On the Cluster Size Distribution for Percolation on Some General Graphs

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Abstract

We show that for any Cayley graph, the probability (at any p) that the cluster of the origin has size n decays at a well-defined exponential rate (possibly 0). For general graphs, we relate this rate being positive in the supercritical regime with the amenability/nonamenability of the underlying graph.

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1 Introduction

Percolation is perhaps the most widely studied statistical physics model for modeling random media. In addition, it is a source of many challenging open problems and beautiful conjectures which are easy to state but often are very difficult to settle; see [13] for a survey and introduction. The classical literature concentrates on studying the model on Euclidean lattices $\mathbb{Z}^d$, $d \geq 2$ and on trees. However in recent years, there has been a great deal of interest in studying

percolation on other infinite, locally finite, connected graphs; see [10, 9, 8, 15, 16, 14, 17, 27].

Our first theorem states that for any Cayley graph, the probability that the cluster of the origin has size n decays at a well-defined exponential rate. For $\mathbb{Z}^d$, this is Theorem 6.78 in [13].

Throughout this paper, C will denote the connected component of a fixed vertex (the origin for Cayley graphs) for Bernoulli percolation.

Theorem 1 *If G is a Cayley graph, then*

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}_p \left(|C| = n\right)$$

exists for every $p \in (0,1)$.

Our method for proving this result combines a randomized version of the usual method using subadditivity (as in for $\mathbb{Z}^d$) together with a proof that any two finite subgraphs of G have disjoint translates that are at distance $\leq \delta$ from each other where δ is an appropriate function of the sizes of the subgraphs. One expects perhaps that one should be able to take δ being a constant, depending only on the graph. See Question 3 for the statement of this problem.

Remark: Interesting, as we point out later, there is a concept of an ordered group who's definition is as follows:

Definition 1 *A group G with a linear ordering $\leq^*$ is called an* (right) ordered group *if for every $a \leq^* b$ we have $ag \leq^* bg$ for all $g \in G$.*

For such groups, the proof of Theorem 6.78 in [13] can be extended. However, for general groups, it seems that this proof cannot be applied.

It is of course of interest to know if the limit above is positive or 0. As will be pointed out later, it is positive below the critical value for all transitive graphs and so we restrict discussion to the supercritical regime. In this case, for $\mathbb{Z}^d$, the limit is 0 (see Theorem 8.61 in [13]) while for trees it is positive (although 0

at the critical value). Equation (10.12) in [13]) has an explicit formula for these probabilities for the rooted infinite 3-ary tree.

One of the key issues studied in percolation is the difference in the behavior of percolation depending on whether the underlying graph is *amenable* or *nonamenable* [9, 8, 17, 27]. For example, for amenable transitive graphs, there is uniqueness of the infinite cluster for all values of p while for nonamenable transitive graphs, it is conjectured that there is nonuniqueness of the infinite cluster for some values of p. Here it is also worthwhile to point out that it is well known that properties of other probabilistic models associated with a graph differ depending on whether the graph is amenable or not. Perhaps the most classical of all is the relation with simple random walk on a graph, first studied by Kesten [22] where it was shown that there is a positive spectral gap in the transition operator if and only if the group is nonamenable. Similar relationships have been investigated with respect to other statistical physics models (see e.g. [20, 21, 17, 11]).

For the nonamenable case, we state the following question.

Question 1 *Is it true that for a general transitive nonamenable graph G we have*

$$\mathbb{P}_p\left(|C| = n\right) \leq \exp\left(-\gamma\left(p\right) n\right) \quad \forall \ n \geq 1 \tag{1}$$

for some $\gamma\left(p\right) > 0$ whenever $p \neq p_c\left(G\right)$?

Consider a general weakly nonamenable graph $G := (\mathcal{V}, \mathcal{E})$ (not necessarily transitive) with bounded degree. Using a not so difficult argument of counting *lattice animals*, one can prove that if v_0 is a fixed vertex of G and C is the open connected component of v_0, then for sufficiently large p there is a function $\gamma\left(p\right) > 0$, such that

$$\mathbb{P}_p\left(|C| = n\right) \leq e^{-\gamma(p)\, n} \quad \forall \ n \geq 1\,. \tag{2}$$

In fact, in the appendix by Gábor Pete in [11] (see equation (A.3)), it is

shown by a slightly more involved argument, that the exponential decay (2) holds whenever $p > 1/(1+\kappa')$ where $\kappa' = \kappa'(G, v_0)$ is the anchored Cheeger constant. This is certainly in contrast to the $\mathbb{Z}^d$ case and also, as we will see later in Section 4, to what happens for a large class of transitive amenable graphs.

Using classical branching process arguments, one can conclude that for any infinite regular tree (which are prototypes for transitive nonamenable graphs), we must have an exponential tail bound for the cluster size distribution, when p is not equal to the critical probability.

The assumption of transitivity is however needed for Question 1 to have a positive answer as the following example illustrates. The graph obtained by taking $\mathbb{Z}^d$ and attaching a regular rooted tree with degree $r+1$ at each vertex where r satisfies $p_c\left(\mathbb{Z}^d\right) < \frac{1}{r}$ is a nontransitive, nonamenable graph which possesses an intermediate regime (above the critical value) of sub-exponential decay as next stated in detail.

Theorem 2 *Consider the graph just described and suppose $p_c\left(\mathbb{Z}^d\right) < \frac{1}{r}$.*

(a) If $p \in \left(0, p_c\left(\mathbb{Z}^d\right)\right) \cup \left(\frac{1}{r}, 1\right)$ then there are functions $\phi_1(p) < \infty$ and $\phi_2(p) > 0$, such that for all $n \geq 1$,

$$\exp\left(-\phi_1(p)\, n\right) \leq \mathbb{P}_p\left(n \leq |C| < \infty\right) \leq \exp\left(-\phi_2(p)\, n\right). \tag{3}$$

(b) If $p \in \left(p_c\left(\mathbb{Z}^d\right), \frac{1}{r}\right)$ then there are functions $\psi_1(p) < \infty$ and $\psi_2(p) > 0$, such that for all $n \geq 1$,

$$\exp\left(-\psi_1(p)\, n^{(d-1)/d}\right) \leq \mathbb{P}_p\left(n \leq |C| < \infty\right) \leq \exp\left(-\psi_2(p)\, n^{(d-1)/d}\right). \tag{4}$$

(c) For $p = \frac{1}{r}$ we have constants $c_1 > 0$ and $c_2(\varepsilon) < \infty$ such that every $\varepsilon > 0$ and for all $n \geq 1$,

$$\frac{c_1}{n^{1/2}} \leq \mathbb{P}_p\left(n \leq |C| < \infty\right) \leq \frac{c_2}{n^{1/2-\varepsilon}}. \tag{5}$$

As also explained in Section 5, if $p_c\left(\mathbb{Z}^d\right) > \frac{1}{r}$, this intermediate regime disappears.

An interesting class of graphs to investigate in regard to Question 1 are products of $\mathbb{Z}^d$ with a homogeneous tree.

