THE VOTER MODEL WITH ANTI–VOTER BONDS

NINA GANTERT, MATTHIAS LÖWE, AND JEFFREY E. STEIF

Abstract. We study the voter model with both positive and negative bonds on a general locally finite connected infinite graph. We obtain various results concerning ergodicity of the process.

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1. Introduction

The *voter model* is one of the standard models in the subject of interacting particle systems. See the books [2] and [5] which give accounts of this subject and in particular study the voter model.

We first describe the voter model on a general locally finite, connected graph $G = (V, E)$. (It is always assumed that our graphs have no self–loops or double edges.) This will be a continuous time Markov process $\{\eta_t\}_{t\geq 0}$ on the state space $X := \{0,1\}^V$. (Of course an initial state or initial distribution must be specified.) A typical configuration in X will be denoted by $\eta := \{\eta(x)\}_{x\in V}$ where $\eta(x) \in \{0,1\}$ for each x. V represents the various agents and a 0 or 1 represent two possible *opinions*; $\eta(x)$ is the opinion of agent x. In the standard voter model, each agent x waits an exponential time with parameter 1, chooses a neighbor at random and at that time, the state of x becomes that of the chosen neighbor. This is only an informal (infinitesimal) description of the process. It can be made rigorous using operator theory (see [5]) or, more simply, via the so-called *graphical representation* which uses an infinite family of independent Poisson processes (see [2]). We say that a configuration is a *fixed state* if it is an absorbing state for the process, or, in other words, if the Dirac measure on this configuration is a stationary distribution. It is obvious that the all 0 and all 1 configurations, to be denoted by $\overline{0}$ and $\overline{1}$ from now on, are the only fixed states, giving us more than one stationary distribution (the point masses at these two configurations) for the underlying voter system. It is known (see [5]) that for the d–dimensional integer lattice, for $d = 1, 2$, the only stationary distributions are

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these (plus their convex combinations) while for $d \geq 3$, there are other, so-called nontrivial, stationary distributions. These results are intimately connected to the fact that simple random walk on $\mathbb{Z}^d$ is transient if and only if $d \geq 3$. More generally, these arguments give that for the voter model on a general graph, there are nontrivial stationary distributions (i.e., ones which are not convex combinations of $\delta_{\overline{0}}$ and $\delta_{\overline{1}}$) if and only if the probability that two independent random walkers starting at distinct locations on the graph never meet is strictly positive. (By a random walk on the graph, we mean the Markov chain on V which waits an exponential amount of time with parameter 1 and then moves to a neighbor chosen at random). We will describe the relationship between the voter model and random walk in Section 2.

We point out that there are recurrent graphs (meaning graphs for which simple random walk on them is recurrent) for which the probability that two independent random walkers never meet is strictly positive; an example of such a graph is the "fishbone graph" which is the 2-dimensional lattice with all horizontal edges not sitting on the x-axis removed (see [3]). See also [4] for an earlier example of an irreducible symmetric recurrent Markov chain for which two independent copies don't meet with positive probability. On the other hand, for transient graphs of bounded degree, it is always the case that the probability that two independent random walkers never meet is strictly positive. (In fact, for graphs of bounded degree, random walk is transient if and only if the expected number of times that two independent random walkers meet is finite.)

We now give a variant of the voter model for which we investigate the question of ergodicity of the process. Recall that a Markov process, which has a unique stationary distribution such that for each initial state the distribution at time t converges weakly to this stationary distribution is called *ergodic*.

Consider a graph $G = (V, E)$ where some of the edges of E are (deterministically) declared "positive" and the others are declared "negative". We will call this a *signed graph*. The "voter model on the signed-graph G" will also be a continuous time Markov process on the state space $X := \{0, 1\}^V$. It is defined exactly as the voter model with one important difference. When a vertex x is updating, if it chooses the neighbor y, then the state of x becomes the state of y if the edge between them is positive but becomes the opposite state of y if this edge is negative. If all of the edges are declared to be negative, then this Markov process is called the *anti-voter model*. The anti-voter model on finite graphs was investigated by Donnelly and Welsh in [1]. The same model on infinite graphs was studied by Matloff in [6] and [7]. Here the random walkers are taken to be a more general Markov chain. The case where the edges of $\mathbb{Z}^d$ are chosen to be positive or negative in a random i.i.d. manner was studied by Saada in [8].

As soon as there is at least one negative edge, it is clear that $\overline{0}$ and $\overline{1}$ are no longer fixed states. However, there still may be other fixed states. For the following definition, recall that a cycle in a graph is called *simple* if it does not contain any vertex twice, except the first and last.

Definition 1.1. *A simple cycle in a signed graph is unsatisfied if the number of negative edges in it is odd.*

The following first proposition is very easy but characterizes those signed graphs for which the voter model has a fixed state.

Proposition 1.1. *The voter model on the signed graph G has a fixed state if and only if there are no unsatisfied cycles. In this case, there are precisely two fixed states. (This certainly implies non-uniqueness of the stationary distribution.)*

We now restrict ourselves in the future to signed graphs which contain at least one unsatisfied cycle. We will see later that this in itself is certainly not sufficient to insure uniqueness of the stationary distribution.

Our first theorem studies the question of ergodicity of the voter model on a signed, *recurrent* graph. By a simple random walk on a signed graph, we mean a simple random walk on the graph where we ignore the state of the edges.

Theorem 1.1. *Consider a signed graph G which contains at least one unsatisfied cycle and for which simple random walk on G is recurrent. Then the corresponding voter model on the signed graph G is ergodic.*

As we will see, the proof of this result becomes simpler if we assume the stronger fact that two independent random walkers on G meet with probability 1.

We now are left with the case of a transient signed graph with at least one unsatisfied cycle. We are not able to give a complete characterization for when there is uniqueness of the stationary distribution (as we were able to do in the recurrent case) but we nonetheless have some results. The first is an easy sufficient condition for non-uniqueness of the stationary distribution.

Proposition 1.2. *Assume that there exists a subset W of V such that the induced subgraph on W contains no unsatisfied cycles and that there exists $x \in W$ such that the event that random walk starting from x stays in W forever has positive probability. Then there is more than one stationary distribution for the corresponding voter model.*

Proposition 1.2 has the following immediate corollary.

Corollary 1.1. *If G is a transient signed graph with a finite number of negative edges, then there is more than one stationary distribution for the corresponding voter model.*

Definition 1.2. *If $G = (V, E)$ is a graph, a subset Z of V is said to be unavoidable if simple random walk starting from* any *point hits Z eventually with probability one.*

Note that in this case, we necessarily hit Z infinitely often.

