A crossover for the bad configurations of random walk in random scenery

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March 17, 2011

This paper is dedicated to the memory of Oded Schramm.

Abstract

In this paper we consider a random walk and a random color scenery on $\mathbb{Z}$. The increments of the walk and the colors of the scenery are assumed to be i.i.d. and to be independent of each other. We are interested in the random process of colors seen by the walk in the course of time. Bad configurations for this random process are the discontinuity points of the conditional probability distribution for the color seen at time zero given the colors seen at all later times.

We focus on the case where the random walk has increments 0, $+1$ or -1 with probability ε, $(1-\varepsilon)p$ and $(1-\varepsilon)(1-p)$, respectively, with $p \in [\frac{1}{2}, 1]$ and $\varepsilon \in [0,1)$, and where the scenery assigns the color black or white to the sites of $\mathbb{Z}$ with probability $\frac{1}{2}$ each. We show that, remarkably, the set of bad configurations exhibits a crossover: for $\varepsilon = 0$ and $p \in (\frac{1}{2}, \frac{4}{5})$ all configurations are bad, while for (p, ε) in an open neighborhood of $(1, 0)$ all configurations are good. In addition, we show that for $\varepsilon = 0$ and $p = \frac{1}{2}$ both bad and good configurations exist. We conjecture that for all $\varepsilon \in [0,1)$ the crossover value is unique and equals $\frac{4}{5}$. Finally, we suggest an approach to handle the seemingly more difficult case where $\varepsilon > 0$ and $p \in [\frac{1}{2}, \frac{4}{5})$, which will be pursued in future work.

MSC 2000. 60G10, 82B20.
Key words and phrases. Random walk in random scenery, conditional probability distribution, bad and good configurations, large deviations.

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Acknowledgment. The research of SB was partially funded by EURANDOM in Eindhoven. The research of SB and FdH was partially supported by DFG and NWO through the Dutch-German Bilateral Research Group on “Mathematics of Random Spatial Models from Physics and Biology” (2004–2009). The research of JES was partially supported by the Swedish Research Council and by the Göran Gustafsson Foundation for Research in the Natural Sciences and Medicine. SB acknowledges travel support from the ESF-program “Random Dynamics in Spatially Extended Systems” (2002–2007). FdH and JES are grateful for hospitality at the Mittag-Leffler Institute in Stockholm in the Spring of 2009, when a part of the research on this paper was carried out. We finally thank the referee for a thorough reading and comments on the paper.

1 Introduction

1.1 Random walk in random scenery

We begin by defining the random process that will be the object of our study. Let $X = (X_n)_{n\in\mathbb{N}}$ be i.i.d. random variables taking the values 0, +1 and −1 with probability ε, $p(1-\varepsilon)$ and $(1-p)(1-\varepsilon)$, respectively, with $\varepsilon \in [0,1)$ and $p \in [\frac{1}{2},1]$. Let $S = (S_n)_{n\in\mathbb{N}_0}$ with $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ be the corresponding *random walk* on $\mathbb{Z}$, defined by

$$S_0 := 0 \qquad \text{and} \qquad S_n := X_1 + \cdots + X_n,\ n \in \mathbb{N},$$

i.e., X_n is the step at time n and S_n is the position at time n. Let $C = (C_z)_{z\in\mathbb{Z}}$ be i.i.d. random variables taking the values B (black) and W (white) with probability $\frac{1}{2}$ each. We will refer to C as the *random coloring* of $\mathbb{Z}$, i.e., C_z is the color of site z. The pair (S,C) is referred to as the *random walk in random scenery* associated with X and C.

Let

$$Y := (Y_n)_{n\in\mathbb{N}_0} \qquad \text{where} \qquad Y_n := C_{S_n}$$

be the sequence of colors observed along the walk. We will refer to Y as the *random color record*. This random process, which takes values in the set $\Omega_0 = \{B,W\}^{\mathbb{N}_0}$ and has full support on Ω_0, will be our main object of study. Because the walk may return to sites it has visited before and see the same color, Y has intricate dependencies. An overview of the ergodic properties of Y is given in [3].

We will use the symbol $\mathbb{P}$ to denote the joint probability law of X and C. The question that we will address in this paper is whether or not there exists a version $V(B \mid \eta)$ of the conditional probability

$$\mathbb{P}\big(Y_0 = B \mid Y = \eta \text{ on } \mathbb{N}\big), \quad \eta \in \Omega_0,$$

such that the map $\eta \mapsto V(B \mid \eta)$ is everywhere continuous on Ω_0. It will turn out that the answer depends on the choice of p and ε.

In [4] we considered the pair (X,Y) and identified the structure of the set of points of discontinuity for the analogue of the conditional probability in the last display. However, (X,Y) is much easier to analyze than Y, because knowledge of X and Y fixes the coloring on the support of X. Consequently, the structure of the set of points of discontinuity for (X,Y) is very different from that for Y. The same continuity question arises for the two-sided version of Y where time is indexed by $\mathbb{Z}$, i.e., the random walk is extended to negative times by putting $S_0 = 0$ and $S_n - S_{n-1} = X_n$, $n \in \mathbb{Z}$, with X_n the step at time $n \in \mathbb{Z}$. In the present paper we will restrict ourselves to the one-sided version.

The continuity question has been addressed in the literature for a variety of random processes. Typical examples include Gibbs random fields that are subjected to some transformation, such as projection onto a lower-dimensional subspace or evolution under a random dynamics. It turns out that even simple transformations can create discontinuities and thereby destroy the Gibbs property. For a recent overview, see [1]. Our main result, described in Section 1.4 below, is a contribution to this area.

1.2 Bad configurations and discontinuity points

In this section we view the conditional probability distribution of Y_0 given $(Y_n)_{n\in\mathbb{N}}$ as a map from $\Omega = \{B,W\}^{\mathbb{N}}$ to the set of probability measures on $\{B,W\}$ (as opposed to a map from Ω_0 to this set). Our question about continuity of conditional probabilities will be formulated in terms of so-called *bad configurations*.

Definition 1.1. *Let $\mathbb{P}$ denote any probability measure on Ω_0 with full support. A configuration $\eta \in \Omega$ is said to be a bad configuration if there is a $\delta > 0$ such that for all $m \in \mathbb{N}$ there are $n \in \mathbb{N}$ and $\zeta \in \Omega$, with $n > m$ and $\zeta = \eta$ on $(0,m) \cap \mathbb{N}$, such that*

$$\Big|\mathbb{P}\big(Y_0 = B \mid Y = \eta \text{ on } (0,n)\cap\mathbb{N}\big) - \mathbb{P}\big(Y_0 = B \mid Y = \zeta \text{ on } (0,n)\cap\mathbb{N}\big)\Big| \geq \delta.$$

In words, a configuration η is bad when, no matter how large we take m, by tampering with η inside $[m,n) \cap \mathbb{N}$ for some $n > m$ while keeping it fixed inside $(0,m) \cap \mathbb{N}$, we can affect the conditional probability distribution of Y_0 in a nontrivial way. Typically, δ depends on η, while n depends on m. A configuration that is not bad is called a *good configuration*.

The bad configurations are the discontinuity points of the conditional probability distribution of Y_0, as made precise by the following proposition (see [6], Proposition 6, and [4], Theorem 1.2).

Proposition 1.2. *Let $\mathbb{B}$ denote the set of bad configurations for Y_0.*
(i) For any version $V(B \mid \eta)$ of the conditional probability $\mathbb{P}(Y_0 = B \mid Y = \eta \text{ on } \mathbb{N})$, the set $\mathbb{B}$ is contained in the set of discontinuity points for the map $\eta \mapsto V(B \mid \eta)$.
(ii) There is a version $V(B \mid \eta)$ of the conditional probability $\mathbb{P}(Y_0 = B \mid Y = \eta \text{ on } \mathbb{N})$ such that $\mathbb{B}$ is equal to the set of discontinuity points for the map $\eta \mapsto V(B \mid \eta)$.

1.3 An educated guess

For the random color record, a *naive guess* is that all configurations are bad when $p = \frac{1}{2}$ because the random walk is recurrent, while all configurations are good when $p \in (\frac{1}{2}, 1]$ because the random walk is transient. Indeed, in the recurrent case we obtain new information about Y_0 at infinitely many times, corresponding to the return times of the random walk to the origin, while in the transient case no such information is obtained after a finite time. However, we will see that this naive guess is wrong. Before we state our main result, let us make an *educated guess*:

- (EG1) $\forall p \in [\frac{1}{2}, \frac{4}{5}]\ \forall \varepsilon \in [0,1)\colon\ \mathbb{B} = \Omega$.
- (EG2) $\forall p \in (\frac{4}{5}, 1]\ \forall \varepsilon \in [0,1)\colon\ \mathbb{B} = \emptyset$.

The explanation behind this is as follows.

Fully biased. Suppose that $p = 1$. Then

$$\mathbb{P}\big(Y_0 = Y_1 \mid Y = \eta \text{ on } \mathbb{N}\big) = \varepsilon + (1-\varepsilon)\tfrac{1}{2},$$

where we use that, for any p and ϵ, S_1 and $(Y_n)_{n\in\mathbb{N}}$ are independent. Hence, the color seen at time 0 only depends on the color seen at time 1, so that $\mathbb{B} = \emptyset$. (Note that if $\varepsilon = 0$, then Y is i.i.d.).