Question 2 *Is there exponential decay in the supercritical regime for $\mathbb{Z}^d \times T_r$ where T_r is the homogeneous r-ary tree?*

We now move to the amenable case.

Conjecture 1 *Let $G := (\mathcal{V}, \mathcal{E})$ be a transitive amenable graph. Then there is a sequence $\alpha_n = o(n)$, such that for $p > p_c(G)$*

$$\mathbb{P}_p\left(n \leq |C| < \infty\right) \geq \exp\left(-\eta(p)\,\alpha_n\right) \quad \forall\ n \geq 1\,, \tag{6}$$

where $\eta(p) < \infty$.

It turns out that the argument of Aizenman, Delyon and Souillard [2, 13] for proving this sub-exponential behavior for $\mathbb{Z}^d$ can be successfully carried out for a large class of transitive amenable graphs. For $\mathbb{Z}^d$, the sequence $\{\alpha_n\}$ can be taken to be $\{n^{\frac{d-1}{d}}\}$.

Theorem 3 *If $G := (\mathcal{V}, \mathcal{E})$ is a Cayley graph of a finitely* presented *amenable group with* one end*, then there is a sequence $\alpha_n = o(n)$ such that for $p > p_c(G)$, there is $\eta(p) < \infty$ such that*

$$\mathbb{P}_p\left(n \leq |C| < \infty\right) \geq \exp\left(-\eta(p)\,\alpha_n\right) \quad \forall\ n \geq 1\,. \tag{7}$$

We finally point out that transitivity is a necessary condition in Conjecture 1.

Proposition 4 *There is an amenable nontransitive graph with $p_c < 1$ for which one has exponential decay of the cluster size distribution at all $p \neq p_c$.*

This paper concerns itself mostly with the supercritical case. It therefore seems appropriate to end this introduction with a few comments concerning the subcritical case. It was shown independently in [25] and [1] that for $\mathbb{Z}^d$ in the subcritical regime, the size of the cluster of the origin has a finite expected value. While it seems that the argument in [25] does not work for all transitive graphs as it seems that it is needed that the balls in the graph grow slower than e^{n^γ} for some $\gamma < 1$, it is stated in [27] that the argument in [1] goes through for any transitive graph. Theorem 6.75 in [13] (due to [3]) states that for $\mathbb{Z}^d$, if the expected size of the cluster is finite, then exponential decay of the tail of the cluster size follows. As stated in [3], this result holds quite generally in transitive situations and so, in combination with the statement in [27] referred to above, for all transitive graphs, one has exponential decay of the cluster size in the subcritical regime.

We point out however, not surprisingly, that transitivity is again needed here. An example of a graph which does not have exponential decay in (a portion of) the subcritical regime is obtained by taking the positive integers, planting a binary tree of depth a_k (sufficiently large) at k for $k \geq 1$ and also attaching to the origin a graph whose critical value is say $3/4$. This graph has $p_c = 3/4$ but for some $p < 3/4$, exponential decay fails.

We mention that Questions 3 and 4 which appear later on and arise naturally in our study could also be of interest to people in geometric group theory.

The rest of the paper is organized as follows. In Section 2, we provide all the necessary definitions and notations. In Section 3, we prove Theorem 1. In Section 4, we prove Theorem 3 and Proposition 4. Finally, in Section 5, we prove Theorem 2 as well as study the variant of the example in Theorem 2 obtained by taking $p_c\left(\mathbb{Z}^d\right) > \frac{1}{r}$ instead.

2 Definitions and notations

Let $G = (\mathcal{V}, \mathcal{E})$ be an infinite, connected graph. We will say G is *locally finite* if every vertex has finite degree.

The i.i.d. Bernoulli bond percolation with probability $p \in [0,1]$ on G is a probability measure on $\{0,1\}^{\mathcal{E}}$, such that the coordinate variables are i.i.d. with Bernoulli (p) distribution. This measure will be denoted by $\mathbb{P}_p$. For a given configuration in $\{0,1\}^{\mathcal{E}}$, it is customary to say that an edge $e \in \mathcal{E}$ is *open* if it is in state 1, otherwise it is said to be *closed*. Given a configuration, write $\mathcal{E} = \mathcal{E}_o \cup \mathcal{E}_c$, where $\mathcal{E}_o$ is the set of all open edges and $\mathcal{E}_c$ is the set of all closed edges. The connected components of the subgraph $(\mathcal{V}, \mathcal{E}_o)$ are called the *open connected components* or *clusters*.

One of the fundamental quantities in percolation theory is the critical probability $p_c(G)$ defined by

$$p_c(G) := \inf \left\{ p \in [0,1] \,\middle|\, \mathbb{P}_p(\exists \text{ an infinite cluster }) = 1 \right\} . \tag{8}$$

The percolation model is said to be *subcritical*, *critical* or *supercritical* regime depending on whether $p < p_c(G)$, $p = p_c(G)$ or $p > p_c(G)$ respectively.

For a fixed vertex $v \in \mathcal{V}$, let $C(v)$ be the open connected component containing the vertex v. Let

$$\theta_G^v(p) := \mathbb{P}_p(C(v) \text{ is infinite }) . \tag{9}$$

For a connected graph G, it is easy to show that irrespective of the choice of the vertex v

$$p_c(G) = \inf \left\{ p \in [0,1] \,\middle|\, \theta_G^v(p) > 0 \right\} . \tag{10}$$

Definition 2 *We will say a graph $G = (\mathcal{V}, \mathcal{E})$ is* transitive *if for every pair of vertices u and v there is an automorphism of G, which sends u to v. In other words, a graph G is transitive if its automorphism group* $\texttt{Aut}(G)$ *acts transitively on $\mathcal{V}$.*

Observe that if G is transitive then we can drop the dependency on the vertex v in (9), and then we can write $\theta_G(p) = \mathbb{P}_p(C(v_0) \text{ is infinite })$ for a fixed vertex v_0 of G. $\theta_G(\cdot)$ is called the *percolation function* for a transitive graph G.

We now give definitions of some of the qualitative properties of a graph G which are important for our study.

Definition 3 *Let $G := (\mathcal{V}, \mathcal{E})$ be an infinite, locally finite, connected graph. The* Cheeger constant *of G, denoted by $\kappa(G)$, is defined by*

$$\kappa(G) := \inf \left\{ \frac{|\partial W|}{|W|} \middle| \emptyset \neq W \subseteq \mathcal{V} \text{ and } |W| < \infty \right\} \tag{11}$$

where $\partial W := \left\{ u \notin W \middle| \exists v \in W, \text{ such that } \{u, v\} \in \mathcal{E} \right\}$ is the external vertex boundary*. The graph G is said to be* amenable *if $\kappa(G) = 0$; otherwise it is called* nonamenable*.*

A variant and weaker property than the above is the following.