Theorem 1.2. *Let $G = (V, E)$ be a signed graph of bounded degree. Assume that for some k, the set of vertices which are part of some unsatisfied cycle of length at most k is unavoidable. Then the corresponding voter model is ergodic.*

We will show how from this one obtains the following result, which was proved by Saada in [8].

Corollary 1.2. *Consider the standard graph $\mathbb{Z}^d$. Declare each edge to be positive with probability $p \in (0,1)$ independently (and negative otherwise). Then, with probability 1, the corresponding voter model is ergodic.*

Remark 1.1. *Note however, that on a general signed graph the fact that simple random walk runs through infinitely many unsatisfied cycles is not necessary for ergodicity of the corresponding voter model. This will be shown in Section 8.*

In Section 2, we give the quick proof of Proposition 1.1. In Section 3, we explain the connection, mentioned earlier, between the voter model and simple random walk as well as derive two lemmas. In Section 4, we state and prove a key proposition, Proposition 4.1, which states that for recurrent graphs with an unsatisfied cycle, the one-dimensional marginals converge to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. In Section 5, we give the easier proof of Theorem 1.1 under the stronger assumption that two independent random walkers meet with probability 1; this uses Proposition 4.1. In Section 6, we prove a result, Theorem 6.1, which is valid for all graphs and may be of independent interest. It simply states that ergodicity of the process follows from the assumption that starting from any initial configuration, all of the one-dimensional marginals approach $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. Combined with Proposition 4.1, this will complete the proof of Theorem 1.1.
In Section 7, we move to the transient case and prove Proposition 1.2, Theorem 1.2 and Corollary 1.2. In Section 8 we give an example that justifies Remark 1.1. Section 9 contains some open problems.

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2. Easy characterization of fixed states

In this section, we give the easy proof of Proposition 1.1.

Proof of Proposition 1.1: It is obvious that if there exists an unsatisfied cycle, then there is no fixed state. Conversely, assume that there is no unsatisfied cycle. We will first construct a partition V_1, V_2 of V such that an edge is negative if and only if it is an edge from V_1 to V_2. (This is similar to the fact that a graph is bipartite if and only if there are no cycles of odd length.) To do this, take an arbitrary vertex v and place it in V_1. Now for any vertex w different from v, place w in V_1 if there is a simple path from v to w which has an even number of negative edges; otherwise, place w in V_2. The key observation is that if there are no unsatisfied simple cycles, then there cannot be both a simple path from v to w which has an even number of negative edges and also a simple path from v to w which has an odd number of negative edges. (Otherwise, there would be a possibly non-simple cycle from v to itself which has an odd number of negative edges from which one could obtain a simple cycle with an odd number of negative edges by "loop removal".) Hence the partition into V_1 and V_2 is well defined. It is then trivial to check that the partition $\{V_1, V_2\}$ is independent of the vertex v initially chosen, that there cannot be negative edges within V_1 or within V_2 and that any edge connecting a vertex in V_1 and a vertex in V_2 must be negative. Now that we have this partition, let η be the configuration which is 1 on V_1 and 0 on V_2. It is immediate that η is a fixed state. In addition, we can reverse 1 and 0 obtaining another fixed state. It is also easy to see that these are the only two fixed states; this follows from the uniqueness of the partition $\{V_1, V_2\}$. $\square$

3. Preliminary results

We first describe the duality relationship between the voter model and random walks mentioned in the introduction.

For $x \in V$, let $(X_t^x)_{t\geq 0}$ be a continuous time random walk on G starting from x and let P^x be the corresponding probability measure on path space. (This simply means that the walker waits an exponential time with parameter 1 and then chooses a neighbor in the graph at random to move to.) We sometimes write $(X_t)_{t\geq 0}$ if we don't specify the starting point of the random walk. If A is a subset of V, by a system of independent *coalescing* random walkers starting in A, we mean a process $(X_t^x)_{t\geq 0, x\in A}$ corresponding to the walkers starting at the different locations in A moving independently but when two walkers meet, they stick together forever.

We will use the following fact about the voter model, for which we refer to [2], Chapter 2 or [5], p. 245. A configuration η_t of the voter model at a *fixed* time t can be constructed from the initial configuration η_0 in the following way using our system of independent coalescing random walks starting from all locations of V. Given $(X_t^x)_{t\geq 0, x\in V}$, we let $\widetilde{\eta}_t(x) = \eta_0(y)$ if $X_t^x = y$ and X^x crossed an even number of negative edges during the interval $[0,t]$ and $\widetilde{\eta}_t(x) = 1 - \eta_0(y)$ if $X_t^x = y$ and X^x crossed an odd number of negative edges during the interval $[0,t]$. Then, for *each* t, $(\widetilde{\eta}_t(x), x \in V)$ has the same distribution as the voter model at time t, started at time 0 from the initial configuration η_0. Intuitively, the random walk describes the backtracking of opinions: $X_t^x = y$ means that the opinion of x at time t comes from the opinion of y at time 0. The fact that this is true for the ordinary voter model can be found in the references mentioned above, but the argument immediately applies to the mixed bond case. Note, importantly, that this construction is valid for fixed t, but if $t_1 \neq t_2$, then $\{(\widetilde{\eta}_{t_1}(x), \widetilde{\eta}_{t_2}(x))\}_{x\in V}$ does **not** have the same distribution as the voter model at times (t_1, t_2), that is, $\{(\eta_{t_1}(x), \eta_{t_2}(x))\}_{x\in V}$. However, the equality of the distributions at a fixed time will allow us to prove statements concerning the limiting behavior of the voter model.

We will now need the following technical lemma. For an event B and a random variable τ, we write $P(B|\tau)$ for $E(I_B|\tau)$. If we have a family of events $\{B_t\}_{t\geq 0}$ and τ is a nonnegative random variable, we write $B_{\tau+t}$ for the event $\{\omega | \omega \in B_{\tau(\omega)+t}\}$.

Lemma 3.1. *Let $(B_t)_{t\geq 0}$ be a family of events and c a constant.*
(i) Assume that, for each $\varepsilon > 0$, there is a random variable τ_ε such that $0 < \tau_\varepsilon < \infty$, a.s. and

$$\text{for all } t \geq 0, \quad |P(B_{\tau_\varepsilon + t} | \tau_\varepsilon) - c| \leq \varepsilon \quad a.s. \tag{3.1}$$

Then $P(B_t) \to c$ as $t \to \infty$.
(ii) Assume that there is a random variable T such that $0 < T < \infty$, a.s. and a function g such that

$$\lim_{t\to\infty} P(B_{T+t} | T) = g(T), \quad a.s. \tag{3.2}$$

Then $0 \leq g(T) \leq 1$ a.s. and $P(B_t) \to E(g(T))$ as $t \to \infty$.