Monotonicity. For fixed ε, we expect monotonicity in p: if a configuration is bad for some $p \in (\frac{1}{2}, 1)$, then it should be bad for all $p' \in [\frac{1}{2}, p)$ also. Intuitively, the random walk with parameters (p', ε) is exponentially more likely to return to 0 after time m than the random walk with parameters (p, ε), and therefore we expect that it is easier to affect the color at 0 for (p', ε) than for (p, ε).

Critical value. For a configuration to be good, we expect that the random walk must have a strictly positive speed conditional on the color record. Indeed, only then do we expect that it is exponentially unlikely to influence the color at 0 by changing the color record after time m. To compute the threshold value for p above which the random walk has a strictly positive speed, let us consider the monochromatic configuration "all black". The probability for the random walk with parameters (p, ε) to behave up to time n like a random walk with parameters (q, δ), with $q \in [\frac{1}{2}, 1]$ and $\delta \in [0, 1)$, is

$$e^{-nH((q,\delta)|(p,\varepsilon))},$$

where

$$\begin{aligned} H\big((q,\delta) \mid (p,\varepsilon)\big) := \delta \log\left(\frac{\delta}{\varepsilon}\right) + (1-\delta)\log\left(\frac{1-\delta}{1-\varepsilon}\right) \\ + (1-\delta)\left[q \log\left(\frac{q}{p}\right) + (1-q)\log\left(\frac{1-q}{1-p}\right)\right] \end{aligned}$$

is the relative entropy of the step distribution (q, δ) with respect to the step distribution (p, ε). The probability for the random coloring to be black all the way up to site $(1-\delta)(2q-1)n$ is

$$(\tfrac{1}{2})^{(1-\delta)(2q-1)n}.$$

The total probability is therefore

$$e^{-nC(q,\delta)} \quad \text{with} \quad C(q,\delta) := H\big((q,\delta) \mid (p,\varepsilon)\big) + (1-\delta)(2q-1)\log 2.$$

The question is: For fixed (p, ε) and $n \to \infty$, does the lowest cost occur for $q = \frac{1}{2}$ or for $q > \frac{1}{2}$? Now, it is easily checked that $q \mapsto C(q, \delta)$ is strictly convex and has a derivative at $q = \frac{1}{2}$ that is strictly positive if and only if $p \in [\frac{1}{2}, \frac{4}{5})$, irrespective of the value of ε and δ. Hence, zero drift has the lowest cost when $p \in [\frac{1}{2}, \frac{4}{5}]$, while strictly positive drift has the lowest cost when $p \in (\frac{4}{5}, 1]$. This explains (EG1) and (EG2).

1.4 Main theorem

We are now ready to state our main result and compare it with the educated guess made in Section 1.3 (see Figure 1).

Theorem 1.3. *(i) There exists a neighborhood of* $(1, 0)$ *in the* (p, ϵ)*-plane for which* $\mathbb{B} = \emptyset$*. This neighborhood can be taken to contain the line segment* $(p_*, 1] \times \{0\}$ *with* $p_* = 1/(1 + 5^5 12^{-6}) \approx 0.997$.
(ii) If $p \in (\frac{1}{2}, \frac{4}{5})$ *and* $\varepsilon = 0$*, then* $\mathbb{B} = \Omega$.
(iii) If $p = \frac{1}{2}$ *and* $\varepsilon = 0$*, then* $\mathbb{B} \notin \{\emptyset, \Omega\}$.

Theorems 1.3(ii–iii) prove (EG1) for $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon = 0$, except for $p = \frac{1}{2}$ and $\varepsilon = 0$, where (EG1) fails. We will see that this failure comes from parity restrictions. Theorem 1.3(i) proves (EG2) in a neighborhood of $(1, 0)$ in the (p, ϵ)-plane. We already have seen that $\mathbb{B} = \emptyset$ when $p = 1$ and $\varepsilon \in [0, 1)$. Note that Theorems 1.3(ii–iii) disprove monotonicity in p for $\varepsilon = 0$. We believe this monotonicity to fail only at $p = \frac{1}{2}$ and $\varepsilon = 0$.

To appreciate why in Theorem 1.3(i) we are not able to prove the full range of (EG2), note that to prove that a configuration is good we must show that the color at 0 cannot be affected by *any* tampering of the color record far away from 0. In contrast, to prove that a configuration is bad it suffices to exhibit *just two* tamperings that affect the color at 0. In essence, the conditions on p and ε in Theorem 1.3(i) guarantee that the random walk has such a large drift that it moves away from the origin no matter what the color record is.

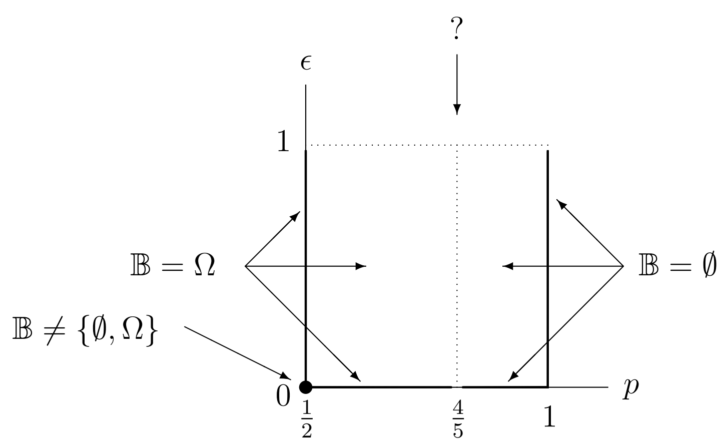

Figure 1: Conjectured behavior of the set $\mathbb{B}$ as a function of p and ϵ. Theorem 1.3 proves this behavior on the left part of the bottom horizontal line and in a neighborhood of the bottom right corner.

We close with (see Figure 1):

Conjecture 1.4. (EG2) *is true.*

Theorem 1.3 is proved in Sections 2–4: (i) in Section 2, (ii) in Section 3 and (iii) in Section 4. It seems that for $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon \in (0,1)$ the argument needed to prove that all configurations are bad is much more involved. In Section 5 we suggest an approach to handle this problem, which will be pursued in future work.

The examples alluded to at the end of Section 1.1 typically have both good and bad configurations. On the other hand, we believe that our process Y has all good or all bad configurations, except at the point $(\frac{1}{2}, 0)$ and possibly on the line segment $\{\frac{4}{5}\} \times [0,1)$. A simple example with such a dichotomy, due to Rob van den Berg, is the following. Let $X = (X_n)_{n\in\mathbb{Z}}$ be an i.i.d. $\{0,1\}$-valued process with the 1's having density $p \in (0,1)$. Let $Y_n = 1\{X_n = X_{n+1}\}$, $n \in \mathbb{Z}$. Clearly, if $p = \frac{1}{2}$, then $Y = (Y_n)_{n\in\mathbb{Z}}$ is also i.i.d., and hence all configurations are good. However, if $p \neq \frac{1}{2}$, then it is straightforward to show that all configurations are bad. See [5], Proposition 3.3.

2 $\mathbb{B} = \emptyset$ for p large and ϵ small

In this section we prove Theorem 1.3(i). The proof is based on Lemmas 2.2–2.4 in Section 2.1, which are proved in Sections 2.2–2.4, respectively. A key ingredient of these lemmas is control of the *cut times* for the walk, i.e., times at which the past and the future of the walk have disjoint supports. Throughout the paper we abbreviate $I_m^n := \{m, \dots, n\}$ for $m, n \in \mathbb{N}_0$ with $m \le n$.

2.1 Proof of Theorem 1.3(i): three lemmas

For $m, n \in \mathbb{N}$ with $m \le n$, abbreviate

$$S_m^n := (S_m, \dots, S_n) \quad \text{and} \quad Y_m^n := (Y_m, \dots, Y_n).$$

The main ingredient in the proof of Theorem 1.3(i) will be an estimate of the number of cut times along S_0^n.

Definition 2.1. *For $n \in \mathbb{N}$, a time $k \in \mathbb{N}_0$ with $k \le n-1$ is a cut time for S_0^n if and only if*

$$S_0^k \cap S_{k+1}^n = \emptyset \quad \textit{and} \quad S_k \ge 0.$$

This definition takes into account only cut times corresponding to locations on or to the right of the origin. Let $CT_n = CT_n(S_0^n) = CT_n(S_1^n)$ denote the set of cut times for S_0^n. Our first lemma reads:

Lemma 2.2. *For $k \in \mathbb{N}_0$, let $\mathcal{E}_k \in \sigma(S_0^k, Y_0^k)$ be any event in the σ-algebra of the walk and the color record up to time k. Then*

$$\mathbb{P}(\mathcal{E}_k \mid k \in CT_n,\, Y_1^n = y_1^n) = \mathbb{P}(\mathcal{E}_k \mid k \in CT_n,\, Y_1^n = \bar{y}_1^n) \tag{2.1}$$

for all $n \in \mathbb{N}$ with $n > k$ and all $y_1^n, \bar{y}_1^n$ such that $y_1^k = \bar{y}_1^k$.