Definition 4 *Let $G := (\mathcal{V}, \mathcal{E})$ be an infinite, locally finite, connected graph. We define the* anchored Cheeger constant *of G with respect to the vertex v_0 by*

$$\kappa'(G, v_0) := \inf \left\{ \frac{|\partial W|}{|W|} \middle| v_0 \in W \subseteq \mathcal{V}, W \text{ connected and } |W| < \infty \right\} \tag{12}$$

where ∂W is defined as above. The graph G is said to be strongly amenable *if $\kappa'(G, v_0) = 0$, otherwise it is called* weakly nonamenable*.*

It is easily argued that for a connected graph G, $\kappa'(G, v_0) = 0$ implies that $\kappa'(G, v) = 0$ for every vertex v and so the definition of strong amenability (or weak nonamenability) does not depend on the choice of the vertex v_0. Of course, the value of the constant $k'(G, v_0)$ may depend on the choice of v_0 in the weakly nonamenable case. It follows by definition that $\kappa(G) \leq \kappa'(G, v_0)$ for any v_0 and so strong amenability implies amenability. On the other hand, it is easy to show that the two notions are not equivalent although if G is transitive then they are equivalent.

A special class of transitive graphs which are associated with finitely generated groups are the so-called *Cayley graphs*.

Definition 5 *Given a finitely generated group $\bar{G}$ and a symmetric generating set $\mathcal{S}$ (symmetric meaning that $\mathcal{S}$=$\mathcal{S}^{-1}$), a graph $G := (\mathcal{V}, \mathcal{E})$ is called the* left-Cayley graph *of $\bar{G}$ obtained using $\mathcal{S}$ if the vertex set of G is $\bar{G}$ and the edge set is $\left\{ \{u, v\} \,\middle|\, v = su \text{ for some } s \in \mathcal{S} \right\}$.*

Similarly we can also define a *right-Cayley graph* of the group $\bar{G}$ obtained using $\mathcal{S}$. Observe that the left- and right-Cayley graphs obtained using the same symmetric generating set are isomorphic, where an isomorphism is given by the group involution $u \mapsto u^{-1}$, $u \in \bar{G}$. It is also easy to see that multiplication on the right by any element in $\bar{G}$ is a graph automorphism of any left-Cayley graph.

If not explicitly mentioned otherwise, by a *Cayley graph* of a finitely generated group $\bar{G}$, we will always mean a left-Cayley graph with respect to some symmetric generating set.

Definition 6 *A group is* finitely presented *if it is described by a finite number of generators and relations.*

Definition 7 *A graph is* one-ended *if when one removes any finite subset of the vertices, there remains only one infinite component. A group is* one-ended *if its Cayley graph is; it can be shown that this is then independent of the generators used to construct the Cayley graph.*

3 Limit of the tail of the cluster size distribution for Cayley graphs

In this section, we prove Theorem 1. Throughout this section o will denote the identity element of our group.

A Cayley graph is said to have *polynomial growth* if the size of a ball is bounded by some polynomial (in its radius). Given a finitely generated group,

its Cayley graph having polynomial growth does not depend on the choice of the finite symmetric generating set. It is well known (see [19]) that the growth of a Cayley graph of polynomial growth is always between $\alpha\, r^k$ and $\frac{1}{\alpha} r^k$, for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$ and that if a Cayley graph is not of polynomial growth, then for any polynomial $p(n)$, the ball of radius n around o is larger than $p(n)$ for all but at most finitely many n.

Let G be a Cayley graph with degree d. Denote by C_x the open component of vertex x; C will stand for the open component of o. As usual, for a (not necessarily induced) subgraph H of G, $E(H)$ is the edge set and $V(H)$ is the vertex set of H. Given some $p \in [0,1]$, let $\pi_n := \mathbb{P}_p\left(|C| = n\right)$.

Lemma 5 *If G is a Cayley graph of linear or of quadratic growth, then*

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{P}_p\left(|C| = n\right)$$

exists for every $p \in (0,1)$.

Proof: If G has quadratic growth then the vertices of G can be partitioned into finite classes, so-called *blocks of imprimitivity*, in such a way that the group of automorphisms restricted to the classes is $\mathbb{Z}^2$, see [28]. Now we can mimic the proof of the claim for $\mathbb{Z}^2$, see [13]: use the subadditive theorem and the fact that for any two connected finite subgraphs of G, one of them has a translate that is disjoint from the other, but at bounded distance from it. For Cayley graphs of linear growth, one can proceed along the same arguments, since a partition into blocks of imprimitivity, as above, exists (see [19]). ■

Before starting the proof of Theorem 1, we first prove the following lemma which gives an important estimate for Cayley graphs with at least cubic growth. Using the simple structure of Cayley graphs of linear or quadratic growth, Lemma 6 is true for every Cayley graph. (In the latter two cases, $(|A| + |B|)^{3/4}$ can be replaced by 1.)

Lemma 6 *Let G be a Cayley graph of at least cubic growth and $A, B \subset G$ be connected subgraphs. Then there is a $\gamma \in \mathrm{Aut}(G)$ such that the translate γA is disjoint from B and* $\mathrm{dist}(\gamma A, B) \leq (|A| + |B|)^{3/4}$.

Proof: Let $\mathcal{A}_n$ be the set of all connected subgraphs of size n in G that contain the o. Fix Γ to be the group whose Cayley graph G is. Note that Γ acts vertex-transitively by right multiplication on G and only the identity of Γ has a fixed point. For a vertex x of G, let $\gamma_x \in \Gamma$ be the (unique) element of Γ that takes o to x. Finally, for a subgraph H of G denote by H' the 1-neighborhood of H (that is, the set of vertices at distance ≤ 1 from H). Note that if H is connected and $|V(H)| > 1$, then $|V(H')| \leq d|V(H)|$, because every point of H has at most $d-1$ neighbors outside of H.

Let $n, m > 1$ and $A \in \mathcal{A}_n$, $B \in \mathcal{A}_m$. Suppose that for some $\gamma \neq \bar{\gamma} \in \Gamma$ there is a point x in A' such that $\gamma B'$ and $\bar{\gamma} B'$ both contain x. Then, by the choice of Γ, $\gamma^{-1}x \neq \bar{\gamma}^{-1}x$. Since $\gamma^{-1}x, \bar{\gamma}^{-1}x \in B'$, we conclude that every $x \in A'$ is contained in at most $|V(B')|$ translates of B' by Γ. Hence there are at most $|V(A')|\,|V(B')|$ translates of B' that intersect A'. Since G has at least cubic growth, so there is a constant $\alpha > 0$ such that, the ball of radius $(n+m)^{3/4}$ around o contains at least $\alpha(n+m)^{9/4}$ points, which is greater than $|V(A')|\,|V(B')| \leq d^2 nm$ for m, n sufficiently large. Therefore there exists a vertex $x_{A,B}$ in this ball of radius $(n+m)^{3/4}$ such that $\gamma_{x_{A,B}} B'$ does not intersect A'. Fix such an $x_{A,B}$. Fix some path $P(A,B)$ of minimal length between A and $x_{A,B}$, denote its length by $|P(A,B)|$. By the choice of $x_{A,B}$ we have $|P(A,B)| \leq (n+m)^{3/4}$. Taking $\gamma := \gamma_{x_{A,B}}^{-1}$ completes the proof. ■

Proof of Theorem 1: For graphs of linear or quadratic growth, the theorem follows from Lemma 5.

