Fortuin-Kasteleyn representations for threshold Gaussian and stable vectors

Malin Palö Forsström [*]   Jeffrey E. Steif [†]

January 8, 2019

Abstract

We study the question of when a $\{0,1\}$-valued threshold process associated to a mean zero Gaussian or a symmetric stable vector corresponds to a *divide and color (DC) process*. This means that the process corresponding to fixing a threshold level h and letting a 1 correspond to the variable being larger than h arises from a random partition of the index set followed by coloring *all* elements in each partition element 1 or 0 with probabilities p and $1-p$, independently for different partition elements. For example, the Ising model with zero external field (as well as a number of other well known processes) is a DC process where the random partition corresponds to the famous FK-random cluster model.

We first determine in general the exact lack of uniqueness of the representing random partition. This amounts to determining the dimensions of the kernels of a certain family of linear operators.

We obtain various results in both the Gaussian and symmetric stable cases. For example, it turns out that all discrete Gaussian free fields yield a DC process when the threshold is zero; this follows quite easily from known facts. For general n-dimensional mean zero, variance one Gaussian vectors with nonnegative covariances, the zero-threshold process is always a DC process for $n = 3$ but this is false for $n = 4$.

The answers in general are quite different depending on whether the threshold level h is zero or not. We show that there is no general monotonicity in h in either direction. We also show that all discrete Gaussian free fields yield DC processes for large thresholds. Moving to $n = 3$, we characterize exactly which mean zero, variance one Gaussian vectors yield DC processes for large h.

In the stable case, among other results, if we stick to the simplest case of a permutation invariant, symmetric stable vector with three variables, we obtain a phase transition in the stability exponent α at the surprising point $1/2$; if the index of stability is larger than $1/2$, then the process yields a DC process for large h while if the index of stability is smaller than $1/2$, then this is not the case.

Keywords and phrases. Random cluster (FK) representions, threshold Gaussian vectors, threshold stable vectors.
MSC 2010 *subject classifications.* Primary 60G15, 60G52

---

[*]Chalmers University of Technology and Gothenburg University, Gothenburg, Sweden. Email: palo@chalmers.se
[†]Chalmers University of Technology and Gothenburg University, Gothenburg, Sweden. Email: steif@chalmers.se

Contents

1   Introduction, notation, summary of results and background

1.1   Introduction

A very simple mechanism for constructing random variables with a (positive) dependency structure is the so-called *generalized divide and color model* introduced in its general form in [12] but having already arisen in many different contexts.

Definition 1.1. A $\{0, 1\}$-valued process $X := (X_i)_{i \in S}$ is a *generalized divide and color model* or *color process* if X can be generated as follows. First choose a random partition π of S according to some arbitrary distribution, and then independently of this and independently for different partition elements in the random partition, assign, with probability p, *all* the variables in a partition element the value 1 and with probability $1 - p$ assign *all* the variables the value 0. This final $\{0, 1\}$-valued process is then called the *color process* associated to π and p. We also say that π and p is a *color representation* of X.

As detailed in [12], many processes in probability theory are color processes; examples are the Ising model with zero external field, the fuzzy Potts model with zero external field, the stationary distributions for the Voter Model and random walk in random scenery.

While certainly the distribution of the color process determines p, it in fact does not determine the distribution of π. This was seen in small cases in [12], and this lack of determination will essentially be completely determined in Section 2.

Since the dependency mechanism in a color process is so simple, it seems natural to ask which $\{0, 1\}$-valued processes fall into this context. We mention that it is trivial to see that any color process has nonnegative pairwise correlations and so this is a trivial necessary condition.

In this paper, our main goal is to study the question of which *threshold Gaussian* and *threshold stable* processes fall into this context. More precisely, in the Gaussian situation, we ask the following question. Given a set of random variables $(X_i)_{i \in I}$ which is jointly Gaussian with mean zero, and given $h \in \mathbb{R}$, is the $\{0, 1\}$-valued process $(X_i^h)_{i \in I}$ defined by

$$X_i^h := I(X_i > h)$$

a color process? In the stable situation, we simply replace the Gaussian assumption by $(X_i)_{i \in I}$ having a symmetric stable distribution. (We will review the necessary background concerning stable distributions in Subsection 1.4.) For the very special case that I is infinite, $h = 0$ and the process is exchangeable, this question was answered positively, both in the Gaussian and stable cases, in [12]. As we will see later (see Theorem 1.15), the set of threshold stable vectors is a much richer class than the set of threshold Gaussian vectors. As such, it is reasonable to study both classes.

Since all the marginals in a color process are necessarily equal, if $h \neq 0$, then a necessary condition in the Gaussian case for $(X_i^h)_{i \in I}$ to be a color process is that all the X_i's have the same variance. Therefore, when considering $h \neq 0$, we will assume that all the (X_i)'s have variance one. However, it will be convenient not to make this latter assumption when considering $h = 0$. For the stable case, we will simply assume that all the marginals are the same.

It has been seen in [12] that $p = 1/2$ and $p \neq 1/2$ (corresponding to $h = 0$ and $h \neq 0$ in the Gaussian setting) behave very differently generally speaking. We will continue to see this difference throughout this paper.

We finally note that the questions looked at here significantly differ from those studied in [12]. In the latter paper, one looked at what types of behavior (ergodic, stochastic domination, etc.) color processes possess while in the present paper, we analyze which random vectors (primarily among threshold Gaussian and threshold stable vectors) are in fact color processes.

1.2 Notation and some standard assumptions

Given a set S, we let $\mathcal{B}_S$ denote the collection of partitions of the set S. We denote $\{1, 2, 3, \ldots, n\}$ by $[n]$ and if $S = [n]$, we write $\mathcal{B}_n$ for $\mathcal{B}_S$. $|\mathcal{B}_n|$ is called the nth Bell number. We denote by $\mathcal{P}_n$ the set of partitions of the integer n.

A random partition of $[n]$ yields a probability vector $q = \{q_\sigma\}_{\sigma \in \mathcal{B}_n} \in \mathbb{R}^{\mathcal{B}_n}$. Similarly, a random $\{0, 1\}$-valued vector $(X_1, \ldots, X_n)$ yields a probability vector $\nu = \{\nu_\rho\}_{\rho \in \{0,1\}^n} \in \mathbb{R}^{\{0,1\}^n}$. The definition of a color process yields immediately, for each n and $p \in [0, 1]$, an affine map $\Phi_{n,p}$ from random partitions of $[n]$, i.e., from probability vectors $q = \{q_\sigma\}_{\sigma \in \mathcal{B}_n}$ to probability vectors $\nu = \{\nu_\rho\}_{\rho \in \{0,1\}^n}$. This map naturally extends to a linear mapping $A_{p,n}$ from $\mathbb{R}^{\mathcal{B}_n}$ to $\mathbb{R}^{\{0,1\}^n}$. See Section 2 for more details.

While perhaps not standard terminology, we adopt the following definition.

Definition 1.2. We call a Gaussian vector *standard* if each marginal has mean zero and variance one.