Proof. (i) Fix $\varepsilon < c$. Choose t large enough such that $P(\tau_\varepsilon \geq t) \leq \varepsilon$. We have

$$P(B_t) = E\left(I_{B_t} I_{\{\tau_\varepsilon < t\}}\right) + E\left(I_{B_t} I_{\{\tau_\varepsilon \geq t\}}\right). \tag{3.3}$$

Considering the conditional expectation of the first term on the r.h.s. of (3.3), we have, due to (3.1),

$$I_{\{\tau_\varepsilon < t\}}(c - \varepsilon) \leq E\left(I_{B_t} I_{\{\tau_\varepsilon < t\}} | \tau_\varepsilon\right) \leq I_{\{\tau_\varepsilon < t\}}(c + \varepsilon). \tag{3.4}$$

Since the second term on the r.h.s. of (3.3) satisfies $E(I_{B_t} I_{\{\tau_\varepsilon \geq t\}}) \leq \varepsilon$, the claim follows from (3.3) and (3.4), after taking expectations in (3.4).
(ii) After some reflection, it is clear that

$$\lim_{t\to\infty} P\left(B_{T+t}|T\right) = g(T), \quad a.s.$$

is equivalent to

$$\lim_{t\to\infty} P\left(B_t|T\right) = g(T), \quad a.s.$$

Now simply apply the bounded convergence theorem. □

4. Recurrence implies convergence of the one-dimensional marginals

In this section, we prove that under the assumptions of recurrence and the existence of an unsatisfied cycle, the one-dimensional marginals converge to $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$. This is stated in the following proposition and is a key step towards proving Theorem 6.1.

Proposition 4.1. *Assume that simple random walk on G is recurrent and that there is an unsatisfied cycle in G. Then, for all $x \in V$ and all initial configurations η_0,*

$$\lim_{t\to\infty} P(\eta_t(x) = 1) = \frac{1}{2}\,. \tag{4.1}$$

The proof of Proposition 4.1 will use the following lemma. Before stating this, we introduce some notation. Given a set I which is a finite union of time intervals, let V_I be the event that during the time set I, (X_t) transverses an odd number of negative edges. Abbreviate $V_{[0,t]}$ by V_t.

Lemma 4.1. *Assume that simple random walk on G is recurrent and that there is an unsatisfied cycle in G. Fix a vertex v on this cycle. Then, for each $x \in V$, there exists an increasing sequence of stopping times T_m such that $T_m < \infty$, for all m, $T_m \to \infty$ as $m \to \infty$, $X_{T_m} = v$, for all m, P^x-a.s. and*

$$P^x(V_{T_m}|T_m) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad in\ L^\infty(P^x) \tag{4.2}$$

For the proof of this lemma, we first need the following two lemmas.

Lemma 4.2. *Let $B_1, B_2, \ldots$ be independent events such that $\sum_i P(B_i) = \infty$ and $\sum_i P(B_i^c) = \infty$. Then*

$$\lim_{n\to\infty} P\left(\sum_{i=1}^n I_{B_i} \text{ is even}\right) = \frac{1}{2}$$

Proof. Let Y_n be the indicator of the event $\{\sum_{i=1}^n I_{B_i}$ is even $\}$. Then the sequence $(Y_n)_{n=1,2,\ldots}$ is a time-inhomogeneous Markov chain with values in $\{0,1\}$ and transition probabilities $P(Y_{n+1}=1|Y_n=0)=P(B_{n+1})$, $P(Y_{n+1}=1|Y_n=1)=1-P(B_{n+1})$. The claim now follows from the convergence theorem for time-inhomogeneous Markov chains, see for instance [9], Theorem 4.4.1. □

The following is a trivial lemma, whose proof is immediate.

Lemma 4.3. *If U_1 and U_2 are independent nonnegative integer valued random variables such that $|P(U_1$ is odd $)-1/2|\le\varepsilon$, then $|P(U_1+U_2$ is odd $)-1/2|\le\varepsilon$.*

Proof of Lemma 4.1: Fix $x\in V$. Let

$$\widetilde{T}_1 =: \inf\{t\ge 0 : X_t=v, X_s=v \text{ for all } s\in[t+1,t+2]\}$$

and $A_1=V_{[\widetilde{T}_1,\widetilde{T}_1+2]}$. Let

$$\widetilde{T}_{i+1} =: \inf\{t\ge \widetilde{T}_i+3 : X_t=v, X_s=v \text{ for all } s\in[t+1,t+2]\}$$

and $A_{i+1}=V_{[\widetilde{T}_{i+1},\widetilde{T}_{i+1}+2]}$. Note that $P^x(\widetilde{T}_i<\infty, i=1,2,\ldots)=1$. We observe that the random variables $\{I_{A_i}\}_{i=1}^\infty$, $\{\widetilde{T}_{i+1}-\widetilde{T}_i\}_{i=1}^\infty$ are independent, $\{I_{A_i}\}_{i=1}^\infty$ are i.i.d. and $\{\widetilde{T}_{i+1}-\widetilde{T}_i\}_{i=1}^\infty$ are i.i.d. Since X_t can either sit still during $[\widetilde{T}_i,\widetilde{T}_i+2]$ or it can just run one time around the unsatisfied cycle and do nothing else, it is clear that $0<P(A_i)<1$. Let $\mathcal{F}_m$ denote the σ–algebra generated by $\{\widetilde{T}_1,\ldots,\widetilde{T}_m\}$. Let $U_1(m)$ be the number of negative edges crossed during the time set $\cup_{i=1}^m[\widetilde{T}_i,\widetilde{T}_i+2]$ and $U_2(m)$ be the number of negative edges crossed during the time set $[0,\widetilde{T}_1]\cup\left(\cup_{i=1}^{m-1}[\widetilde{T}_i+2,\widetilde{T}_{i+1}]\right)$. Since $U_1(m)$ is odd if and only if $\sum\limits_{i=1}^m I_{A_i}$ is odd, Lemma 4.2 (together with the independence mentioned above) implies that $P(U_1(m)$ is odd$|\mathcal{F}_m)\to\frac{1}{2}$ in $L^\infty(P^x)$ as $m\to\infty$. We next observe that conditioned on $\mathcal{F}_m$, $U_1(m)$ and $U_2(m)$ are independent with respect to P^x. Since $V_{\widetilde{T}_m+2}$ occurs if and only if $U_1(m)+U_2(m)$ is odd, we conclude from Lemma 4.3 that

$$P^x(V_{\widetilde{T}_m+2}|\mathcal{F}_m) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad \text{in } L^\infty(P^x) \tag{4.3}$$

and hence that

$$P^x(V_{\widetilde{T}_m+2}|\widetilde{T}_m) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad \text{in } L^\infty(P^x)\,. \tag{4.4}$$

