be the set of vertices within distance r of 0. Given a sequence of integers $\{n_k\}$ going to infinity, define $\{W_k\}$ by

$$W_k := W_{k-1} \cup B(k) \cup W'_{n_k}.$$

For any sequence $\{n_k\}$, it is clear that $W_k \nearrow V$. In addition, it is easy to see that if the n_k's are chosen sufficiently sparsely, then (W_k) will be a Følner sequence. □

Proposition 7.2 *Suppose that $G = (V, E)$ is an infinite, connected graph of bounded degree. Then G is amenable iff there exists a sequence $(W_n)_{n\geq 1}$ of connected finite subsets of V such that (19) holds.*

Proof: We know that there exists a sequence $(W'_n)_{n\geq 1}$ such that (20) holds. Write $W'_n = \cup_{i\in I_n} W'_{n,i}$, where $W'_{n,i}$, $i \in I_n$ are the connected components of W'_n.

We claim that there exists a sequence $(i_n)_{n\geq 1}$, $i_n \in I_n$, such that $W_n = W'_{n,i_n}$ satisfies (19). To justify this claim, suppose otherwise. Then there exists $\epsilon > 0$ and a strictly increasing sequence $(n_j)_{j\geq 1}$, $\lim_{j\to\infty} n_j = \infty$, such that for all j and all $i \in I_{n_j}$, we would have $|\partial W'_{n_j,i}| \geq \epsilon |W'_{n_j,i}|$.

Let D be the maximal degree of G. Then

$$|\partial W'_{n_j}| \geq \frac{1}{D} \sum_{i\in I_{n_j}} |\partial W'_{n_j,i}| \geq \frac{\epsilon}{D} \sum_{i\in I_{n_j}} |W'_{n_j,i}| = \frac{\epsilon}{D} |W'_{n_j}|,$$

which contradicts (20). □

The conditions in the two propositions above cannot be combined, since we know that strong amenability is distinct from amenability. However, Proposition 7.1 has the following analogue for strong amenability.

Proposition 7.3 *Suppose that $G = (V, E)$ is an infinite, locally finite, connected graph. Then G is strongly amenable iff there exists a sequence $(W_n)_{n \geq 1}$ of finite connected subsets of V such that $W_n \nearrow V$ and (19) holds.*

Also regarding the notion of strong amenability, it is worth mentioning that instead of using the anchored Cheeger constant, it can be defined in terms of the constant

$$\kappa'(G) := \inf \left\{ \frac{|\partial W|}{|W|} : 0 \in W \subseteq V, W \text{ connected}, \quad |W| < \infty \right\}.$$

Clearly $\kappa(G) \leq \kappa'(G) \leq \kappa^*(G)$, with strict inequalities being possible. Moreover, the value of $\kappa'(G)$ depends on the choice of 0, while that of $\kappa^*(G)$ does not. For these reasons, $\kappa^*(G)$ is generally a better constant than $\kappa'(G)$ when one is interested in estimating critical points and other numerical features of the graph. But note that $\kappa^*(G) = 0$ iff $\kappa'(G) = 0$, so that $\kappa'(G) = 0$ is an equivalent definition of strong amenability. This observation provides the following corollary to Proposition 7.2.

Proposition 7.4 *Suppose that G is an infinite, locally finite, transitive graph. Then G is amenable iff it is strongly amenable.*

The following is a mean ergodic theorem for stationary processes on amenable transitive graphs. It might be well known but we include it here, since we could not find it in the literature.

Theorem 7.5 *Let $G = (V, E)$ be an infinite locally finite transitive amenable graph, and 0 be a fixed element of V. Let $(S, \mathcal{S})$ be a measurable space, and μ be an automorphism invariant probability measure on $(S^V, \mathcal{S}^V)$ such that $\int (\eta_0)^2 d\mu(\eta) < \infty$. If $\{F_n\}_{n \geq 1}$ is a Følner sequence for G, such that $F_n \nearrow V$, then*

$$\lim_{n \to \infty} \frac{1}{|F_n|} \sum_{v \in F_n} \eta_v = E[\eta_0 | \mathcal{I}]$$

in $L^2(S^V, \mathcal{S}^V, \mu)$, where $\mathcal{I}$ is the σ–algebra of invariant events in $\mathcal{S}^V$.