We next define

$$f(m) := \sup_{n\geq m} \max_{y_1^n} \max_{\substack{A\subseteq I_0^{m-1}\\ |A|\geq \frac{m}{2}}} \mathbb{P}(CT_n \cap A = \emptyset \mid Y_1^n = y_1^n), \qquad m \in \mathbb{N}, \tag{2.2}$$

Our second and third lemma read:

Lemma 2.3. *If* $\lim_{m\to\infty} mf(m) = 0$*, then* $\mathbb{B} = \emptyset$*.*

Lemma 2.4. $\limsup_{m\to\infty} \frac{1}{m} \log f(m) < 0$ *for* (p, ε) *in a neighborhood of* $(1, 0)$ *containing the line segment* $(p_*, 1] \times \{0\}$*.*

Note that Lemma 2.4 yields the exponential decay of $m \mapsto f(m)$, which is much more than is needed in Lemma 2.3. Note that Lemmas 2.3–2.4 imply Theorem 1.3(i).

Lemma 2.2 states that, conditioned on the occurrence of a cut time at time k, the color record after time k does not affect the probability of any event that is fully determined by the walk and the color record up to time k. Lemma 2.3 gives the following sufficient criterion for the non-existence of bad configurations: for any set of times up to time m of cardinality at least $\frac{m}{2}$, the probability that the walk up to time $n \geq m$ has no cut times in this set, even when conditioned on the color record up to time n, decays faster than $\frac{1}{m}$ as $m \to \infty$, uniformly in n and in the color record that is being conditioned on. Lemma 2.4 states that for p and ε in the appropriate range, the above criterion is satisfied.

A key formula in the proof of Lemmas 2.2–2.4 is the following. Let $R(s_1^n)$ denote the range of s_1^n (i.e., the cardinality of its support), and write $s_1^n \sim y_1^n$ to denote that s_1^n and y_1^n are *compatible* (i.e., there exists a coloring of $\mathbb{Z}$ for which s_1^n generates y_1^n). Below we abbreviate $\mathbb{P}(S_1^n = s_1^n)$ by $\mathbb{P}(s_1^n)$.

Proposition 2.5. *For all* $n \in \mathbb{N}$*,*

$$\mathbb{P}(S_1^n = s_1^n, Y_1^n = y_1^n) = \mathbb{P}(s_1^n) \, \left(\tfrac{1}{2}\right)^{R(s_1^n)} 1\{s_1^n \sim y_1^n\}.$$

The factor $(\frac{1}{2})^{R(s_1^n)}$ arises because if $s_1^n \sim y_1^n$, then y_1^n fixes the coloring on the support of s_1^n.

2.2 Proof of Lemma 2.2

Proof. Write $\mathbb{P}(\mathcal{E}_k \mid k \in CT_n,\, Y_1^n = y_1^n) = N_k/D_k$ with (use Proposition 2.5)

$$N_k := \sum_{x=0}^{n} \sum_{s_1^n} 1\{s_k = x\}\, 1\{k \in CT_n(s_1^n)\}\, \mathbb{P}(s_1^n) \, \left(\tfrac{1}{2}\right)^{R(s_1^n)} 1\{s_1^n \sim y_1^n\}\, 1\{\mathcal{E}_k\},$$

$$D_k := \sum_{x=0}^{n} \sum_{s_1^n} 1\{s_k = x\}\, 1\{k \in CT_n(s_1^n)\}\, \mathbb{P}(s_1^n) \, \left(\tfrac{1}{2}\right)^{R(s_1^n)} 1\{s_1^n \sim y_1^n\}.$$

Abbreviate $\{S_k^n > x\}$ for $\{S_l > x \;\forall\, k \leq l \leq n\}$, etc. Note that if $k \in CT_n(s_1^n)$, then we have $1\{s_1^n \sim y_1^n\} = 1\{s_1^k \sim y_1^k\} 1\{s_{k+1}^n \sim y_{k+1}^n\}$ and $R(s_1^n) = R(s_1^k) + R(s_{k+1}^n)$. It follows that

$$N_k = \sum_{x=0}^{n} \sum_{s_1^k} 1\{s_k = x\}\, 1\{s_1^k \le x\}\, \mathbb{P}(s_1^k)\ \left(\tfrac{1}{2}\right)^{R(s_1^k)}\ 1\{s_1^k \sim y_1^k\} 1\{\mathcal{E}_k\}$$
$$\times \sum_{s_{k+1}^n} 1\{s_{k+1}^n > x\}\, \mathbb{P}(s_{k+1}^n \mid S_k = x)\ \left(\tfrac{1}{2}\right)^{R(s_{k+1}^n)}\ 1\{s_{k+1}^n \sim y_{k+1}^n\}$$
$$= C_{k,n}(y_{k+1}^n) \sum_{x=0}^{n} \sum_{s_1^k} 1\{s_k = x\}\, 1\{s_1^k \le x\}\, \mathbb{P}(s_1^k)\ \left(\tfrac{1}{2}\right)^{R(s_1^k)}\ 1\{s_1^k \sim y_1^k\}\, 1\{\mathcal{E}_k\}$$

with (shift S_k back to the origin)

$$C_{k,n}(y_{k+1}^n) := \left[\sum_{s_1^{n-k}} 1\{s_1^{n-k} > 0\}\, \mathbb{P}(s_1^{n-k})\ \left(\tfrac{1}{2}\right)^{R(s_1^{n-k})}\ 1\{s_1^{n-k} \sim y_{k+1}^n\}\right].$$

Likewise, we have

$$D_k = C_{k,n}(y_{k+1}^n) \sum_{x=0}^{n} \sum_{s_1^k} 1\{s_k = x\}\, 1\{s_1^k \le x\}\, \mathbb{P}(s_1^k)\ \left(\tfrac{1}{2}\right)^{R(s_1^k)}\ 1\{s_1^k \sim y_1^k\}.$$

The common factor $C_{k,n}(y_{k+1}^n)$ cancels out and so N_k/D_k only depends on y_1^k. Therefore, as long as $y_1^k = \bar{y}_1^k$, we have the equality in (2.1). □

2.3 Proof of Lemma 2.3

Proof. Since $f(m) \le \frac{1}{2}$ for all large m, we will assume that all the values of m arising in the proof below satisfy this.

For $n \in \mathbb{N}$ and y_1^n and $\bar{y}_1^n$, define

$$\Delta^n(y_1^n, \bar{y}_1^n) := \mathbb{P}(Y_0 = B \mid Y_1^n = y_1^n) - \mathbb{P}(Y_0 = B \mid Y_1^n = \bar{y}_1^n).$$

We will show that if $\lim_{n\to\infty} m f(m) = 0$, then

$$\lim_{m\to\infty} \sup_{n\ge m} \max_{\substack{y_1^n, \bar{y}_1^n \\ y_1^{m-1} = \bar{y}_1^{m-1}}} \mid \Delta^n(y_1^n, \bar{y}_1^n) \mid = 0, \tag{2.3}$$

and hence $\mathbb{B} = \emptyset$ by Definition 1.1.

In what follows, we

$$\text{fix } m, n \in \mathbb{N} \text{ with } m \le n \text{ and } y_1^n, \bar{y}_1^n \text{ with } y_1^{m-1} = \bar{y}_1^{m-1} \tag{2.4}$$

and abbreviate $\Delta = \Delta^n(y_1^n, \bar{y}_1^n)$. Define

$$A = A_m^n(y_1^n, \bar{y}_1^n) := \Big\{k \in I_0^m \colon\ \mathbb{P}(k \in CT_n \mid Y_1^n = y_1^n) - \mathbb{P}(k \in CT_n \mid Y_1^n = \bar{y}_1^n) \ge -2f(m)\Big\}.$$

Using Lemma 2.2, we will show that

$$|A| \geq \tfrac{m}{2} \tag{2.5}$$

and

$$|\Delta| \leq 2f(m)(m+1). \tag{2.6}$$

The argument we will give works for any choice of y_1^n and $\bar{y}_1^n$ subject to (2.4) (with the corresponding A and Δ). Together with $\lim_{n\to\infty} mf(m) = 0$, (2.6) will prove Lemma 2.3. □

2.3.1 Proof of (2.5)

Proof. Write $B := I_0^{m-1} \backslash A = \{b_1, \ldots, b_{m-|A|}\}$. We will show that $f(m) \leq \frac{1}{2}$ and $|B| > \frac{m}{2}$ are incompatible. Indeed, by the definition of A, we have

$$\mathbb{P}(b_i \in CT_n \mid Y_1^n = y_1^n) - \mathbb{P}(b_i \in CT_n \mid Y_1^n = \bar{y}_1^n) < -2f(m), \qquad i = 1, \ldots, m-|A|.$$