Standing assumption. Whenever we consider a Gaussian or symmetric stable vector, we will assume it is nondegenerate in the sense that for all $i \neq j$, $P(X_i \neq X_j) = 1$.

Some further notation which we will use is the following.

$\nu_{x_1, \ldots, x_n}$ or $\nu(x_1, \ldots, x_n)$
    will denote the probability that $\{X_1 = x_1, \ldots, X_n = x_n\}$ for a $\{0, 1\}$-valued process $(X_1, \ldots, X_n)$.

$\nu_{0^k}(h)$ or $\nu_h(0^k)$
    as an illustration, will denote, given a Gaussian or stable vector $(X_1, \ldots, X_n)$

4

the probability that the h-threshold process is identically zero; i.e., the probability that $P(X_1 \leq h, X_2 \leq h, \ldots, X_k \leq h)$. More generally, for $h \in \mathbb{R}$, we let $\nu_h = \mathcal{L}(X^h)$.

$q_{13,2}$

as an illustration, will denote, given a random partition with $n = 3$, the probability that 1 and 3 are in the same partition and 2 is in its own partition.

If we have a partition of a set of more than three elements, $q_{13,2}$ will then mean the above but with regard to the induced (marginal) random partition of $\{1, 2, 3\}$.

$N(0, A)$

will denote a Gaussian vector with mean zero and covariance matrix A.

1.3 Description of results

In Section 2, the linear operator $A_{p,n}$ defined above is studied in detail. The range is characterized and we obtain formulas for the rank and hence the nullity in all cases. The formula differs depending on whether $p = 1/2$ or $p \neq 1/2$. The notions of a *formal solution* and a *nonnegative solution* will be given in that section. A nontrivial kernel of $A_{p,n}$ corresponds exactly to nonuniqueness of a formal solution. A nontrivial kernel is also very closely related to, but not exactly the same as, nonuniqueness of a nonnegative solution. Nonuniqueness of a nonnegative solution corresponds exactly to a color process arising from more than one random partition. We then restrict to the situation where we have invariance under all permutations of $[n]$ and we again characterize the range and give the rank and hence nullity of the appropriate operator.

In Section 3, we present positive results concerning the question of the existance of a color representation for the threshold zero case for discrete Gaussian free fields and more generally for Gaussian vectors whose covariance matrices are so-called *inverse Stieltjes*, meaning that the off-diagonal elements of the inverse covariance matrix are nonpositive. (See subsubsection 3.2.1 for the definition of a discrete Gaussian free field.) This essentially follows from the known fact that the signs of a discrete Gaussian free field conditioned on their absolute values is an Ising Model with nonnegative interaction constants depending on the conditioned absolute values. The latter fact has been observed in [6]. However, it turns out that a threshold zero Gaussian process can be a color process even if its covariance matrix is not inverse Stieltjes. We also relate the class of *inverse Stieltjes* vectors with the set of tree-indexed Gaussian Markov chains.

In Section 4, we provide an alternative proof that threshold zero tree-indexed Gaussian Markov chains are color processes using the Ornstein-Uhlenbeck process. This proof has the advantage that the method leads to our first result for stable vectors, namely that a threshold zero tree-indexed symmetric stable Markov chain is also a color process; in this case, we however use subordinators.

In Section 5, we view our Gaussian vectors from a more geometric perspective and obtain a number of negative (and some positive) results for thresholds $h \neq 0$. In this section, we will obtain our first example where we have a nontrivial phase transition in h. This will be elaborated on in more detail in Theorem 5.9 but we state perhaps what is the main import of that result.

Theorem 1.3. *There exists a four-dimensional standard Gaussian vector X so that X^h is a color process for small positive h but is not a color process for large h.*

Remark 1.4. Given the above it is natural to ponder over the possible monotonicity properties in h. Proposition 5.5 implies that there is no three-dimensional Gaussian vector with such a phase transition among those that are not fully supported, while simulations indicate that there is also no fully supported three-dimensional Gaussian vector with such a phase transition. On the other hand, Corollary 8.5(iii) tells us that there are three-dimensional Gaussian vectors which are not color processes for small h but are color processes for large h. This together with the previous result rules out any type of monotonicity, in either direction. Perhaps however monotonicity holds (in one direction) for fully supported vectors. See the open questions section.

Returning to the threshold zero case, we recall that Proposition 2.12 in [12] implies that for any three-dimensional Gaussian vector with nonnegative correlations, the corresponding zero threshold process is a color process. Our next result says that this is not necessarily the case for four-dimensional Gaussian vectors.

Theorem 1.5. *There exists a four-dimensional standard Gaussian vector X with nonnegative correlations so that X^0 is not a color process. X can be taken to either be fully supported or not.*

In Section 6, we extend the study of the example given in the proof of the previous theorem to the stable case and derive as a consequence some surprising properties of integrals, which we state here. Two ingredients which are used in the proof is [8] and [12].

Theorem 1.6. *The following two integrals, to be taken in the Cauchy principal value sense as they are not Lebesgue integrable, are independent of α, the first having value $\pi^2/6$ and the second having value $\pi^2/4$.*

$$\lim_{\varepsilon \to 0} \int_{(\varepsilon, \pi/2-\varepsilon) \cup (\pi/2+\varepsilon, \pi-\varepsilon)} \frac{\log\left(|\cos\theta|^\alpha + |\sin\theta|^\alpha + |\cos\theta + \sin\theta|^\alpha\right)}{\alpha \cos\theta \sin\theta} d\theta,$$

$$\lim_{\varepsilon \to 0} \int_{(\varepsilon, \pi/2-\varepsilon) \cup (\pi/2+\varepsilon, \pi-\varepsilon)} \frac{\log\left(\frac{|\sin\theta|^\alpha}{2} + |\cos\theta + (\frac{1}{2})^{\frac{1}{\alpha}}\sin\theta|^\alpha\right)}{\alpha \cos\theta \sin\theta} d\theta.$$

In Section 7, we consider the large h Gaussian case. We show that any Gaussian vector which is not fully supported does not have a color a representation for large h; see Corollary 7.3. On the other hand, we show that *all* discrete Gaussian free fields (with constant variance) have color representations for large h.

Theorem 1.7. *If $X := (X_1, X_2, \ldots, X_n)$ is a discrete Gaussian free field which is standard Gaussian, then X^h is a color process for all sufficiently large h.*

In Section 8, we obtain detailed results concerning the existence of a color representation when the threshold $h \to 0$ and when $h \to \infty$ in the general Gaussian case when $n = 3$. In the fully supported case, we have the following result which gives an exact characterization of which Gaussian vectors have a color representation for large h. Note that if two of the covariances are zero, then we trivially have a color representation for all h.

Theorem 1.8. *Let X be a fully supported three-dimensional standard Gaussian vector with $Cov(X_i, X_j) = a_{ij} \in [0, 1)$ for $1 \le i < j \le 3$ and positive definite covariance matrix $A = (a_{ij})$.*

If $a_{ij} > 0$ for all $i < j$, then X^h has a color representation for sufficiently large h if and only if one of the following (nonoverlapping) conditions holds.