Taking $T_m=\widetilde{T}_m+2$, (4.2) follows. □

Proof of Proposition 4.1: Fix $x\in V$. We show that for all initial configurations η_0,

$$\lim_{t\to\infty} P(\widetilde{\eta}_t(x)=1)=\frac{1}{2}\,. \tag{4.5}$$

Let v be an arbitrary vertex on some unsatisfied cycle. Next, choose a sequence of stopping times (T_m) as in Lemma 4.1. Using Lemma 3.1 (i), it is enough to show that for each $\varepsilon > 0$, there is m large enough such that for all initial configurations η_0,

$$\text{for all } t \geq 0, \quad \left| P(\widetilde{\eta}_{T_m+t}(x) = 1 | T_m) - \frac{1}{2} \right| \leq \varepsilon \quad P^x - a.s. \tag{4.6}$$

We write $V_{m,t}$ for the event $V_{[T_m, T_m+t]}$. Recall V_t denotes $V_{[0,t]}$. Then,

$$\begin{aligned} &P\left(\widetilde{\eta}_{T_m+t}(x) = 1 | T_m\right) = \\ &\quad P^x\left(V_{T_m} \cap \left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\})\right) | T_m\right) \\ &\quad + P^x\left(V^c_{T_m} \cap \left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\})\right) | T_m\right) . \end{aligned}$$

Since $X^x_{T_m} = v$, for each m, we can apply the strong Markov property and conclude that, for each m, the random variables $(V_{m,t}, X^x_{T_m+t})$ and V_{T_m} are conditionally independent w.r.t. P^x, given T_m. Since $(V_{m,t}, X^x_{T_m+t})$ is also independent of T_m, we obtain

$$\begin{aligned} &P(\widetilde{\eta}_{T_m+t}(x) = 1 | T_m) = \\ &\quad P^x(V_{T_m} | T_m) P^x\left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\})\right) \\ &\quad + P^x(V^c_{T_m} | T_m)) P^x\left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\})\right) . \end{aligned}$$

Due to Lemma 4.1, $P^x(V_{T_m}|T_m)$ and $P^x(V^c_{T_m}|T_m)$ converge to $\frac{1}{2}$ in $L^\infty(P^x)$ as $m \to \infty$. Since $(V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\})$ and $(V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\})$ are complementary events, we conclude that (4.6) holds true. ☐

5. Proof of Theorem 1.1 when two walkers meet

It turns out that the proof of Theorem 1.1 simplifies somewhat if we make the stronger assumption that two independent random walkers meet with probability 1. We give this simplified proof in this section.

Proof of Theorem 1.1 in the case where the probability that two independent random walkers meet is 1*:* Fix a finite number of locations $x_1, x_2, \ldots, x_k$ in V and consider coalescing simple random walks starting at vertices $x_1, x_2, \ldots, x_k$. Let T be the first time they all meet and let $A = A(x_1, \ldots, x_k)$ be the event that at time T either all the walkers crossed an even number of negative edges or that all the walkers crossed an odd number of negative edges.

We will show that no matter what the initial configuration η_0 is,

$$\lim_{t \to \infty} P(\eta_t(x_i) = 1, i = 1, \ldots, k) = \frac{P(A)}{2}. \tag{5.1}$$

Since this doesn't depend on the initial configuration and a probability measure μ on $\{0,1\}^V$ is determined by the probability it gives to having 1's at any finite number of specified locations, this clearly implies ergodicity of the process. Instead of (5.1), we will show

$$\lim_{t\to\infty} P(\widetilde{\eta}_t(x_i) = 1, i = 1,\ldots,k) = \frac{P(A)}{2}. \tag{5.2}$$

Denote by X_T the location of the coalesced walkers at time T, i.e. $X_T = X_T^{x_i}, 1 \le i \le k$. Let $A^+ = A^+(x_1,\ldots,x_k)$ be the event that at time T all the walkers crossed an even number of negative edges and $A^- = A^-(x_1,\ldots,x_k)$ be the event that at time T all the walkers crossed an odd number of negative edges. Of course, A is the disjoint union of A^+ and A^-. Then, for $t \ge 0$,

$$\begin{aligned} & P(\widetilde{\eta}_{T+t}(x_i) = 1, i = 1,\ldots,k|T) \\ = \; & P(A^+|T) \int\limits_{x\in V} P(\widetilde{\eta}_t(x) = 1)\mu^+(dx) + P(A^-|T) \int\limits_{x\in V} P(\widetilde{\eta}_t(x) = 0)\mu^-(dx), \end{aligned}$$

where μ^+ is the conditional distribution of X_T, given T and conditioned on A^+, and μ^- is the conditional distribution of X_T, given T and conditioned on A^-. Due to Proposition 4.1 and the bounded convergence theorem, both $\int\limits_{x\in V} P(\widetilde{\eta}_t(x) = 1)\mu^+(dx)$ and $\int\limits_{x\in V} P(\widetilde{\eta}_t(x) = 0)\mu^-(dx)$ (which are measurable functions of T since μ^+ and μ^- are) converge to $\frac{1}{2}$ a.s. as $t \to \infty$, resulting in

$$P(\widetilde{\eta}_{T+t}(x_i) = 1, i = 1,\ldots,k|T) \underset{t\to\infty}{\longrightarrow} \frac{1}{2}P(A^+|T) + \frac{1}{2}P(A^-|T) = \frac{1}{2}P(A|T), \quad a.s.$$

and (5.2) follows by Lemma 3.1 (ii). $\square$

6. A general characterization of ergodicity and Theorem 1.1

In this section, we prove a result (applicable to both recurrent and transient graphs) which states that to insure uniqueness of the stationary distribution, it suffices to look at convergence for the one-dimensional marginal distributions. This is stated in the following Theorem 6.1. Observe that Theorem 1.1 follows immediately from Theorem 6.1 together with Proposition 4.1.