Define $B_i := \{b_1, \ldots, b_i\}$, $i = 1, \ldots, m-|A|$, with the convention that $B_0 = \emptyset$. Estimate, writing $FCT_n(B)$ to denote the first cut time for S_0^n in B,

$$\begin{aligned}
&\mathbb{P}(CT_n \cap B \neq \emptyset \mid Y_1^n = y_1^n) - \mathbb{P}(CT_n \cap B \neq \emptyset \mid Y_1^n = \bar{y}_1^n) \\
&\quad = \sum_{i=1}^{m-|A|} \left[\mathbb{P}(FCT_n(B) = b_i \mid Y_1^n = y_1^n) - \mathbb{P}(FCT_n(B) = b_i \mid Y_1^n = \bar{y}_1^n)\right] \\
&\quad = \sum_{i=1}^{m-|A|} \mathbb{P}\big(CT_n \cap B_{i-1} = \emptyset \mid b_i \in CT_n,\, Y_1^n = y_1^n\big) \\
&\qquad\qquad \times [\mathbb{P}(b_i \in CT_n \mid Y_1^n = y_1^n) - \mathbb{P}(b_i \in CT_n \mid Y_1^n = \bar{y}_1^n)] \\
&\quad < -2f(m) \sum_{i=1}^{m-|A|} \mathbb{P}\big(CT_n \cap B_{i-1} = \emptyset \mid b_i \in CT_n,\, Y_1^n = y_1^n\big) \\
&\quad \leq -2f(m) \sum_{i=1}^{m-|A|} \mathbb{P}\big(b_i \in CT_n,\, CT_n \cap B_{i-1} = \emptyset \mid Y_1^n = y_1^n\big) \\
&\quad = -2f(m)\,[1 - \mathbb{P}(B \cap CT_n = \emptyset \mid Y_1^n = y_1^n)]\ ,
\end{aligned}$$

where in the third line we have used Lemma 2.2. This inequality can be rewritten as

$$2f(m) < \mathbb{P}(CT_n \cap B = \emptyset \mid Y_1^n = y_1^n)(1 + 2f(m)) - \mathbb{P}(CT_n \cap B = \emptyset \mid Y_1^n = \bar{y}_1^n).$$

By (2.2), the right-hand side is at most $f(m)(1 + 2f(m))$ when $|B| > \frac{m}{2}$, which gives a contradiction because $f(m) \leq \frac{1}{2}$. □

2.3.2 Proof of (2.6)

Proof. Write

$$\tilde{\Delta} := \mathbb{P}\big(Y_0 = B,\, CT_n \cap A \neq \emptyset \mid Y_1^n = y_1^n\big) - \mathbb{P}\big(Y_0 = B,\, CT_n \cap A \neq \emptyset \mid Y_1^n = \bar{y}_1^n\big).$$

Using (2.2) in combination with (2.5), we may estimate

$$\Delta \leq \tilde{\Delta} + f(m).$$

Let $A = \{a_1, \ldots, a_{|A|}\}$ denote the elements of A in increasing order, and define $A_i := \{a_1, \ldots, a_i\}$, $i = 1, \ldots, |A|$, with the convention that $A_0 = \emptyset$. Then, using Lemma 2.2, we have

$$\begin{aligned}
\tilde{\Delta} &= \sum_{i=1}^{|A|} \Big[\mathbb{P}\big(Y_0 = B,\, FCT_n(A) = a_i \mid Y_1^n = y_1^n\big) - \mathbb{P}\big(Y_0 = B,\, FCT_n(A) = a_i \mid Y_1^n = \bar{y}_1^n\big)\Big] \\
&= \sum_{i=1}^{|A|} \Big[\mathbb{P}\big(Y_0 = B,\, CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, \mathbb{P}(a_i \in CT_n \mid Y_1^n = y_1^n) \\
&\qquad\qquad - \mathbb{P}\big(Y_0 = B,\, CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = \bar{y}_1^n\big)\, \mathbb{P}(a_i \in CT_n \mid Y_1^n = \bar{y}_1^n)\Big] \\
&= \sum_{i=1}^{|A|} \mathbb{P}\big(Y_0 = B,\, CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, D_i,
\end{aligned}$$

where

$$D_i := \mathbb{P}(a_i \in CT_n \mid Y_1^n = y_1^n) - \mathbb{P}(a_i \in CT_n \mid Y_1^n = \bar{y}_1^n).$$

In the third line we have used the fact that $\{CT_n \cap A_{i-1} = \emptyset\} = \{A_{i-1} \cap CT_{a_i} = \emptyset\} \in \sigma(S_0^{a_i}, Y_0^{a_i})$ (the σ-algebra generated by $S_0^{a_i}, Y_0^{a_i}$) on the event $\{a_i \in CT_n\}$, so that Lemma 2.2 applies. The definition of the set A implies that $D_i \geq -2f(m)$ for all i.

Hence, by using Lemma 2.2 once more, we obtain

$$
\begin{aligned}
&\tilde{\Delta} \\
&\le \sum_{i=1}^{|A|} 1\{D_i \ge 0\}\, \mathbb{P}\big(Y_0 = B,\, CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, D_i \\
&\le \sum_{i=1}^{|A|} 1\{D_i \ge 0\}\, \mathbb{P}\big(CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, D_i \\
&= \sum_{i=1}^{|A|} \mathbb{P}\big(CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, D_i \\
&\qquad\qquad + \sum_{i=1}^{|A|} 1\{D_i < 0\}\, \mathbb{P}\big(CT_n \cap A_{i-1} = \emptyset \mid a_i \in CT_n,\, Y_1^n = y_1^n\big)\, (-D_i) \\
&\le \sum_{i=1}^{|A|} \big[\mathbb{P}\big(a_i \in CT_n,\, CT_n \cap A_{i-1} = \emptyset \mid Y_1^n = y_1^n\big) - \mathbb{P}\big(a_i \in CT_n,\, CT_n \cap A_{i-1} = \emptyset \mid Y_1^n = \bar{y}_1^n\big)\big] \\
&\qquad\qquad + 2f(m)|A| \\
&= \mathbb{P}(CT_n \cap A \ne \emptyset \mid Y_1^n = y_1^n) - \mathbb{P}(CT_n \cap A \ne \emptyset \mid Y_1^n = \bar{y}_1^n) + 2f(m)|A| \\
&\le f(m) + 2f(m)m.
\end{aligned}
$$

Thus, we find that $\Delta \le 2f(m)(m+1)$, where the upper bound does not depend on the choice of configurations made in (2.4). Exchanging y_1^n and $\bar{y}_1^n$, we obtain the same bound for $|\Delta|$. Hence, we have proved (2.6). □

2.4 Proof of Lemma 2.4

Proof. For simplicity, we will only consider m-values that are a multiple of 6. The proof is easily adapted to intermediate m-values.

We first state the following fairly straightforward lemma, where we note that $\{S_m^n > \frac{2m}{3}\} = \{S_l > \frac{2m}{3}\ \forall\, m \le l \le n\}$.

Lemma 2.6. *For $m, n \in \mathbb{N}$ with $m \le n$,*

$$\{|CT_n \cap I_0^{m-1}| \le \tfrac{m}{2}\} \subseteq \{S_m^n > \tfrac{2m}{3}\}^c. \tag{2.7}$$

Proof. Note that each cut time k corresponds to a cut point S_k, and so the set $CT_n \cap I_0^{m-1}$ of cut times corresponds to a set $CP_n(m)$ of cut points. On the event $\{S_m^n > \frac{2m}{3}\}$, the interval $I_0^{\frac{2m}{3}}$ is fully covered by S_0^{m-1}. For each $x \in I_0^{\frac{2m}{3}}$, we look at the steps of the random walk entering or exiting x from the right:

- If $x \in CP_n(m)$, then during the time interval I_0^{n-1} there is at least one step exiting x to the right.

- If $x \notin CP_n(m)$, then during the time interval I_0^{n-1} there are at least two steps exiting x to the right and one step entering x from the right (since there must be a return to x from the right).

Since each step refers to a single point x only, and S_0^{m-1} goes along at most m edges (and exactly m edges when $\varepsilon = 0$), we get that

$$m \geq \left|CP_n(n) \cap I_0^{\frac{2m}{3}}\right| + 3\left|I_0^{\frac{2m}{3}} \setminus CP_n(n)\right| = 3(\tfrac{2m}{3} + 1) - 2\left|CP_n(n) \cap I_0^{\frac{2m}{3}}\right|.$$

Hence $|CP_n(n) \cap I_0^{\frac{2m}{3}}| > \frac{m}{2}$. Still on the event $\{S_m^n > \frac{2m}{3}\}$, the cut times corresponding to $CP_n(n) \cap I_0^{\frac{2m}{3}}$ occur before time $m-1$, and so

$$\left|CT_n \cap I_0^{m-1}\right| \geq \left|CP_n(n) \cap I_0^{\frac{2m}{3}}\right|.$$

Hence $|CT_n \cap I_0^{m-1}| > \frac{m}{2}$, and so (2.7) is proved. □

For $A \subseteq I_0^{m-1}$ such that $|A| \geq \frac{m}{2}$, we have

$$\{CT_n \cap A = \emptyset\} \subseteq \left\{\left|CT_n \cap I_0^{m-1}\right| \leq \tfrac{m}{2}\right\}.$$

Therefore, by (2.7),

$$\{CT_n \cap A = \emptyset\} \subseteq \left\{\exists k\colon\ m \leq k \leq n-1,\, S_k = \tfrac{2m}{3},\, S_{k+1}^n > \tfrac{2m}{3}\right\} \cup \{S_n \leq \tfrac{2m}{3}\}. \quad (2.8)$$

2.4.1 Estimate of the probabilities of the events in (2.8)

In this subsection, we obtain upper bounds on the probabilities of the two events on the right-hand side of (2.8) when conditioned on Y_1^n. The upper bounds will appear in (2.13) and (2.14) below. In Section 2.4.2, we use these estimates to finish the proof of Lemma 2.4.