Assume now that our group has at least cubic growth and so the ball of radius r has volume $\geq \alpha r^3$ with some $\alpha > 0$ by the facts about Cayley graphs that we mentioned earlier. Fix Γ as in the proof of the previous lemma.

The generalized subadditive limit theorem (see Theorem II.6 in the Appendix

of [13]) gives the result if we can show that

$$\pi_{m+n} \geq \pi_m \pi_n c^{(m+n)^{3/4} \log_2(m+n)} \tag{13}$$

whenever m and n are sufficiently large, where $0 < c = c(d,p) < 1$ is some constant depending only on d and p.

We will first show that

$$2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}\pi_{m+n}$$

$$\geq \sum_{A\in\mathcal{A}_n}\sum_{B\in\mathcal{A}_m} \mathbb{P}_p\left(C=A\right)\mathbb{P}_p\left(C_{x_{A,B}}=\gamma_{x_{A,B}}(B)\right)p^{(n+m)^{3/4}}(1-p)^{2d(n+m)^{3/4}}$$

where $c_1(d)$ is a constant depending only on d. We will then show that the theorem will follow easily from here.

To prove the above inequality let $A \in \mathcal{A}_n$, $B \in \mathcal{A}_m$. Define $x_{A,B}$ and $\gamma_{x_{A,B}}$ as in the proof of Lemma 6. Let $U(A,B)$ be defined as the union of three graphs: $U(A,B) := A \cup \gamma_{x_{A,B}} B \cup P(A,B)$. Fix some arbitrary $\tilde{X}(A,B) \subset U(A,B)$ set of vertices not containing o such that the subgraph $K(A,B) := U(A,B) \setminus \tilde{X}(A,B)$ is connected and $|V(K(A,B))| = n+m$. Then let $X(A,B)$ be the subgraph of $U(A,B)$ consisting of the edges incident to some element of $\tilde{X}(A,B)$.

For fixed $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_m$ we obtain

$$\begin{aligned} & \mathbb{P}_p\left(C=K(A,B)\right) \\ \geq\ & \mathbb{P}_p\left(C=A\right)\mathbb{P}_p\left(C_{x_{A,B}}=\gamma_{x_{A,B}}(B)\middle|\, C=A\right)p^{|P(A,B)|}(1-p)^{2d|P(A,B)|} \end{aligned}$$

by first opening the edges of $P(A,B)$, closing the other edges incident to the inner vertices of $P(A,B)$ but not in $A \cup \gamma_{x_{A,B}}(B)$, and finally closing every edge incident to some element of $X(A,B)$, whenever it is necessary. The events $\{C=A\}$ and $\{C_{x_{A,B}} = \gamma_{x_{A,B}}(B)\}$ are independent because they are determined by disjoint sets of edges, since $x_{A,B}$ was chosen such that A' and $\gamma_{x_{A,B}}(B')$ are

disjoint. Hence the previous inequality can be rewritten as

$$\begin{aligned} & \mathbb{P}_p\left(C = K(A,B)\right) \\ \geq \; & \mathbb{P}_p\left(C = A\right)\mathbb{P}_p\left(C_{x_{A,B}} = \gamma_{x_{A,B}}(B)\right) p^{(m+n)^{3/4}}(1-p)^{2d(m+n)^{3/4}} \end{aligned} \tag{14}$$

also using $|P(A,B)| \leq (n+m)^{3/4}$.

Now we will show that a given $K \in \mathcal{A}_{m+n}$ can be equal to $K(A,B)$ for at most $2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}$ pairs (A,B), where $c_1(d)$ is a constant depending only on d. First, given m and n, $U(A,B)$ determines (A,B) up to a factor

$$2^{(\log_2(m+n)+1+\log_2 d)(m+n)^{3/4}} m\,.$$

This is because of the following reason. An upper bound for the number of choices for the edges of $P(A,B) \setminus (A \cup \gamma_{x_{A,B}} B)$ from $U(A,B)$ is

$$|E(U(A,B))|^{(m+n)^{3/4}} \leq (2d(m+n))^{(m+n)^{3/4}} = 2^{(\log_2(m+n)+1+\log_2 d)(m+n)^{3/4}}\,,$$

using $|P(A,B)| \leq (n+m)^{3/4}$ and $|E(U(A,B))| \leq (n+m+(n+m)^{3/4})d$. If we delete the edges of $P(A,B) \setminus (A \cup \gamma_{x_{A,B}} B)$ from $U(A,B)$, we get back $A \cup \gamma_{x_{A,B}} B$. This has two components, so one of them is A and the other one is $\gamma_{x_{A,B}} B$. The set $\gamma_{x_{A,B}} B$ may coincide for at most $|V(B)| = m$ many different B's (all being Γ-translates of $\gamma_{x_{A,B}} B$ to o, using again the choice of Γ). We conclude that the number of (A,B) pairs that give the same $U(A,B)$ is at most $2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m$. Now, $X(A,B)$ is $U(A,B) \setminus E(K(A,B))$ without its isolated points (points of degree 0) and so for a given $K \in \mathcal{A}_{m+n}$,

$$\begin{aligned} & |\{(A,B) : K(A,B) = K\}| \\ \leq \; & 2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m\, |\{U(A,B) : K(A,B) = K\}| \\ = \; & 2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m\, |\{X(A,B) : K(A,B) = K\}|\,. \end{aligned}$$

We will bound the cardinality of the set on the right side, with this fixed

K. Given A and B, $X(A,B)$ is such a graph that the union $K(A,B) \cup X(A,B)$ is connected, and $|V(X(A,B))| \leq d(n+m)^{3/4}$ (since X is contained in the 1-neighborhood of $\tilde{X}$, and $|\tilde{X}| \leq (n+m)^{3/4}$). To find an upper bound for the number of possible $X(A,B)$'s with these two properties (and hence where possibly $K(A,B) = K$), we first specify the vertices of $K(A,B)$ that are also in $X(A,B)$ (at most $\binom{n+m}{d(n+m)^{3/4}}$ possibilities). If $X(A,B)$ has k components, with some arbitrary fixed ordering of the vertices of G, let x_i be the first element of $K(A,B) \cap X(A,B)$ in the i'th component. Then for each x_i choose the size of the component of $X(A,B)$ that contains it. There are at most $2^{d(n+m)^{3/4}+1}$ total ways to do this because the number of ways to express an integer k as the ordered sum of positive integers (which would be representing the sizes of the different components) is at most 2^k and then we can sum this up from 1 to $d(n+m)^{3/4}$ corresponding to the different possible sizes for the vertex size of $X(A,B)$. Next, we finally choose the components themselves. It is known that the number of lattice animals on ℓ vertices is at most $7^{2d\ell}$ (see (4.24) in [13]) which gives us a total bound of $7^{2d^2(n+m)^{3/4}}$ for the number of ways to choose all the components. Note that we did not have to choose x_i, since x_i is determined by $X(A,B) \cap K(A,B)$ as soon as we know the components of the x_j for all $j < i$. Calculations similar to the above can be found in [29]. We obtain an upper bound of

$$2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m \binom{n+m}{d(n+m)^{3/4}} 2^{d(n+m)^{3/4}+1} 7^{2d^2(n+m)^{3/4}}$$

for the number of all possible pairs (A,B) that define the same $K = K(A,B)$ for some connected subgraph K with $n+m$ vertices, whenever m and n are not too small. Bounding the binomial coefficient by $(n+m)^{d(n+m)^{3/4}}$, it easy to see that this is at most $2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}$ for some constant $c_1(d)$.