(i) $\mathbf{1}^T A^{-1} > \mathbf{0}$

(ii) $\min_i \mathbf{1}^T A^{-1}(i) = 0$

(iii) $\min_i \mathbf{1}^T A^{-1}(i) < 0$ and $\mathbf{1}^T A^{-1}\mathbf{1} < 2$.

Furthermore, if exactly one of the covariances is equal to zero, then X^h does not have a color representation for large h.

The assumption in (i) of Theorem 1.8, i.e. that $\mathbf{1}^T A^{-1} > \mathbf{0}$, is sometimes called the *Savage condition* (with respect to the vector $\mathbf{1} = (1, 1, \ldots, 1)$). When $A = (a_{ij})$ is the covariance matrix of a (nontrivial) two-dimensional standard Gaussian vector, then $\mathbf{1}^T A^{-1}(1) = \mathbf{1}^T A^{-1}(2) = (1 + a_{12})^{-1} > 0$, and hence the Savage condition always holds in this case. If $A = (a_{ij})$ is the covariance matrix of a three-dimensional standard Gaussian vector, then one can show that

$$\mathbf{1}^T A^{-1}(1) = \frac{(1 + a_{23} - a_{12} - a_{13})(1 - a_{23})}{\det A} \tag{1}$$

and it follows that the Savage condition holds if and only if

$$1 + 2 \min_{i<j} a_{ij} > \sum_{i<j} a_{ij}. \tag{2}$$

When $\mathbf{1}^T A^{-1} \ge \mathbf{0}$, we will refer to this as the *weak Savage condition*. This for example holds for all discrete Gaussian free fields.

The rest of the results we describe in this section concern the stable (non-Gaussian) case. To justify their study, we would want to know that the collection of threshold processes which can be obtained from stable vectors is not the same as those which can be obtained from Gaussian vectors. In fact, as described later on in this subsection, the former is a much larger class, thereby justifying the study of the stable case.

In Section 9, we first look at the $n = 2$ case. While it is trivial that having a color representation is equivalent to having a nonnegative correlation when $n = 2$, in the stable case it is not obvious, even when $n = 2$, which spectral measures yield a threshold vector with a nonnegative correlation. This contrasts with the Gaussian case where nonnegative correlation in the threshold process is simply equivalent to the Gaussian vector having a nonnegative correlation.

The following result from [10] is relevant in this context. Theorem 4.6.1 in [10] and its proof yield Part (i) and Theorem 4.4.1 in [10] (see also (4.4.2) on p. 188 there) easily yields (ii). We denote the standard one-dimensional symmetric α-stable distribution with scale one by $S_\alpha(1, 0, 0)$; see the next subsection for precise definitions.

Theorem 1.9. *Let $\alpha \in (0, 2)$ and let X be a symmetric 2-dimensional α-stable random vector with spectral measure Λ, which is such that both marginals have distribution $S_\alpha(1, 0, 0)$. Then the following hold.*

(i) Λ has support only in the first and third quadrants if and only if $X_1^{h_1}$ and $X_2^{h_2}$ are nonnegatively correlated for all $h_1, h_2 \in \mathbb{R}$.

(ii) If Λ has some support strictly inside the first quadrant, then X_1^h and X_2^h have strictly positive correlation for all sufficiently large h.

Interestingly, (i) is not equivalent to having X_1^h and X_2^h being nonnegatively correlated for all $h \in \mathbb{R}$. While we have not obtained a characterization of this in terms of the spectral measure, the following natural example shows that one does not need to have the spectral measure supported only in the first and third quadrants.

Proposition 1.10. *Let $S_1, S_2 \sim S_\alpha(1, 0, 0)$ be independent and let $a \in (0, 1)$. Set*
$$\begin{cases} X_1 = aS_1 + (1 - a^\alpha)^{1/\alpha} S_2 \\ X_2 = -aS_1 + (1 - a^\alpha)^{1/\alpha} S_2. \end{cases}$$

(This insures that $X_1, X_2 \sim S_\alpha(1, 0, 0)$.) Then the following are equivalent.

(i) X_1^0 and X_2^0 have nonnegative correlation

(ii) $a \leq 2^{-1/\alpha}$

(iii) X_1^h and X_2^h have nonnegative correlations for all h.

We now study the question of the existence of a color representation in the symmetric stable case when $h \to \infty$. Our first result shows that there is a fairly large class for which the answer is affirmative and here the method of proof comes from that used in Theorem 1.7.

Theorem 1.11. *Let X be a symmetric stable distribution whose spectral measure has some support properly inside each orthant. Furthermore, assume that*

$$2 \int_{\mathbb{S}^{n-1}} (\mathbf{x}^{(2)} \vee 0)^\alpha \, d\Lambda(\mathbf{x}) < 1 \tag{3}$$

where $\mathbf{x}^{(2)}$ denotes the second largest coordinate of the vector x. Then X^h is a color process for all sufficiently large h.

The integral condition in (3) will hold for example if the spectral measure is supported sufficiently close to the coordinate axes.

Next, we surprisingly obtain, in the simplest nontrivial stable vector with $n = 3$, a certain phase transition in the stability exponent where the critical point is $\alpha = 1/2$. We state it here although relevant definitions will be given later on.

Theorem 1.12. *Let $\alpha \in (0, 2)$ and let S_0, S_1, S_2, S_3 be i.i.d. each with distribution $S_\alpha(1, 0, 0)$. Furthermore, let $a \in (0, 1)$ and for $i = 1, 2, 3$, define*

$$X_i = aS_0 + (1 - a^\alpha)^{1/\alpha} S_i$$

and $X_\alpha := (X_1, X_2, X_3)$. ($X_\alpha$ is then a symmetric α-stable vector which is invariant under permutations; it is one of the simplest such vectors other than an i.i.d. process.)

(i) If $\alpha > 1/2$, then X^h is a color process for all sufficiently large h.

(ii) If $\alpha < 1/2$, then X^h is not a color process for any sufficiently large h.

The critical value of $1/2$ above was independent of the parameter a, as long as $a \in (0, 1)$. If we however move to a family which has two parameters, but is still symmetric and transitive, we can obtain a phase transition at any point in $(0, 2)$.

Theorem 1.13. *Let $a, b \in (0, 1)$ satisfy $2a^2 + 2b^2 < 1$. Let $c_1 = c_1(a, b) \in (0, 2)$ be the unique solution to $2a^{c_1} + 2b^{c_1} = 1$ and $c_2 = c_2(a, b) := \log 2 / |\log a - \log b| \in (0, \infty]$.*

For $\alpha \in (c_1, 2)$, let $S_1, S_2, \ldots, S_7$ be i.i.d. with $S_1 \sim S_\alpha(1, 0, 0)$ and define

$$\begin{cases} X_1 := aS_1 + bS_2 + bS_4 + aS_5 + (1 - 2a^\alpha - 2b^\alpha)^{1/\alpha} S_7 \\ X_2 := aS_2 + bS_3 + bS_5 + aS_6 + (1 - 2a^\alpha - 2b^\alpha)^{1/\alpha} S_7 \\ X_3 := bS_1 + aS_3 + aS_4 + bS_6 + (1 - 2a^\alpha - 2b^\alpha)^{1/\alpha} S_7. \end{cases}$$

Then $X_\alpha := (X_1, X_2, X_3)$ is a symmetric α-stable vector which is invariant under all permutations, and the following holds.