Write

$$\begin{aligned} &\mathbb{P}\big(\exists k\colon\ m \leq k \leq n-1,\, S_k = \tfrac{2m}{3},\, S_{k+1}^n > \tfrac{2m}{3} \mid Y_1^n = y_1^n\big) \\ &\quad = \sum_{k=m}^{n-1} \mathbb{P}\big(S_k = \tfrac{2m}{3},\, S_{k+1}^n > \tfrac{2m}{3} \mid Y_1^n = y_1^n\big) = \sum_{k=m}^{n-1} \frac{N_k}{D_k}, \end{aligned} \quad (2.9)$$

with (recall Proposition 2.5)

$$N_k := N_k(y_1^n) = \sum_{s_1^n} 1\{s_k = \tfrac{2m}{3}\}\, 1\{s_{k+1}^n > \tfrac{2m}{3}\}\, \mathbb{P}(s_1^n) \left(\tfrac{1}{2}\right)^{R(s_1^n)} 1\{s_1^n \sim y_1^n\},$$

$$D_k := D_k(y_1^n) = \sum_{s_1^n} \mathbb{P}(s_1^n) \left(\tfrac{1}{2}\right)^{R(s_1^n)} 1\{s_1^n \sim y_1^n\}.$$

Estimate

$$\begin{aligned} N_k \leq &\sum_{s_1^k} 1\{s_k = \tfrac{2m}{3}\}\, \mathbb{P}(s_1^k)\, 1\{s_1^k \sim y_1^k\} \\ &\times \sum_{s_{k+1}^n} 1\{s_{k+1}^n > \tfrac{2m}{3}\}\, \mathbb{P}(s_{k+1}^n \mid S_k = \tfrac{2m}{3}) \left(\tfrac{1}{2}\right)^{R(s_{k+1}^n)} 1\{s_{k+1}^n \sim y_{k+1}^n\}. \end{aligned}$$

Here, the bound arises by noting that $1\{s_1^n \sim y_1^n\} \le 1\{s_1^k \sim y_1^k\}1\{s_{k+1}^n \sim y_{k+1}^n\}$ and estimating $R(s_1^n) \ge R(s_{k+1}^n)$. Thus, shifting S_k back to the origin, we get

$$N_k \le \mathbb{P}\big(S_k = \tfrac{2m}{3}, S_1^k \sim y_1^k\big)\, C_{k,n}(y_{k+1}^n) \tag{2.10}$$

with

$$C_{k,n}(y_{k+1}^n) = \sum_{s_1^{n-k}} 1\{s_1^{n-k} > 0\}\, \mathbb{P}(s_1^{n-k})\, \left(\tfrac{1}{2}\right)^{R(s_1^{n-k})}\, 1\{s_1^{n-k} \sim y_{k+1}^n\}.$$

Next, estimate

$$D_k \ge \sum_{s_1^k} 1\{s_1^k \le s_k\}\, \mathbb{P}(s_1^k)\, \left(\tfrac{1}{2}\right)^{R(s_1^k)}\, 1\{s_1^k \sim y_1^k\}$$
$$\times \sum_{s_{k+1}^n} 1\{s_{k+1}^n > s_k\}\, \mathbb{P}(s_{k+1}^n \mid S_k = s_k)\, \left(\tfrac{1}{2}\right)^{R(s_{k+1}^n)}\, 1\{s_{k+1}^n \sim y_{k+1}^n\}.$$

Here, the bound arises by restricting S_1^n to the event

$$\{k \in CT_n\} = \{S_1^k \le S_k\} \cap \{S_{k+1}^n > S_k\},$$

noting that $1\{S_1^n \sim y_1^n\} = 1\{S_1^k \sim y_1^k\}1\{S_{k+1}^n \sim y_{k+1}^n\}$ on this event, and inserting $R(s_1^n) = R(s_1^k) + R(s_{k+1}^n)$. Thus, shifting S_k back to the origin, we get

$$D_k \ge \mathbb{E}\left(\left(\tfrac{1}{2}\right)^{R(S_1^k)}\, 1\{S_1^k \le S_k\}\, 1\{S_1^k \sim y_1^k\}\right)\, C_{k,n}(y_{k+1}^n). \tag{2.11}$$

Combining the upper bound on N_k in (2.10) with the lower bound on D_k in (2.11), and canceling out the common factor $C_{k,n}(y_{k+1}^n)$, we arrive at

$$\mathbb{P}\big(S_k = \tfrac{2m}{3},\, S_{k+1}^n > \tfrac{2m}{3} \mid Y_1^n = y_1^n\big) \le \frac{\mathbb{P}(S_k = \tfrac{2m}{3}, S_1^k \sim y_1^k)}{\mathbb{E}\left(\left(\tfrac{1}{2}\right)^{R(S_1^k)}\, 1\{S_1^k \le S_k\}\, 1\{S_1^k \sim y_1^k\}\right)}. \tag{2.12}$$

Note that this bound is uniform in n.

The numerator of (2.12) is bounded from above by $\mathbb{P}(S_k = \frac{2m}{3})$, while the denominator of (2.12) is bounded from below by $(\frac{1}{2})^k \mathbb{P}(S_k = k) = (\frac{p(1-\varepsilon)}{2})^k$, where we note that $S_1^k \sim y_1^k$ for all y_1^k on the event $\{S_k = k\}$. Hence, by (2.9), we have

$$\mathbb{P}\big(\exists k\colon\ m \le k \le n-1,\, S_k = \tfrac{2m}{3},\, S_{k+1}^n > \tfrac{2m}{3} \mid Y_1^n = y_1^n\big) \le \sum_{k=m}^{n-1} \frac{\mathbb{P}(S_k = \frac{2m}{3})}{(\frac{p(1-\varepsilon)}{2})^k}. \tag{2.13}$$

The bound in (2.13) controls the first term in the right-hand side of (2.8).

Since $\mathbb{P}(Y_1^n = y_1^n) \ge \mathbb{P}(Y_1^n = y_1^n, S_n = n) = (\frac{p(1-\varepsilon)}{2})^n$, we have

$$\mathbb{P}(S_n \le \tfrac{2m}{3} \mid Y_1^n = y_1^n) \le \frac{\mathbb{P}(S_n \le \frac{2m}{3})}{(\frac{p(1-\varepsilon)}{2})^n} \le C\, \frac{\mathbb{P}(S_n = \frac{2m}{3})}{(\frac{p(1-\varepsilon)}{2})^n}, \tag{2.14}$$

provided n is even (which is necessary when $\varepsilon = 0$ because we have assumed that $\frac{2m}{3}$ is even). Here, the constant $C = C(p, \epsilon) \in (1, \infty)$ comes from an elementary large deviation estimate, for which we must assume that

$$(2p-1)(1-\varepsilon) > \tfrac{2}{3}. \tag{2.15}$$

The bound in (2.14) controls the second term in the right-hand side of (2.8).

2.4.2 Completion of the proof

In this section we finally complete the proof of Lemma 2.4.

Combining (2.13–2.14) and recalling (2.2) and (2.8), we obtain the estimate

$$f(m) \leq (C+1) \sum_{k=\frac{m}{2}}^{\infty} \frac{\mathbb{P}(S_{2k} = \frac{2m}{3})}{(\frac{p(1-\varepsilon)}{2})^{2k}}. \tag{2.16}$$

Since there exists a $C' = C'(p, \varepsilon) \in (1, \infty)$ such that, for $k \geq \frac{1}{2}m$,

$$\mathbb{P}(S_{2k} = \tfrac{2m}{3}) \leq C' \mathbb{P}(S_{2k} = \tfrac{4k}{3}),$$

we see that $\limsup_{m\to\infty} \frac{1}{m} \log f(m) < 0$ as soon as

$$\limsup_{m\to\infty} \frac{1}{m} \log \mathbb{P}(S_m = \tfrac{2m}{3}) < \log \big(\tfrac{p(1-\varepsilon)}{2}\big). \tag{2.17}$$

Note that (2.15) holds for (p, ε) in a neighborhood of $(1, 0)$ containing the line segment $(p_*, 1] \times \{0\}$.