Proof: Let Γ be the set of automorphisms of G, with the topology of pointwise convergence. It is well known (see, e.g., [57]) that Γ is a locally compact group with a subbasis for its topology given by the countable collection of open-compact sets $\Gamma_{u,v} = \{\gamma \in G : \gamma(u) = v\}$, $u, v \in V$.

In [55] (see also [4]), it is shown that G being amenable and transitive implies that Γ is unimodular, meaning that a left Haar measure, $m(\cdot)$, is also a right Haar measure. Choose a representative, g_v, from each $\Gamma_{0,v}$. Since $\Gamma_{u,v} = g_v \Gamma_{0,0} g_u^{-1}$, it follows that

$$m(\Gamma_{u,v}) \text{ does not depend on } u, v \in V. \tag{21}$$

Set $\Gamma_v = \Gamma_{0,v}$ and

$$\tilde{F}_n := \{\gamma \in \Gamma : \gamma^{-1}(0) \in F_n\} = \cup_{v \in F_n} \Gamma_v^{-1}.$$

Then each $\tilde{F}_n$ is a compact set and $\tilde{F}_n \nearrow \Gamma$. Moreover, using (21) and the fact that the F_n's are a Følner sequence for G, one can show that $\{\tilde{F}_n\}_{n \geq 1}$ is a Følner sequence for Γ in the sense that

$$\lim_{n \to \infty} \frac{m(K\tilde{F}_n \triangle \tilde{F}_n)}{m(\tilde{F}_n)} = 0,$$

for any compact set K in Γ. A proof of this claim is contained in the proof of the "if" part of [4, Theorem 3.9] (in this proof the authors assume that the set K contains the identity; to see that no generality is lost see [23, Lemma 4.2]).

We can now apply [23, Corollary 3.4] to get

$$\lim_{n \to \infty} \frac{1}{m(\tilde{F}_n)} \int_{\tilde{F}_n} \eta_{\gamma^{-1}(0)} dm(\gamma) = E[\eta_0 | \mathcal{I}],$$

in L^2. ([23, Corollary 3.4] does not identify the limit in this fashion, but by applying it to the random variables $\eta_0 \, \mathbf{1}_A$, with $A \in \mathcal{I}$, and taking expectations, one can see that the limit fits the definition of $E[\eta_0 | \mathcal{I}]$.)

By decomposing $\tilde{F}_n$ into a union of sets $\Gamma_v^{-1} = \Gamma_{v,0}$ and using (21), one can easily show that for any n,

$$\frac{1}{m(\tilde{F}_n)} \int_{\tilde{F}_n} \eta_{\gamma^{-1}(0)} dm(\gamma) = \frac{1}{|F_n|} \sum_{v \in F_n} \eta_v,$$

giving us the result. □

The mean ergodic theorem above can be extended to quasi-transitive graphs, by combining the arguments in its proof with [4, Lemma 3.10 and Proposition 3.6]. The result can be stated as follows, where we are using notation introduced in the proof above.

Theorem 7.6 *Let $G = (V, E)$ be an infinite locally finite quasi-transitive amenable graph, and $\{0_1, ..., 0_L\}$ be a complete set of representatives in V of the orbits of the automorphism group of G. Let $(S, \mathcal{S})$ be a measurable space, and μ be an automorphism invariant probability measure on $(S^V, \mathcal{S}^V)$ such that $\int (\eta_{0_i})^2 d\mu(\eta) < \infty$, $i = 1, ..., L$. If $\{F_n\}_{n\geq 1}$ is a Følner sequence for G, such that $F_n \nearrow V$, then*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{v\in F_n} \eta_v = \sum_{i=1}^{L} \frac{E[\eta_{0_i}|\mathcal{I}]}{m(\Gamma_{0_i})},$$

in $L^2(S^V, \mathcal{S}^V, \mu)$, where $\mathcal{I}$ is the σ–algebra of invariant events in $\mathcal{S}^V$, and $m(\cdot)$ is normalized in such a way that $\sum_{i=1,...,L}(m(\Gamma_{0_i}))^{-1} = 1$.

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Olle Häggström
Mathematical Statistics, Chalmers University of Technology, Gothenburg.
`olleh@math.chalmers.se`
`http://www.math.chalmers.se/~olleh/`

Roberto H. Schonmann
Department of Mathematics, University of California, Los Angeles.

rhs@math.ucla.edu
http://www.math.ucla.edu/~rhs/

Jeffrey E. Steif
Department of Mathematics, Chalmers University of Technology, Gothenburg.
steif@math.chalmers.se
http://www.math.chalmers.se/~steif/