(i) If $c_2 \le c_1$, then, for all $\alpha \in (c_1, 2)$, X_α^h is a color process for all sufficiently large h.

(ii) If $c_2 \ge 2$, then, for all $\alpha \in (c_1, 2)$, X_α^h is not a color process for any sufficiently large h.

(iii) If $c_2 \in (c_1, 2)$, then, for all $\alpha \in (c_1, c_2)$, X_α^h is not a color process for any sufficiently large h while for all $\alpha \in (c_2, 2)$, X_α^h is a color process for all sufficiently large h.

In particular, for any $\alpha_c \in (0, 2)$ and $\epsilon < \alpha_c$, we can choose a and b so that $c_1 = \epsilon$ and $c_2 = \alpha_c$, in which case X_α is defined for all $\alpha \in (\epsilon, 2)$ and where the question of whether the large h threshold is a color process has a phase transition at α_c.

Remark 1.14. The case $a > b = 0$, which is not included above, corresponds to the fully symmetric case studied in Theorem 1.12.

In Section 10, we look at a somewhat different question. We ask which $\{0, 1\}$-valued random vectors $(X_1, \ldots, X_n)$, which are $\{0, 1\}$-symmetric, arise as the zero threshold of a Gaussian or stable vector. The answer turns out to be completely different in the two cases, which perhaps is not surprising since the stable vectors have a much larger parameter space.

We first point out that given any symmetric n-dimensional Gaussian vector X and any $\alpha \in (0, 1)$, there is an n-dimensional α-stable symmetric vector Y so that X^0 and Y^0 have the same distribution. Given X, we simply let $Y := W^{1/2} X$ where $W \sim S_{\alpha/2}(2(\cos \pi\alpha/4)^{2/\alpha}, 1, 0)$. By [10], p. 78, Y is an α-stable symmetric vector, and clearly X^0 and Y^0 have the same distribution since $W \ge 0$ as $\alpha/2 < 1$. In fact, more generally, if we let $TD_{\alpha,n}$ be the set of distributions on $\{0, 1\}^n$ which can be obtained by taking the zero threshold of some symmetric α-stable random vector, then for any $\alpha' > \alpha$ we have that

$$TD_{\alpha',n} \subseteq TD_{\alpha,n}. \tag{4}$$

9

Next, we give a simple example of a $\{0,1\}$-symmetric process with $n = 4$ which cannot be represented by a zero-threshold Gaussian process. Let (X_1, X_2, X_3, X_4) be uniformly distribution on the set of eight configurations which have three 1's and one -1 or three -1's and one 1. One quickly checks that (X_1, X_2, X_3, X_4) is pairwise independent. Hence if it corresponded to a zero-threshold Gaussian process, all of the six correlations would have to be zero in which case (X_1, X_2, X_3, X_4) would be i.i.d.

The above process is not a color process, but there are even color processes which cannot be represented by a zero-threshold Gaussian process when $n = 4$ based on dimension counting. The dimension of the space of standard Gaussians with $n = 4$ is 6-dimensional. On the other hand, Theorem 2.2(ii) implies that the dimension of the set of color processes is 7-dimensional when $n = 4$.

When $n = 2$, it is clear that all $\{0,1\}$-symmetric processes arise from some zero-threshold Gaussian vector. When $n = 3$, the relevant sets are each 3-dimensional and one can in fact show that, unlike when $n = 4$, the set of $\{0,1\}$-symmetric processes corresponds bijectively to the set of zero threshold Gaussian processes.

For stable vectors, the situation is completely different as stated in the following theorem.

Theorem 1.15. *Given any probability measure μ on $\{0,1\}^n$ and any $\epsilon > 0$, there exists an n-dimensional stable vector X such that the distribution of X^0 is within ϵ of μ. If desired, we could scale things so that each marginal has scale one. Moreover, if μ has a $\{0,1\}$-symmetry, then X can be taken to be a symmetric stable.*

Finally, we mention the following interesting application of Theorem 1.15 which again distinguishes stable vectors from Gaussian vectors. We know this result is false for $n = 3$ in the sense that if $n = 3$ then $\alpha_c = 0$.

Theorem 1.16. *(i). For all $n \geq 4$, there exists a critical value $\alpha_c(n) \in (0,2]$ so that for all $\alpha \in (\alpha_c(n), 2)$, the zero-threshold process for any fully transitive and symmetric α-stable vector of length n whose marginals have scale one is a color process provided the threshold process has pairwise nonnegative correlations while for all $\alpha \in (0, \alpha_c)$, there exists a fully transitive symmetric α-stable vector of length n whose marginals have scale one and whose zero-threshold has pairwise strictly positive correlations but does not have a color representation.*
(ii). For all $n \geq 4$ and $\epsilon > 0$, there exists $\alpha_c(n, \epsilon) < 2$ such that for all $\alpha \in (\alpha_c(n, \epsilon), 2)$, the zero-threshold process for any fully transitive and symmetric α-stable vector of length n whose marginals have scale one whose distance (in some fixed metric) to an i.i.d. standard normal vector is at least ϵ is a color process provided the threshold process has pairwise nonnegative correlations.

Finally, in Section 11, we state a number of open questions.

1.4 Background on symmetric stable vectors

We refer the reader to [10] for the theory of stable distributions and will just present here the background needed for our results.

Definition 1.17. A random vector $X := (X_i)_{1 \leq i \leq d}$ in $\mathbb{R}^d$ has a *stable* distribution if for all n, there exist $a_n > 0$ and b_n so that if $(X^1, \ldots, X^n)$ are n i.i.d.

copies of X, then

$$\sum_{1 \leq i \leq n} X^i \overset{\mathcal{D}}{=} a_n X + b_n.$$

It is known that for any stable vector, there exists $\alpha \in (0, 2]$ so that $a_n = n^{1/\alpha}$. The Gaussian case corresponds to $\alpha = 2$. Ignoring constant random variables, a stable random variable (i.e., with $d = 1$ above) has four parameters, (1) $\alpha \in (0, 2]$ which is called the *stability exponent*, (2) $\beta \in [-1, 1]$ which is called the *asymmetry parameter*, (3) σ which is a *scale parameter* and (4) μ which is a *shift parameter*. When $\alpha = 2$, there is no β parameter, μ corresponds to the mean and σ corresponds to the standard deviation divided by $\sqrt{2}$, an irrelevant scaling. The distribution of this random variable is denoted by $S_\alpha(\sigma, \beta, \mu)$. More precisely, $S_\alpha(\sigma, \beta, \mu)$ is defined by its characteristic function $f(\theta)$, which, for $\alpha \neq 1$ is

$$\exp\left(-\sigma^\alpha |\theta|^\alpha (1 - i\beta(\operatorname{sgn}\theta)\tan(\pi\alpha/2)) + i\mu\theta\right).$$

See [10] for the formula when $\alpha = 1$. One should be careful and keep in mind that different authors use different parameterizations for the family of stable distributions. Throughout this paper, we will only consider symmetric stable random variables corresponding to $\beta = \mu = 0$ and sometimes often assume $\sigma = 1$. The above then simplifies to a random variable having distribution $S_\alpha(\sigma, 0, 0)$ which means its characteristic function is $f(\theta) = e^{-\sigma^\alpha |\theta|^\alpha}$. In the symmetric case, this formula is also valid for $\alpha = 1$.