Since every $K(A,B)$ is in $\mathcal{A}_{m+n}$, the first inequality below follows from this

last estimation on the overcount. The second one is a consequence of (14)

$$
\begin{aligned}
& 2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}\pi_{m+n} \\
\geq \quad & \sum_{A\in\mathcal{A}_n}\sum_{B\in\mathcal{A}_m} \mathbb{P}_p\left(C=K(A,B)\right) \\
\geq \quad & \sum_{A\in\mathcal{A}_n}\sum_{B\in\mathcal{A}_m} \mathbb{P}_p\left(C=A\right)\mathbb{P}_p\left(C_{x_{A,B}}=\gamma_{x_{A,B}}(B)\right)p^{(n+m)^{3/4}}(1-p)^{2d(n+m)^{3/4}} \\
\geq \quad & \pi_n\pi_m\beta^{(n+m)^{3/4}} \\
\geq \quad & \pi_n\pi_m\beta^{(n+m)^{3/4}\log_2(m+n)}\,,
\end{aligned}
$$

where $\beta := p(1-p)^{2d} \in (0,1)$, whenever m and n are large enough. This yields Equation (13) with an appropriate choice of $c(d,p)$ as desired and proves the theorem. ■

Remarks: The following claim seems intuitively clear, but "continuity" arguments that work for Z^d (or more generally, for so-called ordered groups) fail for arbitrary groups. If it were true, then the proof of Theorem 1 would become significantly simpler: the subadditive theorem could be applied almost right away.

Question 3 *Let G be a transitive graph. Is there a constant c depending on G such that for any finite subgraphs A and B there is an automorphism γ such that γA and B are disjoint and at distance c from each other?*

Our Lemma 6 only shows that there exists a γ such that A and γB are at distance $\leq (|A|+|B|)^{3/4}$. As observed by Iva Kozáková (personal communication), one cannot have a positive answer to Question 3 with $c=1$ for all groups. An example showing this is the free product of a cycle of length 3 and a cycle of length 4, with A and B equal to cycles of length 3 and 4 respectively.

It is worth noting that for a Cayley graph of a so-called *ordered group*, the proof of Theorem 1 is rather straightforward. This is primarily because of the remarks made above. In this case the proof is really a generalization of the proof for $\mathbb{Z}^d$. Interesting enough one can also show that on the infinite regular

tree with degree 3 (with is not a Cayley graph of an ordered group) such an argument does not work. Still, Theorem 1 holds of course for it and there is in fact an affirmative answer to Question 3 in this case.

4 Sub-exponential decay for certain transitive amenable graphs in the supercritical regime

While Question 1 and Conjecture 1 propose a characterization of amenability via cluster size decay in the supercritical regime (assuming, for completeness, the widely believed conjecture [10], that $p_c < 1$ whenever G grows faster than linear), a conjecture of Pete suggests that this sharp contrast vanishes from a slightly different point of view. Instead of the size of the cluster, consider the size of its boundary. It is known from Kesten and Zhang [23] that when $G = \mathbb{Z}^d$, for all $p > p_c$, there exists a k such that the probability that the exterior boundary of the k-closure (see Definition 8) of a finite supercritical cluster has size $\geq n$ decays exponentially in n. (This is not true without taking the closure, as also shown in [23] for $p \in (p_c, 1 - p_c)$.) This led Pete to conjecture that for any transitive graph and supercritical p, there exists a constant $k = k(p)$ such that $\mathbb{P}(n < |\partial_k^+ C(o)| < \infty) \leq \exp(-cn)$, where $\partial_k^+ C(o)$ denotes the exterior boundary of the k-closure of the cluster of o. See [26] for applications.

Before starting on the proof of Theorem 3, we prove the following (technical) lemma which will be needed in the proof.

Lemma 7 *Let G be an amenable Cayley graph. Then there is a sequence $\{W_n\}_{n\geq 1}$ of subsets of $\mathcal{V}$ such that for every $n \geq 1$ the induced graph on W_n is connected and*

$$\lim_{n\to\infty} \frac{|\partial W_n|}{|W_n|} = 0. \tag{15}$$

Moreover, $\sup_n \frac{|W_{n+1}|}{|W_n|} < \infty$.

Proof: For groups of linear or quadratic growth, define W_n to be the ball of radius n and it is immediate. (In fact, for all groups of polynomial growth,

the (nontrivial) facts we mentioned earlier concerning them implies that we can take W_n to be the ball of radius n in these cases as well.)

We now assume that the group as at least cubic growth rate. Since G is amenable, there exists a sequence $\{W_n\}_{n\geq 1}$ of nonempty finite subsets of $\mathcal{V}$ such that for every $n \geq 1$ the induced subgraph on W_n is connected and satisfies equation (15). (In the definition of amenability, the W_n's are not necessarily connected, but it is easy to check that they may be taken to be.) Without loss of generality, we can also assume $|W_n| \leq |W_{n+1}|$.

Now, whenever $|W_{n+1}| / |W_n| > 3$ we will add a new set E in the Følner sequence, after W_n, with the property that E is connected, that $|E|/|W_n| \leq 3$, and such that $|\partial E|/|E| \leq |\partial W_n|/|W_n| + 2d/|W_n|^{1/4}$. The lemma then can be proved by repeating this procedure as long as there are two consecutive sets in the sequence whose sizes have ratio greater than 3.

So all what is left, is to show the existence of such an E. Now, apply Lemma 6 with A and B both chosen to be W_n. Take the union of A, γB, and the path of length $\leq (|A| + |B|)^{3/4}$ between A and $\gamma(B)$. Let the resulting graph be E. Clearly E satisfies the condition about its size. It also satisfies the isoperimetric requirement, because $|\partial E| \leq 2|\partial W_n| + (2d|W_n|)^{3/4}$ and $|E| \geq 2|W_n|$, where d is the degree of a vertex in G. This completes the proof. ■

Proof of Theorem 3: Let $\{W_n\}_{n\geq 1}$ be a sequence of subsets of $\mathcal{V}$ satisfying the conditions of Lemma 7.