By Cramer's theorem of large deviation theory (see e.g. [2], Chapter I), the left-hand side of (2.17) equals $-I(p, \varepsilon)$ with

$$I(p, \varepsilon) := \sup_{\lambda \in \mathbb{R}} \big[\tfrac{2}{3}\lambda - \log M(\lambda; p, \varepsilon)\big], \tag{2.18}$$

where

$$M(\lambda; p, \varepsilon) := \varepsilon + p(1-\varepsilon)e^{\lambda} + (1-p)(1-\varepsilon)e^{-\lambda} \tag{2.19}$$

is the moment-generating function of the increments of S. Due to the strict convexity of $\lambda \mapsto \log M(\lambda; p, \varepsilon)$, the supremum is attained at the unique $\bar{\lambda}$ solving the equation

$$\tfrac{2}{3} = \frac{(\partial/\partial\lambda) M(\lambda; p, \varepsilon)}{M(\lambda; p, \varepsilon)}, \tag{2.20}$$

where we note that $\bar{\lambda} < 0$ because of (2.15). For the special case where $\varepsilon = 0$, an easy calculation gives

$$\bar{\lambda} = \tfrac{1}{2} \log \big(\tfrac{5(1-p)}{p}\big),$$

implying that $I(p, 0) = \log C(p)$ with $C(p) = [5/6p]^{5/6}[1/6(1-p)]^{1/6}$. Hence the inequality in (2.17) reduces to $C(p) > 2/p$, which is equivalent to $p > p^*$ with $p^* = 1/(1 + 5^5 12^{-6})$. The same formulas (2.18–2.20) show that (2.17) holds in a neighborhood of $(1, 0)$. □

3 $\mathbb{B} = \Omega$ for $p \in (\frac{1}{2}, \frac{4}{5})$ and $\epsilon = 0$

Throughout the remainder of this paper (with the sole exceptions of Subsection 4.1 and the claim of independence immediately prior to (3.7)) we use Y_1^∞, $\bar{Y}_1^\infty$ and $\tilde{Y}_1^\infty$ to represent specific sequences rather than random sequences. This abuse of notation will nowhere cause harm.

In this section we prove Theorem 1.3(ii). The proof is based on the following observations valid for a random walk that cannot pause ($\varepsilon = 0$).

(I) On a color record of the type $[WWBB]^M$, $M \in \mathbb{N}$, the walk cannot turn. Indeed, a turn forces the same color to appear in the color record two units of time apart.

(II) Any color record Y_1^{m-1} up to time $m \in \mathbb{N}$ can be seen in a unique way along a stretch of coloring of the type $[WWBB]^M$ with $M \geq m$. Indeed, on such a stretch each site has a W-neighbor and a B-neighbor, so once the starting or ending point of the walk is fixed it is fully determined by Y_1^{m-1}.

We prove Theorem 1.3(ii) by showing the following claim:

- For any Y_1^∞, $p \in (\frac{1}{2}, \frac{4}{5})$ and $m \in \mathbb{N}$, we can find $\bar{Y}_m^\infty$ and $\widetilde{Y}_m^\infty$ such that

$$\lim_{n\to\infty} \left|\mathbb{P}\big(C_0 = W \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) - \mathbb{P}\big(C_0 = W \mid Y_1^{m-1} \vee \widetilde{Y}_m^n\big)\right| = 2p - 1. \tag{3.1}$$

In view of Definition 1.1, this claim will imply that Y_1^∞ is bad.

Proof. Fix $m \in \mathbb{N}$.

1. We begin with the choice of $\bar{Y}_m^n$. For $L \in \mathbb{N}$, let

$$\begin{aligned} \bar{Y}_m^n := [WWBB]^m\, WBB\, [WWBB]^{2m}\, WBB\, [WWBB]^{2m+1} \cdots \\ \cdots WBB\, [WWBB]^{2m+L-1}\, WBB\, [WWBB]^{2m+L}. \end{aligned} \tag{3.2}$$

The interest in this color record relies on three facts:

(1) For $l = 0, \ldots, L$, on the color record $[WWBB]^{2m+l}$ the walk cannot turn (see (I) above).

(2) On $\bar{Y}_m^n$, the isolated W's at the beginning of the WBB's play the role of *pivots*, since the walk can only turn there as is easily checked. We call W_0 the pivot W seen at time $5m$ (this is the first pivot) and W_l, $l = 1, \ldots, L$, the subsequent pivots seen at times

$$t(l) := 5m + \sum_{j=0}^{l-1} [3 + 4(2m + j)] = k(2k + 8m + 1) + 5m, \qquad k = 1, \ldots, L.$$

(3) Since the length of the color record $[WWBB]^{2m+l}$ increases with l, if the walk does not turn on pivot W_l, then it cannot turn on any later pivot. Indeed, going straight through W_l means that the coloring has an isolated W surrounded by two B's, and this color stretch is impossible to cross at any later time with any color record of the type $[WWBB]^M$, $M \in \mathbb{N}$.

The first color record $[WWBB]^m$ serves to prevent W_0 from being in the coloring seen by the walk up to time $m - 1$, because the walk cannot turn between time m and time $5m$ (see (I) above). The total time is

$$n = n(L) = L(2L + 8m + 5) + 13m + 2.$$

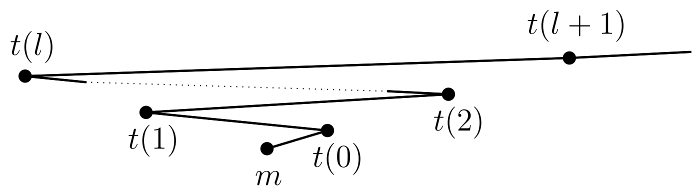

Figure 2: A walk in $LT_l \cap EU$. The last turn occurs at time $t(l)$. Depending on the parity of l, the walk between time m and time $t(l)$ starts its zigzag motion either to the right (as drawn) or to the left (l is odd in this picture).

The above three facts imply that the behavior of the walk from time m to time n (i.e., the increments $X_{m+1}, \dots, X_n$), leading to $\bar{Y}_m^n$ as its color record, can be characterized by the first pivot W_l, if any, where the walk makes no turn. There are $L+2$ possibilities, including the ones where there is a turn at every pivot or at no pivot. This characterization is up to a 2-fold symmetry in the direction of the last step of the walk, which can be either upwards or downwards (this is the same symmetry as $X \to -X$). Note that, except for the case where the walk makes no turn from time m to time n, the behavior of the walk from time 1 to time m (i.e., the increments $X_2, \dots, X_m$) is fully determined (up to the 2-fold symmetry) by $\bar{Y}_1^n$ (see (II) above). This is because $l \mapsto t(l+1) - t(l)$ is increasing, so that $t(l+1) - t(l) \geq t(1) - t(0) = 3 + 8m > 5m$.

Our goal will be to prove that for large L the walk, conditioned on $Y_1^{m-1} \vee \bar{Y}_m^n$, with a high probability turns on every pivot and ends by moving upwards. To that end we define the following events for the walk up to time n:

- $LT_l := \{$the walk turns on pivots $W_0, W_1, \dots, W_l$ and does not turn on pivots $W_{l+1}, \dots, W_L\}$ ('last turn on l'), $l = 0, \dots, L$.
- $NT := \{$the walk does not turn on any pivot$\}$ ('no turn').
- $EU := \{S_n = S_{n-1} + 1\}$ ('end upwards').
- $ED := \{S_n = S_{n-1} - 1\}$ ('end downwards').

Using these events, we may write

$$\begin{aligned} 1 &= \mathbb{P}\big(NT, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) + \mathbb{P}\big(NT, ED \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \\ &\quad + \sum_{l=0}^{L} \left[\mathbb{P}\big(LT_l, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) + \mathbb{P}\big(LT_l, ED \mid Y_1^{m-1} \vee \bar{Y}_m^n\big)\right]. \end{aligned} \tag{3.3}$$

Now, on the event LT_l, the length of the coloring seen by the walk from time 1 to time n is

$$n - t(l) + 1 = \sum_{j=l}^{L} [3 + 4(2m + j)] = (L - l + 1)(2L + 2l + 8m + 3).$$

Only two walks from time m to time n are in LT_l and these are reflections of each other (one in EU and one in ED). For either of these two walks, we have that $|S_{t(l)} - S_{t(0)}| = u(l)$, where

$$u(l) := \sum_{j=1}^{l} (-1)^{l-j}[t(j) - t(j-1)] = (1+8m)\,1\{l \text{ odd}\} + 2l.$$

It is easily checked that any walk in $EU \cap LT_l$ ends a distance at least $2v(l,L)$ above any walk in $ED \cap LT_l$, with (see Figure 2)

$$\begin{aligned} v(l,L) &:= n(L) - t(l) - u(l) + (-1)^{l-1}4m - m \\ &= (L-l)(2L+2l+8m+5) + 2l + 3m + 2 - 1\{l \text{ odd}\}. \end{aligned}$$

Hence we have, using the fact that all walks in LT_l visit the same number of colors,

$$\mathbb{P}\big(LT_l, ED \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \leq \left(\frac{1-p}{p}\right)^{v(l,L)} \mathbb{P}\big(LT_l, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big). \tag{3.4}$$

Since $(1-p)/p < 1$ (because $p > \frac{1}{2}$) and $\lim_{L\to\infty} \inf_{0\leq l \leq L} v(l,L) = \infty$, it follows that for L large the probability of $LT_l \cap ED$ is negligible with respect to the probability of $LT_l \cap EU$ uniformly in l.