The structure of stable vectors is much more complicated than for Gaussian vectors which we illustrate in three ways.

First, the set of two-dimensional Gaussian vectors, where each marginal has mean zero and variance one, is one-dimensional, parameterized by the correlation. However, for fixed α, the set of *symmetric* two-dimensional stable vectors (symmetric means that the distribution is invariant under $\mathbf{x} \mapsto -\mathbf{x}$) where each marginal has distribution $S_\alpha(1, 0, 0)$ is infinite dimensional, corresponding essentially to the set of probability measures on the unit circle which are invariant under $\mathbf{x} \mapsto -\mathbf{x}$.

Secondly, if (X_1, X_2) is jointly Gaussian with $X_1 \overset{\mathcal{D}}{=} X_2$, then we know that $(X_1, X_2) \overset{\mathcal{D}}{=} (X_2, X_1)$; this is false in general for symmetric stable random vectors.

Thirdly, if (X_1, X_2) is jointly Gaussian, then either (i) (X_1^h, X_2^h) is nonnegatively correlated for all h (which corresponds to X_1 and X_2 being nonnegatively correlated) or (ii) (X_1^h, X_2^h) is nonpositively correlated for all h (which corresponds to X_1 and X_2 being nonpositively correlated); this is false in general for symmetric stables in that the thresholds can be positively correlated for some h and negatively correlated for other h. This will be further discussed in Section 9.

Finally, a random vector in $\mathbb{R}^d$ has a symmetric stable distribution with stability exponent α if and only if its characteristic function $f(\theta)$ has the form

$$f(\theta) = \exp(-\int_{\mathbb{S}^{d-1}} |\theta \cdot \mathbf{x}|^\alpha \, d\Lambda(\mathbf{x}))$$

for some finite measure Λ on $\mathbb{S}^{d-1}$ which is invariant under $\mathbf{x} \mapsto -\mathbf{x}$. Λ is called the *spectral measure* corresponding to the α-stable vector. For $\alpha \in (0, 2)$ fixed, different Λ's yield different distributions. This is not true for $\alpha = 2$.

In a number of cases, we will have a symmetric α-stable vector $X := (X_1, \ldots, X_d)$ which is obtained by having

$$X = A(Y_1, \ldots, Y_m)$$

where A is a $d \times m$ matrix and $Y = (Y_1, \ldots, Y_m)$ are i.i.d. random variables with distribution $S_\alpha(1, 0, 0)$. In such a case, there is a simple formula for the spectral measure Λ for X. Consider the columns of A as elements of $\mathbb{R}^d$, denoted by $\mathbf{x}_1, \ldots, \mathbf{x}_m$. Then Λ is obtained by placing, for each $i \in [m]$, a mass of weight $\|\mathbf{x}_i\|_2^\alpha / 2$ at $\pm \mathbf{x}_i / \|\mathbf{x}_i\|_2$. See p. 69 in [10].

2 The induced linear operator

2.1 General set up

In order to understand when one has uniqueness of a color representation, it is natural to consider signed measures on $\mathcal{B}_n$ so that one can place the question in a vector space context where uniqueness (essentially) corresponds to the kernel of the associated operator being trivial (see [12]). This is done as follows.

We let $\Phi_{n,p}$ be the map taking random partitions of $[n]$ to probability vectors on $\{0, 1\}^n$; this just sends the random partition to its corresponding color process. For each n and p, we have a natural linear mapping $A_{p,n}$ from $\mathbb{R}^{\mathcal{B}_n}$ to $\mathbb{R}^{\{0,1\}^n}$ which extends $\Phi_{n,p}$. We can identify $\mathcal{B}_n$ with the natural basis for $\mathbb{R}^{\mathcal{B}_n}$ and hence $A_{p,n}$ is uniquely determined by describing the image of each $\sigma \in \mathcal{B}_n$. Given $\sigma \in \mathcal{B}_n$ and a binary string $\rho \in \{0, 1\}^n$, we write $\sigma \lhd \rho$ if ρ is constant on the partition elements of σ. We clearly have

$$A_{p,n}(\sigma)_\rho := A_{p,n}(\rho, \sigma) := \begin{cases} p^{c(\sigma,\rho)}(1-p)^{\|\sigma\|-c(\sigma,\rho)} & \text{if } \sigma \lhd \rho \\ 0 & \text{otherwise} \end{cases}$$

where $\|\sigma\|$ is equal to the number of partition elements in the partition σ and $c = c(\sigma, \rho)$ is the number of partition elements on which ρ is 1. For $\rho \in \{0, 1\}^n$, we write $-\rho$ to denote the binary string where the zeros and ones in ρ are switched, i.e. $-\rho = 1 - \rho$.

In [12], the dimension of the kernel of $A_{p,n}$, and hence of the range, was determined for a few small values of n. The following result gives the formula for these in general, and also gives an explicit description of the range of $A_{p,n}$ for all n and p. The analysis of the $p = 1/2$ case is much easier and we do that in the next subsection while the $p \neq 1/2$ case is done in the subsequent subsection. In the last subsection, we look at the situation where there is invariance with respect to permutations of $[n]$.

We end this subsection with a few more elementary remarks. $A_{p,n}$ sends nonnegative vectors to nonnegative vectors and if $q = (q_\sigma)_{\sigma \in \mathcal{B}_n}$ is sent to ν under $A_{p,n}$, then $\sum_{\rho \in \{0,1\}^n} \nu_\rho = \sum_{\sigma \in \mathcal{B}_n} q_\sigma$.

Definition 2.1. Given n and p, if $\nu \in \mathbb{R}^{\{0,1\}^n}$ corresponds to a probability vector (i.e., the coordinates of ν are nonnegative and add to one), an element $q \in \mathbb{R}^{\mathcal{B}_n}$ so that

$$A_{p,n}q = \nu \tag{5}$$

is called a *formal solution* while a nonnegative element $q \in \mathbb{R}^{\mathcal{B}_n}$ so that $A_{p,n}q = \nu$ is called a *nonnegative solution*.

Of course, there is a nonnegative solution if and only if q is a color process. It will however be convenient in the analysis to allow formal solutions (which are not necessarily nonnegative) which one might show afterwards are in fact nonnegative solutions.