For a finite set $W \subseteq V(G)$, let $\partial_{\mathtt{Ext}} W$ be the set of $v \in \partial W$ for which there exists a path from v to ∞ which lies (other than v) in $V(G) \setminus (W \cup \partial W)$. It is easy to see that if the induced graph on W is connected, then for any vertex $w \in W$ the set $\partial_{\mathtt{Ext}} W$ is a *minimal cutset* between w and ∞. From [6, 29] we know that, since we are assuming the graph G is a Cayley graph of a finitely presented group with one end, there exists a positive integer t_0, such that any

minimal cutset Π between any vertex v and ∞ must satisfy

$$\forall A, B \text{ with } \Pi = A \cup B, \mathrm{dist}_G(A, B) \leq t_0. \tag{16}$$

Letting $U^t := \left\{ v \in V(G) \,\middle|\, \mathrm{dist}_G(v, U) \leq t \right\}$ for any $U \subseteq V(G)$, and $t \in \mathbb{N}$, it is not hard to deduce from the above that for any connected finite subset of vertices W, we have that the induced subgraph on $(\partial_{\mathtt{Ext}} W)^{t_0}$ is connected. In particular, it follows that for each $n \geq 1$ the induced graph on $(\partial_{\mathtt{Ext}} W_n)^{t_0}$ is connected, and further by using (15) we get

$$\lim_{n \to \infty} \frac{\left| (\partial_{\mathtt{Ext}} W_n)^{t_0} \right|}{|W_n|} = 0\,. \tag{17}$$

Now the proof by Aizenman, Delyon and Souillard [2] as given in [13] (see page 218), essentially goes through when we replace a "n-ball" of $\mathbb{Z}^d$ by W_n, and the "boundary of a n-ball" by $(\partial_{\mathtt{Ext}} W_n)^{t_0}$, leading to the sub-exponential bound (7). The point of Lemma 7 is that we need to obtain the claim in the theorem for all n; without Lemma 7, we could only make the conclusion for a sequence of n going to ∞. ■

Remarks: Note that to carry out the above proof, we do not need that (16) holds for all minimal cutsets but only for some fixed Følner sequence, i.e. for a sequence of connected W_n's satisfying (15). Thus a positive answer to the following question would imply Theorem 3 for an arbitrary amenable group.

Definition 8 *The* k-closure *of a graph G is defined to be the graph on the vertex set of G with an edge between two vertices if and only if their distance in G is at most k.*

Hence (16) is equivalent to saying that any minimal cutset Π of G is connected in the t_0-closure of G.

Question 4 *Does every amenable graph have a Følner sequence $\{W_n\}$ (that is, a sequence satisfying Equation (15)), such that for some k the k-closure of ∂W_n*

is connected for every n?

In [29], an example of a Cayley graph (coming from the so-called the lamplighter group) with one end, whose "usual" Følner sequence does not satisfy the above property for its minimal cutsets, is given. As a consequence, the lamplighter group is not finitely presented (which was first shown in [7]), and hence 3 does not apply to this case. However, the lamplighter group is not a counterexample to Question 4, as shown by the following construction. Consequently, it has subexponential decay of cluster size probabilities in the supercritical regime.

Example 1 Recall that, informally, the lamplighter group G is defined as follows. An element of G is a labeling of $\mathbb{Z}$ with labels "on" or "off", with only finitely many on, together with one specified element of $\mathbb{Z}$, the position of the lamplighter. Take the element when we move the lamplighter one step to the right (corresponding to multiplication from the right by the element with all the lamps off and the lamplighter at 1), and the element when we switch the lamp where the lamplighter is (corresponding to the element when the lamplighter is in 0 and the lamp there is the only one on), as a set of generators for the right-Cayley graph that we consider now. This way we defined multiplication for any two elements. See e.g. [29] for a more formal definition.

Given $x \in G$, let $\pi(x)$ be the position of the lamplighter.

To construct the desired Følner sequence W_n, let B_n be the set of elements x with $\pi(x) \in [1, n]$ and all the lamps outside $[1, n]$ are off. The B_n form the "usual" Følner sequence that we referred to earlier. We shall add paths to B_n to get W_n, in the following way. For each element x of the inner boundary of B_n we will define a path P_x. Note that since x is on the boundary of B_n, $\pi(x)$ is either 1 or n. If $\pi(x) = n$, and if the rightmost lamp that is on is at place $n - k$, P_x will be the following. Start from x, then the lamplighter moves to the $n + k + 1$'th place, switches the lamp on there, then moves back to place $n-k$, switch the lamp, then move to $n+k+1$ again and switch, and then move back to n. The endpoints of P_x are in the boundary of B_n and the interior of

P_x is disjoint from B_n. For those x, where $\pi(x) = 1$, use the above definition but "reflected". Finally, define a path from the point where all lamps are off and the lamplighter is in n to the one where all lamps are off and he is in 1, by sending the lamplighter to $2n$, switch, go to $-n$, switch, back to $2n$, switch, back to $-n$, switch, and then go to 1. Define W_n as the union of B_n and all the P_x, where x is some boundary point of B_n.

We only sketch the proof of that W_n is a Følner sequence with boundaries that are connected in the 2-closure. We leave it for the interested reader to fill out the details.

Look at the 2-closure G_2 of G. Define a graph on the connected components of the inner boundary ∂B_n in this thickening: put an edge between two if some points of the two are connected by a path P_x defined above. One can show that the paths were defined so that this graph is connected. The boundary of a path is clearly connected in the 2-closure, and (one can show that) these path-boundaries are (basically) contained in ∂W_n. One concludes that W_n has a connected boundary in G_2. W_n is Følner, because the paths added (and hence their boundaries, and the boundary of W_n) were constructed so as to have total length constant times 2^n, while W_n has size of order $n2^n$.

We complete this section with the

Proof of Proposition 4:

Let $\mathbb{T}_2^\rho$ be the infinite rooted binary tree, with root ρ. Consider the graph G obtained by attaching an infinite *ray* (a copy of $\mathbb{Z}_+$) at the vertex ρ of the graph $\mathbb{T}_2^\rho$. It is easy to see that G is amenable even though $\mathbb{T}_2^\rho$ is not, but of course is not transitive.

Observe that for the i.i.d. bond percolation on G, the critical probability $p_c(G) = p_c(\mathbb{T}_2) = \frac{1}{2}$. Let C be the open connected component containing the vertex ρ. Fix $0 < p < 1$, it is immediate that under the measure $\mathbb{P}_p$ we have

$$|C| \stackrel{d}{=} X + Y\,,$$

where $X \sim \text{Geometric}\,(p)$ and Y has a distribution same as the total size of a Galton-Watson branching process with progeny distribution $\text{Binomial}\,(2,p)$, and X and Y are independent.