The same reasoning gives the inequality

$$\begin{aligned} &\mathbb{P}\big(LT_l, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \\ &\quad \leq \left(\frac{p}{1-p}\right)^{u(l+1)+5m} \left(\tfrac{1}{2}\right)^{t(l+1)-t(l)} \mathbb{P}\big(LT_{l+1}, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big). \end{aligned} \tag{3.5}$$

Indeed, any walk in $LT_l \cap EU$ covers $t(l+1) - t(l)$ more sites than any walk in $LT_{l+1} \cap EU$, while it is not hard to see that it makes at most $u(l+1) + 5m$ more steps to the right. Since $t(l+1) - t(l) \sim 4l$ and $u(l+1) + 5m \sim 2l$ as $l \to \infty$, and $p/(1-p) < 4$ (because $p < \frac{4}{5}$), we find that

$$\frac{\mathbb{P}(LT_l, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n)}{\mathbb{P}(LT_{l+1}, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n)}$$

decreases exponentially in l for l large. Hence the largest value $l = L$ dominates. Similar estimates allow us to neglect probabilities containing the event NT.

Combining (3.3–3.5), we obtain that, for fixed m,

$$\lim_{L\to\infty} \mathbb{P}\big(LT_L, EU \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) = 1,$$

which immediately yields that, for fixed m,

$$\mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) = \mathbb{P}\big(C_0 = B \mid LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n\big)\,[1 + o(1)], \tag{3.6}$$

where the error $o(1)$ tends to zero as $L \to \infty$.

The key point of (3.6) is that $LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n$ forces the coloring around the origin to look like $\cdots BBWWBBWWBB \cdots$. More specifically, $LT_L, EU, \bar{Y}_m^n$ tells us the coloring on a large region relative to S_m and, after this, Y_1^{m-1} determines the walk from time 1 to time m (relative to S_1). Since S_1 is independent of $\{LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n\}$, we therefore have

$$\mathbb{P}\big(C_0 = B \mid LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n\big) \in \{p, 1-p\}. \tag{3.7}$$

Equations (3.6) and (3.7) tell us that for large n, $\mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^n\big)$ will be very close to p or $1-p$. The idea now will be to modify the extension far away so that an "opposite" type of structure is forced upon us and thereby reverse the p and $1-p$ above.

2. We next move to the choice of $\widetilde{Y}_m^n$. We take

$$\begin{aligned} \widetilde{Y}_m^n := {} & [WWBB]^m\, WBB\, [WWBB]^{2m}\, WBB\, [WWBB]^{2m+1} \cdots \\ & \cdots WBB\, [WWBB]^{2m+L-1}\, [WWBB]^{2m+L}. \end{aligned} \tag{3.8}$$

The difference with $\bar{Y}_m^n$ in (3.2) is that we removed the last pivot W_L and the 2 B's following it (so that $n \to n-3$). The same computations as before give

$$\mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \widetilde{Y}_m^n\big) = \mathbb{P}\big(C_0 = B \mid LT_{L-1}, EU, Y_1^{m-1} \vee \widetilde{Y}_m^n\big)\,[1 + o(1)]. \tag{3.9}$$

Now $LT_{L-1}, EU, Y_1^{m-1} \vee \widetilde{Y}_m^n$ forces the walk to do the exact opposite up to time $t(L-1)$ to what $LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n$ forced it to do, because there is one turn less and the walk still ends upwards. Therefore, by symmetry, the walk from time 1 to time $m-1$ must also do the exact opposite, and so we conclude that, for $q \in \{p, 1-p\}$,

$$\begin{aligned} & \mathbb{P}\big(C_0 = B \mid LT_L, EU, Y_1^{m-1} \vee \bar{Y}_m^n\big) = q \\ \iff \quad & \mathbb{P}\big(C_0 = B \mid LT_{L-1}, EU, Y_1^{m-1} \vee \widetilde{Y}_m^n\big) = 1 - q. \end{aligned} \tag{3.10}$$

Combining (3.6) and (3.9–3.10), we obtain the claim in (3.1). □

4 $\mathbb{B} \notin \{\emptyset, \Omega\}$ for $p = \frac{1}{2}$ and $\epsilon = 0$

In this section we prove Theorem 1.3(iii).

Proof. We will prove that if $p = \frac{1}{2}$ and $\varepsilon = 0$, then

$$\begin{aligned} & Y_1^\infty = B^\infty \text{ is bad,} \\ & Y_1^\infty = BB\, WBB\, [WWBB]\, WBB\, [WWBB]^2\, WBB\, [WWBB]^3 \ldots \text{ is good.} \end{aligned} \tag{4.1}$$

(In the second line, BB is put at the beginning to ensure that the first W may be a pivot.)

4.1 Proof of the first claim in (4.1)

In this subsection Y_1^n and Y_0^{n-1} denote random sequences, and we switch back to specific sequences only in the last display.

Write

$$\mathbb{P}\big(C_0 = W \mid Y_1^n = B^n\big) = \mathbb{P}\big(C_0 = W \mid S_1 = 1, Y_1^n = B^n\big)$$
$$= \mathbb{P}\big(C_{-1} = W \mid Y_0^{n-1} = B^n\big) = \frac{N(n)}{D(n)}$$

with

$$\begin{aligned} N(n) &:= \mathbb{P}\big(C_{-1} = W, Y_0^{n-1} = B^n\big) = \sum_{i\in\mathbb{N}} (\tfrac{1}{2})^{i+2}\, p(n,i,1), \\ D(n) &:= \mathbb{P}\big(Y_0^{n-1} = B^n\big) = \sum_{i,j\in\mathbb{N}} (\tfrac{1}{2})^{i+j+1}\, p(n,i,j), \end{aligned} \tag{4.2}$$

where $p(n,i,j) := \mathbb{P}(\tau_i \geq n, \tau_{-j} \geq n)$ is the probability that simple random walk (with $p = \frac{1}{2}$ and $\varepsilon = 0$) starting from 0 stays between $-j+1$ and $i-1$ (inclusive) prior to time n. To see the second equality in (4.2), let $E_{i,j}$ be the event that there is a B at the origin, and the first W to the right and to the left of the origin are located at i and $-j$, respectively. Then

$$\mathbb{P}\big(Y_0^{n-1} = B^n\big) = \sum_{i,j\in\mathbb{N}} \mathbb{P}(E_{i,j})\, \mathbb{P}\big(Y_0^{n-1} = B^n \mid E_{i,j}\big),$$

which is easily seen to be the claimed sum. The first equality in (4.2) is handled similarly.

Trivially, $p(n,i,j) \geq p(n,i+j-1,1)$ for all $i,j \in \mathbb{N}$, and therefore

$$D(n) \geq \sum_{i\in\mathbb{N}} i(\tfrac{1}{2})^{i+2}\, p(n,i,1). \tag{4.3}$$

Next, using Proposition 21.1 in [7], we easily deduce that

$$p(n,i,1) \sim \big[\cos(\tfrac{\pi}{i+1})\big]^{n-1} \begin{cases} C_i^{\text{even}} & \text{as } n\to\infty \text{ through } n \text{ even}, \\ C_i^{\text{odd}} & \text{as } n\to\infty \text{ through } n \text{ odd}, \end{cases}$$

where $\sim$ means that the ratio of the two sides tends to 1, and

$$C_i^{\text{even}} = \tfrac{4}{i+1} \sin\left(\tfrac{\pi}{i+1}\right) \sum_{\substack{0\leq j<i \\ j \text{ odd}}} \sin\left(\tfrac{\pi(j+1)}{i+1}\right),$$

$$C_i^{\text{odd}} = \tfrac{4}{i+1} \sin\left(\tfrac{\pi}{i+1}\right) \sum_{\substack{0\leq j<i \\ j \text{ even}}} \sin\left(\tfrac{\pi(j+1)}{i+1}\right).$$

From this it follows that

$$\lim_{n\to\infty} \frac{p(n,i+1,1)}{p(n,i,1)} = \infty, \qquad i \in \mathbb{N}. \tag{4.4}$$

Combining (4.2–4.4), we get $\lim_{n\to\infty} N(n)/D(n) = 0$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\big(C_0 = B \mid Y_1^n = B^n\big) = 1. \tag{4.5}$$

On the other hand, an extension of $Y_1^{m-1} = B^{m-1}$ with $\bar{Y}_m^n$ as in Section 3 gives

$$P\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) = \mathbb{P}\big(C_0 = B \mid LT_L, Y_1^{m-1} \vee \bar{Y}_m^n\big)\,[1+o(1)] = \tfrac{1}{2}\,[1+o(1)] \tag{4.6}$$

(recall (3.5–3.7)). Combining (4.5–4.6), we get the first claim in (4.1).

4.2 Proof of the second claim in (4.1)

Pick $L \in \mathbb{N}$ and $m - 1 = L(2L+5) + 2$. Then

$$Y_1^{m-1} = BB\,WBB\,[WWBB]\,WBB\,[WWBB]^2 \cdots WBB\,[WWBB]^L.$$

As in Section 3, a turn on a white pivot forces turns on all previous white pivots. Therefore a walk compatible with Y_1^{m-1} having at least one turn is characterized by the index $k = 0, 1, \dots, L-1$ of its last pivot W_k. The time of the k-th pivot is $3 + \sum_{j=0}^{k-1}[3 + 4(j+1)]$.