Lastly, we explain the relationship between nontriviality of the kernel of $A_{p,n}$ and uniqueness of a color representation (i.e., uniqueness of a nonnegative solution). First observe that $A_{p,n}$ has a nontrivial kernel if and only if for any ν in the range, there are an infinite number of formal solutions. Hence if the kernel is trivial, there is always at most one nonnegative solution. The converse is not true since an i.i.d. process clearly has at most one color representation even if the kernel is nontrivial. However, as also explained in [12], if the kernel is nontrivial, ν is in the range and there exists a nonnegative solution all of whose coordinates are positive, then one has infinitely many nonnegative solutions since we can add a small constant times an element in the kernel. More generally, if ν is in the range and there exists a nonnegative solution q for ν, then there is another nonnegative solution (and then infinitely many) if and only if there is an element q' in the kernel whose negative-valued coordinates are contained in the support of q.

2.2 Formal solutions for the $p = 1/2$ case

Theorem 2.2. *(i). Given a $\{0, 1\}$-symmetric probability vector $\nu = (\nu_\rho)$ on $\{0, 1\}^n$, a formal solution (i.e. a solution to $A_{\frac{1}{2}, n} q = \nu$) is given by*

$$q_\sigma = \begin{cases} 2(\nu_{0_S 1_{S^c}} + \nu_{1_S 0_{S^c}}), & \text{if } \sigma = \{S, S^c\}, \ S \neq \emptyset, [n] \\ 1 - \sum_{\sigma' : |\sigma'| = 2} q_{\sigma'} & \text{if } \sigma = [n] \\ 0 & \text{otherwise.} \end{cases}$$

In addition, this yields a color representation (i.e., a nonnegative solution to $A_{\frac{1}{2}, n} q = \nu$) if and only if

$$\nu_{00\ldots0} + \nu_{11\ldots1} \geq 0.5. \tag{6}$$

(Obviously, this condition is not necessary for there to be some color representation.)

(ii). The range of $A_{\frac{1}{2}, n}$ is

$$\left\{ (\nu_\rho) : \forall \rho, \ \nu_\rho = \nu_{-\rho} \right\}. \tag{7}$$

As a consequence, $A_{\frac{1}{2}, n}$ has rank 2^{n-1} and hence nullity $|\mathcal{B}_n| - 2^{n-1}$.

Proof of (i). It is elementary to check that $\{q_\sigma\}$ above yields a formal solution. Note that one has

$$q_{[n]} = 1 - \sum_{\sigma : |\sigma| = 2} q_\sigma = 1 - \sum_{\{S, S^c\} : 0 < |S| < n} 2(\nu_{0_S 1_{S^c}} + \nu_{1_S 0_{S^c}})$$
$$= 1 - 2(1 - (\nu_{00\ldots0} + \nu_{11\ldots1})).$$

This clearly yields the second statement of (i). $\qquad \square$

13

Proof of (ii). Clearly every element of the range must satisfy the symmetry condition (7) since $p = 1/2$. Using (i) and the fact that the symmetric probability vectors in (i) generate the subspace given by (7), it follows that the range of $A_{\frac{1}{2},n}$ is precisely this set. The statement concerning the rank and nullity follow immediately. $\square$

Remark 2.3. We check what (7) amounts to when $X := (X_i)_{1 \le i \le n}$ is a standard Gaussian vector with covariance matrix (a_{ij}) for which $Y = X^0$ is of course $\{0,1\}$-symmetric. We will see later (21) that $P(Y_i \ne Y_j) = \frac{\arccos a_{ij}}{\pi}$ which is known as Sheppard's formula (see [11]). Next

$$1 - (\nu_{00\ldots n} + \nu_{11\ldots 1}) \le \sum_{i<j} P(Y_i \ne Y_j) = \sum_{i<j} \frac{\arccos a_{ij}}{\pi}$$

$$\le \frac{1}{\pi} \binom{n}{2} \sup_{i,j} \arccos a_{ij}.$$

Hence the inequality needed in Theorem 2.2 will be satisfied whenever

$$\inf_{i,j} a_{ij} \ge \cos\left(\frac{\pi}{n(n-1)}\right) \left(\approx 1 - \frac{\pi^2}{2n^4}\right).$$

2.3 Formal solutions for the $p \ne 1/2$ case

Theorem 2.4. *For $p \notin \{0, 1/2, 1\}$, $A_{p,n}$ has rank $2^n - n$ and hence nullity $|\mathcal{B}_n| - (2^n - n)$. The range of $A_{p,n}$ is equal to*

$$\Big\{\{\nu_\rho\} : \forall i, \ p \sum_{\rho:\rho(i)=0} \nu_\rho = (1-p) \sum_{\rho:\rho(i)=1} \nu_\rho \Big\}. \tag{8}$$

(The vector space defined by (8) is the analogue of the marginal distributions each being $p\delta_1 + (1-p)\delta_0$.) In particular, if ν is a probability vector on $\{0,1\}^n$, all of whose marginals are $p\delta_1 + (1-p)\delta_0$, then ν is in the range of $A_{p,n}$. (Of course, there might be no probability vector $q = (q_\sigma)_{\sigma \in \mathcal{B}_n}$ which maps to ν; i.e. ν need not be a color process.)

Proof.

Step 1. *The rank of $A_{p,n}$ is at least $2^n - n$.*

Proof of Step 1. We will modify our system of equations in such a way that things become more transparent. First, for $\sigma \in \mathcal{B}_n$ and $T \subseteq [n]$, we let σ_T be the restriction of σ to T. Also, let

$$\nu(1^T) := \sum_{\rho:\ \rho(T)\equiv 1} \nu_\rho.$$

We now consider the system of linear equations given by

$$\nu(1^T) = \sum_{\sigma \in \mathcal{B}_n} p^{\|\sigma_T\|} q_\sigma, \qquad T \subseteq [n]. \tag{9}$$

By an inclusion-exclusion argument, one sees that (9) is equivalent to (5). Letting $A'_{p,n}$ be the corresponding $2^n \times |\mathcal{B}_n|$ matrix for this system, it suffices to show that the rank of $A'_{p,n}$ is at least $2^n - n$.

Let $\sigma^{\emptyset}$ be the partition into singletons and for each $T \subseteq [n]$ with $|T| > 1$, let $\sigma^T \in \mathcal{B}_n$ be the unique partition with exactly one non-singleton partition element given by T. If e.g. $n = 5$ we would have that $\sigma^{\{1,2,3\}} = (123, 4, 5)$. One easily verifies that $\|\sigma_S^T\| = |S \backslash T| + (1 \wedge |S \cap T|)$ for $T = \emptyset$ or $|T| > 1$.