Theorem 6.1. *Consider a signed graph G. The corresponding voter model is ergodic if and only if for all $x \in V$ and all initial configurations η_0,*

$$\lim_{t\to\infty} P(\eta_t(x) = 1) = \frac{1}{2}. \tag{6.1}$$

Proof: The "only if" part follows easily from symmetry.
For the "if" direction, fix a finite number of locations $x_1, x_2, \ldots, x_k$ in V. Define a (random) equivalence relation on $\{x_1, x_2, \ldots, x_k\}$ by stipulating that $x_i \sim x_j$ if the random walkers, started from x_i and x_j, eventually coalesce. Each equivalence class containing at least two elements has a (random) coalescence time, which is the first time such that all the walkers in the class have coalesced. (This is not a stopping time.) Let $A_\ell = A_\ell(x_1, \ldots, x_k)$ be the event that there are ℓ equivalence classes and that in each equivalence class containing at least two elements, either all the walkers crossed an even number of negative edges or they crossed an odd number of negative edges at the respective coalescence time of the equivalence class. We will show that no matter what the initial configuration η_0 is,

$$\lim_{t\to\infty} P(\eta_t(x_i) = 1, i = 1, \ldots, k) = \sum_{\ell=1}^{k} P(A_\ell) \left(\frac{1}{2}\right)^{\ell}. \tag{6.2}$$

Since this doesn't depend on the initial configuration and a probability measure μ on $\{0,1\}^V$ is determined by the probability it gives to having 1's at any finite number of specified locations, this clearly implies ergodicity.
We have to decompose according to all possible cases for coalescence and non-coalescence. By a *lexicographically ordered partition* (l.o.p.) of $\{1, \ldots, k\}$, we mean an ordered partition $(I_1, \ldots, I_\ell)$ of $\{1, \ldots, k\}$ such that the smallest elements of $(I_1, \ldots, I_\ell)$ are in increasing order. (Obviously, there is exactly one ordering of any partition which is lexicographically ordered.) For an l.o.p. $(I_1, \ldots, I_\ell)$ of $\{1, \ldots, k\}$, let $D_{I_1,\ldots,I_\ell}$ denote the event that the equivalence classes of the above equivalence relation (where we identify $\{x_1, \ldots, x_k\}$ with $\{1, \ldots, k\}$) are $\{I_1, \ldots, I_\ell\}$, and let $\widetilde{D}_{I_1,\ldots,I_\ell} := D_{I_1,\ldots,I_\ell} \cap A_\ell$. It suffices to show that for each such l.o.p. $(I_1, \ldots, I_\ell)$,

$$\lim_{t\to\infty} P\left(\{\widetilde{\eta}_t(x_i) = 1, i = 1, \ldots, k\} \cap D_{I_1,\ldots,I_\ell}\right) = \left(\frac{1}{2}\right)^{\ell} P\left(\widetilde{D}_{I_1,\ldots,I_\ell}\right). \tag{6.3}$$

We will need the following lemma which we prove afterwards.

Lemma 6.1. *For $x_1, \ldots, x_m \in V$ with $x_i \neq x_j$ for $i \neq j$, assume that the event that for all $1 \leq i < j \leq m$ and all positive t, $X_t^{x_i} \neq X_t^{x_j}$ has positive probability. (Since the walkers are coalescing, this simply means that no two ever meet.) Let $\nu_{x_1,\ldots,x_m}$ be the conditional distribution of $(X_t^{x_1}, \ldots, X_t^{x_m})_{t\geq 0}$ given this event. Let $\nu^s_{x_1,\ldots,x_m}$ be the conditional distribution of $(X_t^{x_1}, \ldots, X_t^{x_m})_{t\geq 0}$ obtained by conditioning on the event that for all $1 \leq i < j \leq m$, $X_s^{x_i} \neq X_s^{x_j}$ (and hence that no two have met by time s) and then letting the walkers after time s, evolve independently without coalescing. Then*

$$\lim_{s\to\infty} d_{TV}\left(\nu^s_{x_1,\ldots,x_m}, \nu_{x_1,\ldots,x_m}\right) = 0, \tag{6.4}$$

where $d_{TV}(\cdot,\cdot)$ denotes the total variation distance.

We now let T be the last time a coalescence occurs. (Note that T is of course not a stopping time.) Using Lemma 3.1 (ii), (6.3) follows if we show that for each l.o.p. $(I_1,\dots,I_\ell)$,

$$\lim_{t\to\infty} P\left(\{\widetilde{\eta}_{T+t}(x_i)=1, i=1,\dots,k\}\cap D_{I_1,\dots,I_\ell}\,|T\right)=\left(\frac{1}{2}\right)^{\ell} P\left(\widetilde{D}_{I_1,\dots,I_\ell}\,|T\right). \quad (6.5)$$

Next for an l.o.p. $(I_1,\dots,I_\ell)$ and $(a_1,\dots,a_\ell)\in\{0,1\}^\ell$, let $\widetilde{D}^{a_1\dots,a_\ell}_{I_1,\dots,I_\ell}$ be the subevent of $\widetilde{D}_{I_1,\dots,I_\ell}$ where for each $j\in\{1,\dots,\ell\}$, the walkers with starting points in I_j transversed (mod 2) a_j negative bonds during $[0,T]$ (recall that we are identifying $(x_1,\dots x_k)$ with $(1,\dots k)$). Now,

$$\begin{aligned}&P\left(\{\widetilde{\eta}_{T+t}(x_i)=1, i=1,\dots,k\}\cap D_{I_1,\dots,I_\ell}\,|T\right)=\\&\qquad\sum_{(a_1,\dots,a_\ell)\in\{0,1\}^\ell} P\left(\widetilde{D}^{a_1,\dots,a_\ell}_{I_1,\dots,I_\ell}\,|T\right)\\&\times\sum_{(y_1,\dots,y_\ell)\in V^\ell, y_i\neq y_j}\nu_{y_1,\dots,y_\ell}[\eta_0(X^{y_i}_t)=1-a_i \text{ for } i=1,\dots,\ell]Q^{a_1,\dots,a_\ell}_{T,I_1,\dots,I_\ell}(y_1,\dots,y_\ell)\end{aligned}$$

where $Q^{a_1,\dots,a_\ell}_{T,I_1,\dots,I_\ell}(\{y_1,\dots,y_\ell\})$ is the conditional probability given T and $\widetilde{D}^{a_1,\dots,a_\ell}_{I_1,\dots,I_\ell}$ that at time T, the positions of the walkers with starting points in $I_1\dots,I_\ell$ are $y_1,\dots,y_\ell$. We claim that for all $(y_1,\dots,y_\ell)\in V^\ell$, $y_i\neq y_j$ for $i\neq j$, and all $(a_1,\dots,a_\ell)\in\{0,1\}^\ell$

$$\lim_{t\to\infty}\nu_{y_1,\dots,y_\ell}[\eta_0(X^{y_i}_t)=a_i \text{ for } i=1,\dots,\ell]=\left(\frac{1}{2}\right)^{\ell} \quad (6.6)$$

from which (6.5) follows from the bounded convergence theorem. To establish this claim, fix such a $(y_1,\dots,y_\ell)\in V^\ell$, $y_i\neq y_j$ for $i\neq j$, and a $(a_1,\dots,a_\ell)\in\{0,1\}^\ell$, let $\varepsilon>0$ and choose by Lemma 6.1 an s so that

$$d_{TV}\left(\nu^s_{y_1,\dots,y_\ell},\nu_{y_1,\dots,y_\ell}\right)<\varepsilon. \quad (6.7)$$