Conditioning on $Y_1^{m-1} \vee \bar{Y}_m^n$ still leaves us the freedom to choose $S_1 \in \{-1,+1\}$ and $S_2 \in \{S_1 - 1, S_1 + 1\}$. Since $p = \frac{1}{2}$, it is easily checked that, conditioned on $Y_1^{m-1} \vee \bar{Y}_m^n$ (and even on the last pivot), S_1 and $S_2 - S_1$ are independent fair coin flips. There are 4 compatible walks with no turn and $4L$ compatible walks with at least one turn. Since $p = \frac{1}{2}$, all these walks have the same probability, but the walks with no turn have a larger cost for the coloring. Let NT and $AOT := [NT]^c$ denote the event that the walk makes no turn, respectively, at least one turn. We claim that

$$\mathbb{P}\big(NT \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \le \frac{1}{L+1}\,\mathbb{P}\big(AOT \mid Y_1^{m-1} \vee \bar{Y}_m^n\big). \tag{4.7}$$

To see how this comes about, recall Proposition 2.5, which says that for an arbitrary walk s_1^{m-1} and an arbitrary extension $\bar{Y}_m^n$,

$$\begin{aligned}
&\mathbb{P}\big(S_1^{m-1} = s_1^{m-1}, Y_1^{m-1} \vee \bar{Y}_m^n\big)\\
&\qquad = \sum_{\bar{s}_1^{n-m+1}} \mathbb{P}(s_1^{m-1} \vee \bar{s}_1^{n-m+1})\,(\tfrac{1}{2})^{R(s_1^{m-1} \vee \bar{s}_1^{n-m+1})}\,1\{s_1^{m-1} \vee \bar{s}_1^{n-m+1} \sim Y_1^{m-1} \vee \bar{Y}_m^n\}.
\end{aligned}$$

(The notation $s_1^{m-1} \vee \bar{s}_1^{n-m+1}$ denotes the walk obtained by appending the second walk to the end of the first walk.) Note that any compatible walk up to time $m-1$ ends either at the right end of the range or at the left end of the range. Let $s_1^{m-1}[0]$ and $s_1^{m-1}[1]$ denote compatible walks with no turn, respectively, at least one turn, either both ending at the right end of the range or both ending at the left end of the range. Then $R(s_1^{m-1}[0] \vee \bar{s}_1^{n-m+1}) \ge R(s_1^{m-1}[1] \vee \bar{s}_1^{n-m+1})$. Moreover, for any $\bar{s}_1^{n-m+1}$ and $\bar{Y}_m^n$, if $s_1^{m-1}[0] \vee \bar{s}_1^{n-m+1} \sim Y_1^{m-1} \vee \bar{Y}_m^n$, then also $s_1^{m-1}[1] \vee \bar{s}_1^{n-m+1} \sim Y_1^{m-1} \vee \bar{Y}_m^n$. Hence

$$\mathbb{P}\big(s_1^{m-1}[0], Y_1^{m-1} \vee \bar{Y}_m^n\big) \le \mathbb{P}\big(s_1^{m-1}[1], Y_1^{m-1} \vee \bar{Y}_m^n\big).$$

Summing over $s_1^{m-1}[0]$ and $s_1^{m-1}[1]$, we obtain (4.7).

Next, on the event AOT, $C_0 = B$ is fully determined by S_1 and S_2. Therefore, by symmetry,

$$\mathbb{P}\big(C_0 = B \mid AOT, Y_1^{m-1} \vee \bar{Y}_m^n\big) = \tfrac{1}{2}.$$

Hence, uniformly in $\bar{Y}_m^n$,

$$\begin{aligned}
&\mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \\
&\quad = \mathbb{P}\big(C_0 = B, AOT \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) + \mathbb{P}\big(C_0 = B, NT \mid Y_1^{m-1} \vee \bar{Y}_m^n\big) \\
&\quad = \tfrac{1}{2} + O(\tfrac{1}{L}).
\end{aligned}$$

Since $L \to \infty$ as $m \to \infty$, the second claim in (4.1) follows. □

5 A possible approach to show that $\mathbb{B} = \Omega$ when $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon \in (0, 1)$

In this section, we explain a strategy for proving that $\mathbb{B} = \Omega$ when $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon \in (0, 1)$. It seems that this case is much more delicate than the case $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon = 0$ treated in Sections 3–4. This strategy will be pursued in future work.

5.1 Proposed strategy of the proof

For $M \in \mathbb{N}$, we use the notation W^M, B^M, $[WB]^M$ etc. to abbreviate

$$\underbrace{WW\cdots W}_{M \text{ times } W}, \quad \underbrace{BB\cdots B}_{M \text{ times } B}, \quad \underbrace{WBWB\cdots WB}_{M \text{ times } WB}, \quad \text{etc.}$$

Fix any configuration Y_1^∞. To try to prove that Y_1^∞ is bad, we do the following:

(1) For $m, k, K \in \mathbb{N}$ with $k \geq 2$, we consider the two color records from time m to time $m + kK$ defined by

$$\bar{Y}_m^{m+kK}(B) := [WB^{k-1}]^K W, \quad \bar{Y}_m^{m+kK}(W) := [BW^{k-1}]^K B.$$

(2) We *expect* that, for any $p \in [\frac{1}{2}, \frac{4}{5})$ and $\varepsilon \in (0, 1)$,

$$\begin{aligned}
&\inf_{m\in\mathbb{N}} \inf_{Y_1^{m-1}} \liminf_{k\to\infty} \liminf_{K\to\infty} \\
&\quad \big|\mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(B)\big) - \mathbb{P}\big(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(W)\big)\big| \\
&\qquad \geq (1-\varepsilon)(1-p),
\end{aligned} \tag{5.1}$$

where $\vee$ denotes the concatenation operation. In view of Definition 1.1, this would imply that Y_1^∞ is a bad configuration, as desired.

The idea behind the above strategy is that $\bar{Y}_m^{m+kK}(B)$ forces the walk to hit many white sites at sparse times from time m onwards. In order to achieve this, the walk can either move out to infinity, in which case the coloring must contain many long black intervals, or the walk can hang around the origin, in which case the coloring must contain a single white site close to the origin with two long black intervals on either side. Since the drift of the random walk is not too large, the best option is to hang around the origin. The single white site, at or next to the origin, is enough for the walk to generate any (!) color record Y_1^{m-1} prior to time m, because the pausing probability is strictly positive. As a result, the conditional probability to see a black origin given $Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(B)$ is closer to 1 than given $Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(W)$. With the latter conditioning, the role of B and W is reversed, and the effect of the conditioning is to have the origin lie in a region containing a single black site separating two long white intervals, so that the conditional probability to see a black origin is closer to 0.

5.2 A few more details

The task is to control the conditional probability $\mathbb{P}(C_0 = B \mid Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(B))$. For that purpose, mark the positions of the walk at the times $m + ki$, $i = 0, \ldots, K$, that correspond to the isolated W's in $\bar{Y}_m^{m+kK}(B)$. By the definition of $\bar{Y}_m^{m+kK}(B)$, two subsequent W's either correspond to the same white site or to two white sites that are separated by a single interval of black sites of length at least 1.

On the event $Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(B)$, let W_0 be the white site visited at time m. Relative to this site, all the white sites in C can be labeled $(W_i)_{i\in\mathbb{Z}}$, with W_{-1} the first white site on the left of W_0, W_1 the first white site on the right of W_0, etc. (see Figure 3). Let $\mathcal{B}_i$ denote the black interval between W_i and W_{i+1}. i_{min} and i_{max} are the indices of the left-most and right-most white sites visited by the walk between times m and $m + kK$.

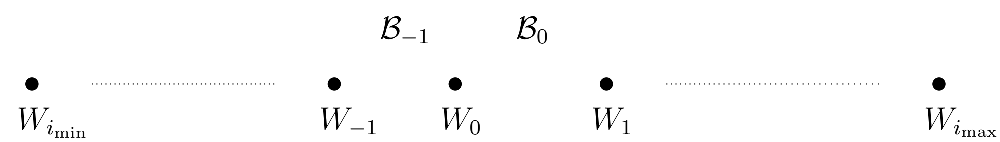

Figure 3: White sites separated by black intervals. W_0 is the white site seen at time m in $Y_1^{m-1} \vee \bar{Y}_1^{m+kK}(B)$.

The above representation allows to obtain an explicit (although complex) formula for the conditional probability $\mathbb{P}(\cdot \mid Y_1^{m-1} \vee Y_m^{m+kK}(B))$ involving classical simple random walk quantities.

Let $\mathcal{E}_i$ denote the event that $\mathcal{B}_i$ is visited between times m and $m + kK$. Then the key fact that needs to be proved is the following:

$$\inf_{Y_1^{m-1}} \liminf_{k\to\infty} \liminf_{K\to\infty} \mathbb{P}\big(\mathcal{E}_{-1} \cap \mathcal{E}_0 \mid Y_1^{m-1} \vee \bar{Y}_m^{m+kK}(B)\big) = 1 \qquad \forall\, m \in \mathbb{N}. \tag{5.2}$$

From (5.2) we are able to prove the desired result (5.1), but the argument needed to prove (5.2) is long and we are still working on trying to complete it.

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