Now for every $p \in (0,1)$ we know that X has an exponential tail. Moreover from classical branching process theory we know that when $p \neq \frac{1}{2}$ we must also have an exponential tail for Y on the event $[Y < \infty]$. This is because when $p < \frac{1}{2}$ the process is subcritical and we can use Lemma 8(a) (given in Section 5); and when $p > \frac{1}{2}$, the process is supercritical, but on the event $[Y < \infty]$ it is distributed according to a subcritical Galton-Watson process (see Theorem I.D.3 on page 52 of [5]). These facts together prove that for every $p \neq \frac{1}{2}$ there is a constant $\lambda\,(p) > 0$, which may depend on p, such that

$$\mathbb{P}_p\,(|C| = n) \leq \exp\,(-\lambda\,(p)\,n) \quad \forall\ n \geq 1\,. \tag{18}$$

■

5 A special non-transitive, nonamenable graph

In this section we will study a particular non-transitive graph, which is also nonamenable; namely, the d-dimensional integer lattice $\mathbb{Z}^d$ with rooted regular trees planted at each vertex of it. More precisely, for each $\mathbf{x} \in \mathbb{Z}^d$, let $\mathbb{T}_r^{\mathbf{x}}$ be an infinite rooted regular tree with degree $(r+1)$ which is rooted at $\mathbf{x}$. Thus each vertex of $\mathbb{T}_r^{\mathbf{x}}$ has degree $(r+1)$ except for the root $\mathbf{x}$, which has degree r. We consider the graph

$$G := \mathbb{Z}^d \bigcup \left(\bigcup_{\mathbf{x} \in \mathbb{Z}^d} \mathbb{T}_r^{\mathbf{x}} \right). \tag{19}$$

Observe that $p_c(G) = \min\left\{p_c\left(\mathbb{Z}^d\right), p_c\left(\mathbb{T}_r^{\mathbf{0}}\right)\right\}$ and recall $p_c\left(\mathbb{T}_r^{\mathbf{0}}\right) = \frac{1}{r}$.

Before we prove Theorem 2, we note that for this particular graph G the two critical points $p_c\left(\mathbb{Z}^d\right)$ and $\frac{1}{r}$ play two different roles. $p_c\left(\mathbb{Z}^d\right)$ is the critical point for the i.i.d. Bernoulli bond percolation on G, but when p is between $p_c\left(\mathbb{Z}^d\right)$

and $\frac{1}{r}$ the cluster size of the origin behaves like the cluster size of the origin for supercritical bond percolation on $\mathbb{Z}^d$. This is of course intuitively clear, because in this region the planted trees are all subcritical. On the other hand when $p > \frac{1}{r}$ then the tree components take over and we have the exponential decay of the cluster size of the origin, conditioned to be finite (see Lemma 8 below).

The following lemma will be needed to prove Theorem 2. This result is classical in the branching process literature, for a proof see [24, 4].

Definition 9 *we say that a nonnegative random variable has an exponential tail if there exists c so that $P(X \geq t) \leq e^{-ct}$ for $t \geq 1$.*

Lemma 8 *Consider a subcritical or critical branching process with progeny distribution N. Let S be the total size of the population starting with one individual.*

(a) If N has an exponential tail and the process is subcritical, then S has an exponential tail.

(b) If the process is critical and N has a finite but non-zero variance, then then there is a constant c (depending on the distribution of N), such that

$$\mathbf{P}\left(S = n\right) \leq \frac{c}{n^{3/2}}\,. \tag{20}$$

Moreover if the offspring distribution is non sub-lattice*, that is, there is no $a \in \mathbb{N}$ and $b \in \mathbb{N} \cup \{0\}$ such that $\mathbf{P}\left(N \in a\mathbb{N} + b\right) = 1$, then*

$$\mathbf{P}\left(S = n\right) \sim \frac{c}{n^{3/2}}\,. \tag{21}$$

Proof of Theorem 2: Let $C_{\mathbb{Z}^d} := C \cap \mathbb{Z}^d$ and $C_{\mathbb{T}_r^{\mathbf{x}}} := C \cap \mathbb{T}_r^{\mathbf{x}}$ for $\mathbf{x} \in \mathbb{Z}^d$. By definition

$$C = \bigcup_{\mathbf{x} \in C_{\mathbb{Z}^d}} C_{\mathbb{T}_r^{\mathbf{x}}}, \tag{22}$$

where the union is a disjoint union. Thus

$$|C| = \sum_{\mathbf{x} \in C_{\mathbb{Z}^d}} \left| C_{\mathbb{T}_r^{\mathbf{x}}} \right| . \tag{23}$$

Moreover, using the special structure of this particular graph G, we observe that under $\mathbb{P}_p$, when conditioned on the random cluster $C_{\mathbb{Z}^d}$, the *tree-components*, $\left\{ C_{\mathbb{T}_r^{\mathbf{x}}} \right\}_{\mathbf{x} \in C_{\mathbb{Z}^d}}$, are independent and identically distributed, each being a family tree of a Galton-Watson branching process with progeny distribution Binomial (r, p).

(a) First of all it is easy to show that for any bounded degree graph the tail of the size of the cluster of any fixed vertex has an exponential lower bound. So the lower bound for our graph follows trivially.

Now for $p < p_c\left(\mathbb{Z}^d\right) = p_c\left(G\right)$ we must have $\mathbb{P}_p\left(|C| < \infty\right) = 1$. The exponential upper bound follows using the representation (23) the following (easy) lemma, who's proof is given later.

Lemma 9 *Let $(S_j)_{j \geq 1}$ be i.i.d. non-negative random variables, which are independent of N, which is a positive integer valued random variable. Let*

$$Z := \sum_{j=1}^{N} S_j \, . \tag{24}$$

Then if S_j's and N have exponential tails, so does Z.

Here we note that by Lemma 8 the summands of the right-hand side of (23) have exponential tail while the random index in the same equation has exponential tail by Theorem 6.75 of Grimmett [13].

For the case $p > \frac{1}{r} > p_c\left(G\right)$, we have $\mathbb{P}_p\left(|C| < \infty\right) < 1$. Observe that

$$[|C| < \infty] = \left[|C_{\mathbb{Z}^d}| < \infty, \text{ and } \left| C_{\mathbb{T}_r^{\mathbf{x}}} \right| < \infty \ \forall \ \mathbf{x} \in C_{\mathbb{Z}^d} \right] . \tag{25}$$

Once again the exponential upper bound follows from the decomposition (23) and using Lemma 9. This is because under the conditional distribution $\mathbb{P}_p\left(\cdot \,\middle|\, |C| < \infty \right)$ summands of the right-hand side of (23) have exponential tail (see Theorem

I.D.3 of [5]), and the index has exponential tail because of the following argument:

$$\begin{aligned}\mathbb{P}_p\left(\left|C_{\mathbb{Z}^d}\right| \geq n \,\middle|\, |C| < \infty\right) &= \frac{\mathbb{P}_p\left(\infty > \left|C_{\mathbb{Z}^d}\right| \geq n \text{ and } \left|C_{\mathbb{T}_r^{\mathbf{x}}}\right| < \infty \ \forall\ \mathbf{x} \in C_{\mathbb{Z}^d}\right)}{\mathbb{P}_p\left(|C| < \infty\right)} \\ &\leq \frac{\left(\mathbb{P}_p\left(\left|C_{\mathbb{T}_r^{\mathbf{0}}}\right| < \infty\right)\right)^n}{\mathbb{P}_p\left(|C| < \infty\right)}.\end{aligned}$$

(b) First we obtain the lower bound for $p \in \left(p_c\left(\mathbb{Z}^d\right), \frac{1}{r}\right)$. We observe that $\mathbb{P}_p\left(\left|C_{\mathbb{T}_r^{\mathbf{x}}}\right| < \infty\right) = 1$ for all $\mathbf{x} \in \mathbb{Z}^d$. So the events $[|C| < \infty]$ and $[|C_{\mathbb{Z}^d}| < \infty]$ are a.s. equal under $\mathbb{P}_p$. But from the decomposition (23) we get $[n \leq |C| < \infty] \supseteq [n \leq |C_{\mathbb{Z}^d}| < \infty]$. So the required lower bound follows from the corresponding lower bound in Theorem 8.61 of Grimmett [13].