For $(b_1,\dots,b_\ell)\in\{0,1\}^\ell$, let $E(b_1,\dots,b_\ell)$ be the event that for each $i\in\{1,\dots,\ell\}$, X^{y_i} transversed (mod 2) b_i negative bonds during $[0,s]$.
Now for $t>s$,

$$\begin{aligned}&\nu^s_{y_1,\dots,y_\ell}[\eta_0(X^{y_i}_t)=a_i \text{ for } i=1,\dots,\ell]=\\&\qquad\sum_{(b_1,\dots,b_\ell)\in\{0,1\}^\ell}\nu^s_{y_1,\dots,y_\ell}(E(b_1,\dots,b_\ell))\\&\times\sum_{(z_1,\dots,z_\ell)\in V^\ell, z_i\neq z_j}\prod_{i=1}^{\ell}P[\eta_0(X^{z_i}_{t-s})=a_i 1_{b_i=0}+(1-a_i)1_{b_i=1}]R^{b_1\dots,b_\ell}_{s,y_1,\dots,y_\ell}((z_1,\dots,z_\ell))\end{aligned}$$

where $R^{b_1,\dots,b_\ell}_{s,y_1,\dots,y_\ell}$ is the distribution of the positions at time s of the ℓ random walkers starting at $y_1,\dots,y_\ell$, conditioned on not having met by time s and conditioned on the event $E(b_1,\dots,b_\ell)$. By (6.1), for all $(z_1,\dots,z_\ell)\in V^\ell$, $z_i\neq z_j$ for $i\neq j$, $(a_1,\dots,a_\ell)\in\{0,1\}^\ell$ and $(b_1,\dots,b_\ell)\in\{0,1\}^\ell$, the above integrand $\prod_{i=1}^\ell P[\eta_0(X^{z_i}_{t-s})=a_i 1_{b_i=0}+(1-a_i)1_{b_i=1}]$ approaches $\left(\frac{1}{2}\right)^\ell$ as $t\to\infty$. By the bounded convergence theorem, $\nu^s_{y_1,\dots,y_\ell}[\eta_0(X^{y_i}_t)=a_i$ for $i=1,\dots,\ell]$ also approaches $\left(\frac{1}{2}\right)^\ell$ as $t\to\infty$. By (6.7), $\nu_{y_1,\dots,y_\ell}[\eta(X^{y_i}_t)=a_i$ for $i=1,\dots,\ell]$ has a lim inf and a lim sup as $t\to\infty$ each within ε of $\left(\frac{1}{2}\right)^\ell$. As $\varepsilon>0$ is arbitrary, this proves (6.6). □

Proof of Lemma 6.1: Let $\mu^s_{x_1,\dots,x_m}$ be the conditional distribution of $(X^{x_1}_t,\dots,X^{x_m}_t)_{t\geq 0}$, conditioned on the event that for all $1\leq i<j\leq m$ and all positive $t\leq s$, $X^{x_i}_t\neq X^{x_j}_t$. We show that both $d_{TV}\left(\mu^s_{x_1,\dots,x_m},\nu_{x_1,\dots,x_m}\right)$ and $d_{TV}\left(\nu^s_{x_1,\dots,x_m},\mu^s_{x_1,\dots,x_m}\right)$ approach 0 as $s\to\infty$. The first follows from the easy abstract fact that if $(\Omega,\mathcal{F},P)$ is an arbitrary probability space, B_s is a decreasing sequence of events (as $s\to\infty$) with $P(\cap_{s\geq 0}B_s)>0$, then

$$\lim_{s\to\infty} d_{TV}\left(P(\cdot|B_s),P(\cdot|\cap_s B_s)\right)=0.$$

Here

$$B_s:=\{\text{no two of the walkers coalesce by time } s\}.$$

For the second, we note that with

$$A_s:=\{\text{at least two of the walkers coalesce after time } s\}$$

we have that

$$\lim_{s\to\infty} P(A_s|B_s)=0$$

since $P(A_s)$ approaches 0 as $s\to\infty$ and $P(B_s)$ does not approach 0 as $s\to\infty$ and then observe that $d_{TV}\left(\nu^s_{x_1,\dots,x_m},\mu^s_{x_1,\dots,x_m}\right)$ approaching 0 follows immediately from this. □

7. The transient case

In this section we prove Proposition 1.2, Theorem 1.2 and Corollary 1.2.

Proof of Proposition 1.2: First, it is a consequence of the martingale convergence theorem that if an x exists with this property, then there exists a vertex y such that a random walker starting from location y stays in W forever with probability larger than 3/4. More precisely, let B be the event that (X_t) stays in W forever. Let $\mathcal{F}_t$ be the σ-field generated by $\{(X_s)_{0\leq s\leq t}\}$, $t\geq 0$. Let $M_t:=E(I_B|\mathcal{F}_t)$, $t\geq 0$. Then (M_t) is a martingale with respect to $(\mathcal{F}_t)$. Due to the martingale convergence theorem, $M_t\to I_B$ for $t\to\infty$, P^x-a.s. for all $x\in V$. Since $P^x(B)>0$ and $M_t\leq P^{X_t}(B)$, the claim follows. Next, as in the proof of Proposition 1.1, choose a configuration η_0 on W

such that an edge within W is negative if and only if the η_0 values of its endpoints are different. Assume $\eta_0(y) = 1$. Extend η_0 to V arbitrarily. Then it is immediate that starting from the configuration η_0, $P(\widetilde{\eta}_t(y) = 1) \geq 3/4$ for all t. Using a convergent subsequence of the Cesaro averages starting from η_0 gives a stationary distribution μ with $\mu(\eta(y) = 1) \geq 3/4$. By symmetry there is a stationary distribution $\tilde{\mu}$ such that $\tilde{\mu}(\eta(y) = 1) \leq 1/4$ which implies non-uniqueness of the stationary distribution. $\square$

Proof of Theorem 1.2: In view of Theorem 6.1, it suffices to prove that for all $x \in V$ and all initial configurations η_0, (6.1) holds. We will do this by modifying the proof of Proposition 1.1. We first need the following lemma which is analogous to Lemma 4.1 and which will be proved afterwards in a similar manner. Recall the definitions of V_I, V_t and $V_{m,t}$ from Section 4.