Consider the equation system

$$\nu(1^S) = \sum_{T\,:\,|T| \neq 1} p^{\|\sigma_S^T\|} q_{\sigma^T}, \qquad S \subseteq [n]$$

and let $A'' = A''_{p,n}$ be the corresponding $2^n \times (2^n - n)$ matrix. Define $B = (B(S, S'))_{S,S' \subseteq [n]}$ by

$$B(S, S') := (-p)^{|S| - |S'|} I(S' \subseteq S).$$

If we order the rows (from top to bottom) and columns (from left to right) of B such that the sizes of the corresponding sets are increasing, then B is a lower triangular matrix with $B(S, S) = 1$ for all $S \subseteq [n]$. In particular, this implies that B is invertible for all $p \in (0, 1)$, and hence A'' and BA'' (also a $2^n \times (2^n - n)$ matrix) have the same rank. Moreover, for any $S, T \subseteq [n]$ with $|T| \neq 1$ we get

$$
\begin{aligned}
(BA'')(S, T) &= \sum_{S' \subseteq S} (-p)^{|S| - |S'|} A''(S', T) \\
&= \sum_{S' \subseteq S} (-p)^{|S| - |S'|} p^{|S' \backslash T| + (1 \wedge |S' \cap T|)} \\
&= (-p)^{|S|} \sum_{S' \subseteq S} (-p)^{-|S'|} p^{|S' \backslash T|} p^{1 \wedge |S' \cap T|} \\
&= (-p)^{|S|} \sum_{S' \subseteq S} (-p)^{-|S' \cap T|} (-1)^{|S' \backslash T|} p^{1 \wedge |S' \cap T|} \\
&= \begin{cases} (-p)^{|S|} \sum_{S' \subseteq S} (-p)^{-|S'|} p^{1 \wedge |S'|} & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

In the case $S \subseteq T$, we can simplify further to obtain

$$
\begin{aligned}
(BA'')(S, T) &= (-p)^{|S|} \sum_{S' \subseteq S} (-p)^{-|S'|} p^{1 \wedge |S'|} \\
&= (-p)^{|S|} \left((1 - p^{-1})^{|S|} \cdot p + (1 - p) \right) \\
&= p(1 - p)^{|S|} + (-p)^{|S|}(1 - p).
\end{aligned}
$$

Note that since $p \neq 1/2$, if $S \subseteq T$, then $(BA'')(S, T) = 0$ if and only if $|S| = 1$. If we order the rows (from top to bottom) and columns (from left to right) of BA'' so that the corresponding sets are increasing in size, it is obvious that the $(2^n - n) \times (2^n - n)$ submatrix of BA'' obtained by removing the rows corresponding to $|S| = 1$ has full rank. This implies that BA'' has rank at least $2^n - n$ which implies the same for A'' since B is invertible. Finally, since A'' is a submatrix of $A'_{p,n}$, we obtain the desired lower bound on the rank of the latter. $\qquad \square$

Step 2. *The rank of $A_{p,n}$ is at most $2^n - n$.*

Proof of Step 2. We first claim that if $\nu = \{\nu_\rho\}_{\rho \in \{0,1\}^n}$ is in the range, then it is in the set defined in (8). To see this, let $\nu = A_{p,n} q$ and fix an i. The expression in the left hand side of (8) becomes

$$p \sum_{\rho:\rho(i)=0} \sum_\sigma A_{p,n}(\rho,\sigma)q_\sigma = p \sum_\sigma q_\sigma \sum_{\rho:\rho(i)=0} A_{p,n}(\rho,\sigma).$$

With σ fixed, let

$$T_\sigma^i : \{\rho : \rho(i) = 0\} \mapsto \{\rho : \rho(i) = 1\}$$

be the bijection which flips ρ on the partition element of σ which contains i. It is clear that for all ρ with $\rho(i) = 0$, we have

$$(1-p)A_{p,n}(T_\sigma^i(\rho),\sigma) = pA_{p,n}(\rho,\sigma)$$

and hence the previous expression is

$$(1-p)\sum_\sigma q_\sigma \sum_{\rho:\rho(i)=0} A_{p,n}(T_\sigma^i(\rho),\sigma) = (1-p)\sum_\sigma q_\sigma \sum_{\rho:\rho(i)=1} A_{p,n}(\rho,\sigma) =$$

$$(1-p)\sum_{\rho:\rho(i)=1} \sum_\sigma A_{p,n}(\rho,\sigma)q_\sigma = (1-p)\sum_{\rho:\rho(i)=1} \nu_\rho.$$

$A_{p,n}$ is mapping into a 2^n-dimensional vector space and each of the n equations in (8) gives one linear constraint. It is easy to see that these n constraints are linearly independent (for example, one can see this by just looking at the number of times each of the vectors $0^k 1^{n-k}$ appears on the two sides). It follows that the rank of $A_{p,n}$ is at most $2^n - n$. $\qquad\square$

With Steps 1 and 2 completed, together with the claim at the start of Step 2, we conclude that the rank is as claimed and the range is characterized as claimed. Finally, the claim concerning probability vectors follows immediately.
$\qquad\square$

Remark 2.5.

(i) The argument for the $p \neq 1/2$ can equally well be carried out with minor modifications for the $p = 1/2$ case but we preferred the simpler argument which even gives more.

(ii) This last proof shows that, when dealing with formal solutions, we only need to use partitions which have at most one nonsingleton partition element. This is in large contrast to the earlier proof of the $p = 1/2$ case where we only needed to use partitions which have at most two partition elements.

(iii) The rank of an operator as a function of its matrix elements is not continuous but it is easily seen to be lower semicontinuous. We see this lack of continuity at $p = 1/2$ as well as of course at $p = 0$ and $p = 1$.

Since we will often deal with the case $n = 3$, it will be convenient to have at our disposal the formula for the unique formal solution when (1) $p \neq 1/2$ and (2) the distribution ν of (X_1, X_2, X_3) satisfies (8) with marginals p as well as the general solution for the case when the distribution is $\{0,1\}$-symmetric (i.e.,

satisfies (7) and hence $p = 1/2$). One can immediately check the validity of the following. The form of the kernel (that there is one free variable and it appears as it does) is discussed in [12]. If $p \notin \{0, 1/2, 1\}$, then $A_{p,3}\, q = \nu$ has a unique solution $(q_\sigma)_{\sigma \in \mathcal{B}_3}$ given by

$$\begin{cases} q_{1,2,3} &= \frac{\nu_{100} - \nu_{011}}{(1-p)p(1-2p)} \\ q_{12,3} &= \frac{(1-p)\nu_{110} - p\nu_{001}}{(1-p)p(1-2p)} \\ q_{13,2} &= \frac{(1-p)\nu_{101} - p\nu_{010}}{(1-p)p(1-2p)} \\ q_{1,23} &= \frac{(1-p)\nu_{011} - p\nu_{100}}{(1-p)p(1-2p)} \\ q_{123} &= 1 - \frac{\nu_1 \nu_{000} - \nu_0 \nu_{111}}{(1-p)p(1-2p)} \end{cases} \tag{10}$$

and if $p = 1/2$, then every solution $q = (q_\sigma)_{\sigma \in \mathcal{B}_3}$ of $A_{\frac{1}{2},3}\, q = \nu$ has the form

$$\begin{cases} q_{123} &= 1 - 4(\nu_{001} + \nu_{010} + \nu_{100}) + t \\ q_{12,3} &= 4\nu_{001} - t \\ q_{1,23} &= 4\nu_{100} - t \\ q_{13,2} &= 4\nu_{010} - t \\ q_{1,2,3} &= 2t, \end{cases} \tag{11}$$

where $t \in \mathbb{R}$ is a free variable.