Now to get the upper bound note that for every fixed $L > 0$ such that $n/L \in \mathbb{N}$,

$$\mathbb{P}_p\left(n \leq |C| < \infty\right) \leq \mathbb{P}_p\left(\tfrac{n}{L} \leq |C_{\mathbb{Z}^d}| < \infty\right) + \mathbf{P}\left(\sum_{j=1}^{n/L} S_j \geq n\right) \tag{26}$$

where $(S_j)_{j\geq 1}$ are i.i.d. random variables distributed as the total size of a subcritical Galton-Watson branching process with Binomial (r, p) progeny distribution. This again follows from the decomposition (23)

Now from Lemma 8 we get that $\mu := \mathbf{E}\left[S_1\right] < \infty$ and moreover the moment generating function $\mathfrak{M}_{S_1}(s) := \mathbf{E}\left[\exp\left(sS_1\right)\right] < \infty$ for some $s > 0$. Thus using the large deviation estimate Lemma 9.4 of [12] we will get an exponential upper bound for the second summand on the right hand side of (26), by choosing $L > \mu$. But by Theorem 8.65 of Grimmett [13] we get that the first term on the right-hand side of (26) must satisfy an upper bound of the form

$$\mathbb{P}_p\left(\tfrac{n}{L} \leq |C_{\mathbb{Z}^d}| < \infty\right) \leq \exp\left(-\tfrac{\eta(p)}{L^{(d-1)/d}} n^{(d-1)/d}\right), \tag{27}$$

where $\eta(p) > 0$. This proves the required upper bound.

(c) Finally we will prove the polynomial bounds when $p = \frac{1}{r}$. First to get the upper bound, we observe that for any $0 < \beta < 1$ we have

$$\mathbb{P}_p\left(n \leq |C| < \infty\right) \leq \mathbb{P}_p\left(\lfloor n^\beta \rfloor \leq |C_{\mathbb{Z}^d}| < \infty\right) + \mathbf{P}\left(\sum_{j=1}^{\lfloor n^\beta \rfloor} \bar{S}_j \geq n\right) \tag{28}$$

where $\left(\bar{S}_j\right)_{j\geq 1}$ are i.i.d. random variables distributed as the size of a critical Galton-Watson branching process with Binomial$\left(r, \frac{1}{r}\right)$ progeny distribution. This estimate along with Lemma 8(b) yields the desired polynomial upper bound.

To get the lower bound, we note that as in case (b), we also have $\left|C_{\mathbb{T}_r^{\mathbf{x}}}\right| < \infty$ a.s. with respect to $\mathbb{P}_p$, for all $\mathbf{x} \in \mathbb{Z}^d$ and so $[|C| < \infty]$ and $[|C_{\mathbb{Z}^d}| < \infty]$ are a.s. equal. Thus

$$\begin{aligned}
\mathbb{P}_p\left(n \leq |C| < \infty\right) &= \mathbb{P}_p\left(|C| \geq n, |C_{\mathbb{Z}^d}| < \infty\right) \\
&\geq \mathbb{P}_p\left(|C_{\mathbb{Z}^d}| < \infty \text{ and } \left|C_{\mathbb{T}_r^{\mathbf{0}}}\right| \geq n\right) \\
&= \left(1 - \theta_{\mathbb{Z}^d}(p)\right) \mathbb{P}_p\left(\left|C_{\mathbb{T}_r^{\mathbf{0}}}\right| \geq n\right) \\
&= \frac{c''}{n^{1/2}}
\end{aligned}$$

where $c'' > 0$ is a constant. The last equality follows from Lemma 8(b). ■

Remark: The above theorem does not cover the case $p = p_c\left(\mathbb{Z}^d\right)$ and for that we would need exact tail behavior of the cluster-size distribution for critical i.i.d. bond percolation on $\mathbb{Z}^d$. Unfortunately, except for $d = 2$ (see Theorem 11.89 of [13]) and for large d (see [18]), such results are largely unknown.

We now provide a proof of Lemma 9 which is presumably well known.

Proof of Lemma 9: By assumption, there exists $\gamma > 1$ such that the random variable N has a generating function $\phi_N(s) := \mathbf{E}\left[s^N\right]$ which is finite for all $s < \gamma$. Similarly, there exists $c > 0$, such that the moment generating function $\mathfrak{M}_{S_1}(\lambda) := \mathbf{E}\left[\exp\left(\lambda S_1\right)\right] < \infty$ for all $\lambda < c$. By the Lebesgue dominated

convergence theorem, $\mathfrak{M}_{S_1}(\lambda) \to 1$ as $\lambda \downarrow 0$ and so we can find $\lambda_0 > 0$ such that $1 \leq \mathfrak{M}_{S_1}(\lambda_0) < \gamma$.

Now by definition (24), the moment generating function of Z is given by

$$\mathfrak{M}_Z(s) = \phi(\mathfrak{M}_{S_1}(s)) .$$

So in particular $\mathfrak{M}_Z(\lambda_0) < \infty$. Then by Markov inequality we get

$$\mathbf{P}(Z > z) \leq \mathfrak{M}_Z(\lambda_0) \exp(-\lambda_0 z) ,$$

which completes the proof. ■

The following theorem covers the case when $p_c(\mathbb{Z}^d) > \frac{1}{r}$, in which case $p_c(G) = \frac{1}{r}$. It is not surprising that the intermediate regime of sub-exponential decay does not appear in this case. The proof of this theorem is quite similar to that of Theorem 2 so we omit the proof.

Theorem 10 *Suppose $p_c(\mathbb{Z}^d) > \frac{1}{r}$ and let C be the open connected component of the origin $\mathbf{0}$ of G.*

(a) For $p \neq \frac{1}{r}$ we have

$$\exp(-\nu_1(p)\, n) \leq \mathbb{P}_p(n \leq |C| < \infty) \leq \exp(-\nu_2(p)\, n) \quad \forall n \geq 1 , \tag{29}$$

where $\nu_1(p) < \infty$ and $\nu_2(p) > 0$.

(b) For $p = \frac{1}{r}$ the lower bound in equation (5) holds with the same constant $c_1 > 0$, and the upper bound holds for every $\varepsilon > 0$ but with possibly a different constant $c_2' \equiv c_2'(\varepsilon) < \infty$.

Remark: Once again, the case $p = p_c(\mathbb{Z}^d) = \frac{1}{r}$ is left open, because of similar reason as mentioned in the remark after the proof of Theorem 2. (Of course, one would be surprised if there were any d and r (other than $d = r = 2$) where the above held.)

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