Lemma 7.1. *Under the assumptions of Theorem 1.2, there exists, for each $x \in V$, an increasing sequence of stopping times T_m such that $T_m < \infty$, for all m, $T_m \to \infty$ as $m \to \infty$, P^x-a.s. and*

$$P^x(V_{T_m} | T_m, X_{T_m}) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad \text{in } L^\infty(P^x). \tag{7.1}$$

Exactly as in the proof of Proposition 1.1, using Lemma 3.1 (i), it is enough to show that for each $\varepsilon > 0$, there is m large enough such that for all initial configurations η_0, (4.6) holds and to verify this, we prove the stronger fact that for large m

$$\text{for all } t \geq 0, \quad \left| P(\widetilde{\eta}_{T_m+t}(x) = 1 | T_m, X_{T_m}) - \frac{1}{2} \right| \leq \varepsilon \quad P^x - a.s. \tag{7.2}$$

Now this is proved almost as (4.6) was proved. We modify the first displayed equation after (4.6) by replacing all the conditionings on T_m by conditionings on both T_m and X_{T_m}. The strong Markov property implies that, for each m, the random variables $(V_{m,t}, X^x_{T_m+t})$ and V_{T_m} are conditionally independent w.r.t. P^x, given T_m and X_{T_m}. We then obtain

$$\begin{aligned}
&P(\widetilde{\eta}_{T_m+t}(x) = 1 | T_m, X_{T_m}) = \\
&P^x(V_{T_m} | T_m, X_{T_m}) P^x\left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\}) | T_m, X_{T_m} \right) \\
&+ P^x(V^c_{T_m} | T_m, X_{T_m})) P^x\left((V_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 0\}) \cup (V^c_{m,t} \cap \{\eta_0(X^x_{T_m+t}) = 1\}) | T_m, X_{T_m} \right) .
\end{aligned}$$

Due to Lemma 7.1, $P^x(V_{T_m} | T_m, X_{T_m})$ and $P^x(V^c_{T_m} | T_m, X_{T_m})$ converge to $\frac{1}{2}$ in $L^\infty(P^x)$ as $m \to \infty$, and we conclude that (7.2) holds true. $\square$

Proof of Lemma 7.1: This will be proved by modifying the proof of Lemma 4.1. Choose k such that the set of vertices which are part of some unsatisfied cycle of length at most k is unavoidable. Denote this set of vertices by $\mathcal{G}$.

Let

$$\widetilde{T}_1 =: \inf\{t \geq 0 : X_t \in \mathcal{G}, X_s = X_t \text{ for all } s \in [t+1, t+2]\}$$

and $A_1 = V_{[\widetilde{T}_1, \widetilde{T}_1+2]}$. Let

$$\widetilde{T}_{i+1} =: \inf\{t \geq \widetilde{T}_i + 3 : X_t \in \mathcal{G}, X_s = X_t \text{ for all } s \in [t+1, t+2]\}$$

and $A_{i+1} = V_{[\widetilde{T}_{i+1}, \widetilde{T}_{i+1}+2]}$. Since $\mathcal{G}$ is unavoidable, we have $P^x(\widetilde{T}_i < \infty, i = 1, 2, \ldots) = 1$. Denote the σ–algebra generated by $\{\widetilde{T}_1, \ldots, \widetilde{T}_m, X_{\widetilde{T}_1}, \ldots, X_{\widetilde{T}_m}\}$ by $\mathcal{F}_m$. We observe that for every m, $A_1, \ldots, A_m$ are conditionally independent given $\mathcal{F}_m$. Observe also that since G is of bounded degree, if we look at all graphs obtained by choosing a $v \in V$ and taking the induced subgraph generated by all vertices within distance k of v, we only get a finite number of graphs (up to isomorphism). Since X_t can either sit still during $[\widetilde{T}_i, \widetilde{T}_i+2]$ or it can just run one time around an unsatisfied cycle of length at most k and do nothing else, the above graph property observation implies that there exists a positive number $\delta > 0$ such that for all m and for any $i \in \{1, \ldots, m\}$, $P(A_i|\mathcal{F}_m) \in [\delta, 1-\delta]$. Let $U_1(m)$ be the number of negative edges crossed during the time set $\cup_{i=1}^m [\widetilde{T}_i, \widetilde{T}_i + 2]$ and $U_2(m)$ be the number of negative edges crossed during the time set $[0, \widetilde{T}_1] \cup \left(\cup_{i=1}^{m-1} [\widetilde{T}_i + 2, \widetilde{T}_{i+1}]\right)$. Since $U_1(m)$ is odd if and only if $\sum_{i=1}^{m} I_{A_i}$ is odd, Lemma 4.2, together with the above, implies that $P(U_1(m) \text{ is odd}|\mathcal{F}_m) \to \frac{1}{2}$ in $L^\infty(P_x)$ as $m \to \infty$. We next observe that conditioned on $\mathcal{F}_m$, $U_1(m)$ and $U_2(m)$ are independent with respect to P^x. Since $V_{\widetilde{T}_m+2}$ occurs if and only if $U_1(m) + U_2(m)$ is odd, we conclude from Lemma 4.3 that

$$P^x(V_{\widetilde{T}_m+2}|\mathcal{F}_m) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad \text{in } L^\infty(P^x) \tag{7.3}$$

from which we can conclude

$$P^x(V_{\widetilde{T}_m+2}|\widetilde{T}_m, X_{\widetilde{T}_m}) \underset{m\to\infty}{\longrightarrow} \frac{1}{2} \quad \text{in } L^\infty(P^x) \tag{7.4}$$

Taking $T_m = \widetilde{T}_m + 2$, (7.1) follows. □

Proof of Corollary 1.2: For $d = 2$, this follows from Theorem 1.1 since clearly with probability 1, there will exist some unsatisfied cycle. For $d \geq 3$, this follows, using Theorem 1.2, from the fact that the set of points which belong to an unsatisfied cycle of length 4 is with probability 1 an unavoidable set. To show this latter fact, let $\mathcal{G}$ be the set of vertices which are part of some unsatisfied cycle of length 4 and consider the following inductive construction of the random walk, together with a declaration of being positive or negative for a *subset* of the edges. Let $x \in \mathbb{Z}^d$ be the starting point of the random walk. Choose a cycle of length 4 containing x, and declare, independently, each of the four edges to be positive (with probability p) and negative otherwise. Consider the stopping times τ_i where the random walk visits a vertex v_i for the first time, and v_i is contained in a cycle of length 4 whose edges have not been declared yet. Choose arbitrarily (according to a prespecified deterministic rule) such a cycle of length 4 and declare independently, and independently of everything done before, each of the four edges to be positive (with probability p) and negative otherwise. Then $X_{\tau_i} = v_i$ and $P^x(\tau_i < \infty, \forall i) = 1$ (the latter follows immediately if one considers, e.g., hitting times of spheres around the starting point). The events $\{X_{\tau_i} \in \mathcal{G}\}$, $i = 1, 2, \ldots$ are independent and $P(X_{\tau_i} \in \mathcal{G}) \geq 4p^3(1-p) > 0$. The Borel–Cantelli lemma implies that $P\left(X_{\tau_i} \in \mathcal{G} \text{ for infinitely many } i\right) = 1$. Finally, Fubini's theorem implies that a.s. $\mathcal{G}$ is an unavoidable set. □

8. An Example with few unsatisfied cycles

In this section we will give an example of a signed graph such that the corresponding voter model is ergodic, but simple random walk almost surely runs around only finitely many unsatisfied cycles.

Example 8.1. The signed graph $G = (V, E)$ will be the rooted binary tree T_2 with some additional edges. We will choose V to be the vertex set of T_2 which we enumerate by $v_{n,k}, k = 1, \ldots, 2^n, n = 0, 1, 2, \ldots$. Here the index n stands for the distance of the corresponding vertex to the root $v_{0,1}$ and vertex $v_{n,k}$ has children $v_{n+1,2k-1}$ and $v_{n+1,2k}$ (and parents are connected to children by a bond). We will also write $|x|$ to indicate the height of vertex x; hence for $x = v_{n,k}$ we have that $|x| = n$. Additionally to these edges we have the following edges: Choose a strictly increasing sequence $(a(n))_{n\in\mathbb{N}_0}$ of positive integers with a(0)=0. At distance $a(n), n \geq 1$, of the root, connect the left half of the descendents of a vertex at level $a(n-1)$ to the right half of the descendents of the same vertex by connecting each of the pairs $(v_{a(n),k}, v_{a(n),2^{a(n)-a(n-1)-1}+k}), k = j2^{a(n)-a(n-1)-1} + j'$, where $0 \leq j \leq 2^{a(n-1)+1} - 1$ is even and $j' = 1, 2, \ldots 2^{a(n)-a(n-1)-1}$ with a new edge. On this edge set E define signs as follows: The edges $(v_{a(n),k}, v_{a(n)+1,2k-1}), k = 1, \ldots 2^{a(n)}$ are declared to be negative while all other edges (in particular all new edges) are positive.

Denote by L_n the vertices at height $a(n)$, i.e. all vertices v at distance $a(n)$ from the root. Note that in order for simple random walk to run through an unsatisfied cycle, there must be an n such that the walker hits L_{n-1} after having hit L_n. Hence by Borel–Cantelli, if $a(n)$ goes to infinity sufficiently fast, simple random walk almost surely will only go through finitely many unsatisfied cycles.

On the other hand one can show, that for all $x \in V$ and for all initial configurations η_0,

$$\lim_{t\to\infty} P(\eta_t(x) = 1) = \frac{1}{2};$$

Theorem 6.1 then implies ergodicity. Indeed, fix $x \in V$ and denote by i_0 the smallest i such that $|x| \leq a(i)$ and for $i \geq i_0$ introduce the stopping times S_i as the first hitting time for X_t^x of L_i, i.e.

$$S_i := \inf\{t > 0 : X_t^x \in L_i\}.$$

Next, let

$$T_{i_0} := \inf\{t : t \in \bigcup_{i=i_0}^{\infty}\{S_i + 1\}, |X_s^x| = |X_t^x| \text{ for all } s \in [t-1, t]\}$$

and for $i \geq i_0$ let

$$T_{i+1} := \inf\{t > T_i : t \in \bigcup_{i=i_0}^{\infty}\{S_i + 1\}, |X_s^x| = |X_t^x| \text{ for all } s \in [t-1, t]\}.$$

Using this definition of T_i we will verify (7.1). After that, one can proceed by copying the proof of Theorem 1.2 after Lemma 7.1. For $i \geq i_0$ define

$$\tau_i := \sup\{t < T_{i+1} : |X_t^x| = |X_{T_i}^x|\}$$

and for $m \geq i_0$

$$\mathcal{F}_m := \sigma\left(\{T_i, X_{T_i}^x, i = i_0, \ldots, m\} \cup \{\tau_i, X_{\tau_i}^x, i = i_0, \ldots, m-1\}\right).$$

For $i \geq i_0$ put $A_i := V_{[\tau_i, T_{i+1}]}$ with $V_\bullet$ defined as in Section 4. Then it is easy to see that the $A_i, i = i_0, \ldots, m-1$ are i.i.d. and independent of $\mathcal{F}_m$. Hence $P(A_i|\mathcal{F}_m) = P(A_i) \in (0, 1)$. Now following the proof of Lemma 7.1 we conclude that indeed (7.1) holds true.

9. Open Questions

(1) Is the converse of Proposition 1.2 true?
(2) Can we assign positive and negative signs to the edges on $\mathbb{Z}^3$ such that the corresponding voter model is ergodic, but almost surely simple random walk only goes around finitely many unsatisfied cycles?
(3) Is the fact that almost surely simple random walk goes around infinitely many unsatisfied cycles sufficient for ergodicity of the process?
(4) Does the uniqueness of the stationary distribution imply ergodicity?

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(Nina Gantert) Institut für Mathematische Stochastik, Universität Karlsruhe, Englerstr.2, 76128 Karlsruhe, Germany
E-mail address, Nina Gantert: `N.Gantert@math.uni-karlsruhe.de`

(Matthias Löwe) Institut für Mathematische Statistik, Fachbereich Mathematik und Informatik, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
E-mail address, Matthias Löwe: `maloewe@math.uni-muenster.de`

(Jeffrey E. Steif) Department of Mathematics, Chalmers University of Technology, S-412 96 Gothenburg, Sweden and School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-1060
E-mail address, Jeffrey E. Steif: `steif@math.chalmers.se`