2.4 The fully invariant case

It is sometimes natural to consider situations where one has some further invariance property. One natural case is the following. The symmetric group S_n acts naturally on $\mathcal{B}_n$, $\{0,1\}^n$, $\mathcal{P}(\mathcal{B}_n)$, $\mathcal{P}(\{0,1\}^n)$, $\mathbb{R}^{\mathcal{B}_n}$ and $\mathbb{R}^{\{0,1\}^n}$ where $\mathcal{P}(X)$ denotes the set of probability measures on X. (Of course $\mathcal{P}(\mathcal{B}_n) \subseteq \mathbb{R}^{\mathcal{B}_n}$ and the action on the former is just the restriction of the action on the latter; similarly for $\mathcal{P}(\{0,1\}^n) \subseteq \mathbb{R}^{\{0,1\}^n}$.) To understand uniqueness of a color representation when we restrict to S_n-invariant probability measures, it is natural to again extend to the vector space setting, which is done as follows.

Let $Q_n^{\mathrm{Inv}} := \{q \in \mathbb{R}^{\mathcal{B}_n} : g(q) = q \ \forall g \in S_n\}$ and $V_n^{\mathrm{Inv}} := \{\nu \in \mathbb{R}^{\{0,1\}^n} : g(\nu) = \nu \ \forall g \in S_n\}$. We next let $A_{p,n}^{\mathrm{Inv}}$ be the restriction of $A_{p,n}$ to Q_n^{Inv}. It is elementary to check that $A_{p,n}^{\mathrm{Inv}}$ maps into V_n^{Inv} and furthermore, it is easy to check (see for example Proposition 3.11 in [12] for something related) that

$$A_{p,n}^{\mathrm{Inv}}(Q_n^{\mathrm{Inv}}) = A_{p,n}(\mathbb{R}^{\mathcal{B}_n}) \cap V_n^{\mathrm{Inv}}. \tag{12}$$

Recalling that P_n is the set of partitions of the integer n, we have an obvious mapping from $\mathcal{B}_n$ to P_n, denoted by $\sigma \mapsto \pi(\sigma)$. $\mathbb{R}^{P_n}$ can then be canonically identified with Q_n^{Inv} via $(q_\pi)_{\pi \in P_n}$ is identified with $(q_\sigma)_{\sigma \in \mathcal{B}_n}$ where $q_\sigma = \frac{q_{\pi(\sigma)}}{a_{\pi(\sigma)}}$ where a_π is the number of σ's for which $\pi(\sigma) = \pi$. We divide by a_π in order that probability vectors will be identified with probability vectors. In an analogous way, V_n^{Inv} can be canonically identified with $\mathbb{R}^{n+1}$ in such a way that probability vectors are identified with probability vectors; namely, $(\nu_i)_{0 \le i \le n}$ is identified with $(\nu_\rho)_{\rho \in \{0,1\}^n}$ where $\nu_\rho = \frac{\nu_{\|\rho\|}}{\binom{n}{\|\rho\|}}$ and $\|\rho\|$ is the number of ones in the binary string ρ.

The following is the analogue of Theorem 2.4. Again, in [12], this was done for some small values of n.

17

Theorem 2.6. *(i). For $p \notin \{0, 1/2, 1\}$, $A_{p,n}^{\mathrm{Inv}}$ has rank n and hence nullity $|P_n| - n$. The range of $A_{p,n}^{\mathrm{Inv}}$ (after identifying V_n^{Inv} with $\mathbb{R}^{n+1}$) is*

$$\left\{ (\nu_0, \ldots, \nu_n) : \nu_n = \frac{p}{1-p} \sum_{k=0}^{n-1} \frac{n-k}{n} \nu_k - \sum_{k=0}^{n-2} \frac{k+1}{n} \nu_{k+1} \right\}. \tag{13}$$

(ii) $A_{\frac{1}{2},n}^{\mathrm{Inv}}$ has rank $\lfloor n/2 \rfloor + 1$ and hence nullity $|P_n| - \lfloor n/2 \rfloor - 1$. The range of $A_{\frac{1}{2},n}$ (after identifying V_n^{Inv} with $\mathbb{R}^{n+1}$) is

$$\left\{ (\nu_0, \ldots, \nu_n) : \nu_i = \nu_{n-i} \ \forall i = 1, \ldots, n \right\}. \tag{14}$$

Proof. (i). Denoting by U_n the subset of $\mathbb{R}^{n+1}$ satisfying (13), we claim that

$$U_n = A_{p,n}^{\mathrm{Inv}}(Q_n^{\mathrm{Inv}}). \tag{15}$$

Since U_n is clearly an n-dimensional subspace of $\mathbb{R}^{n+1}$, the proof of (i) will then be done.

To see this, first take $\nu^{\mathrm{Inv}} \in V_n^{\mathrm{Inv}}$ and let ν be the corresponding element in U_n. We first need to show that (8) is satisfied ν^{Inv}. Fixing any i, we have

$$p \sum_{\rho:\rho(i)=0} \nu_\rho^{\mathrm{Inv}} = p \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{\nu_k}{\binom{n}{k}} = p \sum_{k=0}^{n-1} \frac{n-k}{n} \nu_k$$

and

$$(1-p) \sum_{\rho:\rho(i)=1} \nu_\rho^{\mathrm{Inv}} = (1-p) \sum_{k=0}^{n-2} \binom{n-1}{k} \frac{\nu_{k+1}}{\binom{n}{k+1}} + (1-p)\nu_n$$

$$= (1-p) \sum_{k=0}^{n-2} \frac{k+1}{n} \nu_{k+1} + (1-p)\nu_n.$$

Hence, since $\nu \in U_n$, (8) holds. In view of (12), this shows $\subseteq$ in (15) holds.

Now fix $\nu^{\mathrm{Inv}} \in A_{p,n}^{\mathrm{Inv}}(Q_n^{\mathrm{Inv}})$. Clearly $\nu^{\mathrm{Inv}} \in V_n^{\mathrm{Inv}}$ and by Theorem 2.4, (8) holds. The above computation shows that the corresponding $\nu \in \mathbb{R}^{n+1}$ satisfies (13) and hence is in U_n. This shows $\supseteq$ in (15) holds as well.

(ii). Denoting now by U_n the subset of $\mathbb{R}^{n+1}$ satisfying (14), we claim that

$$U_n = A_{\frac{1}{2},n}^{\mathrm{Inv}}(Q_n^{\mathrm{Inv}}). \tag{16}$$

Since U_n is clearly an $(\lfloor n/2 \rfloor + 1)$-dimensional subspace of $\mathbb{R}^{n+1}$, the proof of (ii) will then be done. However, in view of (7) in Theorem 2.4 and (12), this is immediate. $\qquad \square$

3 Stieltjes matrices, discrete Gaussian free fields and tree-indexed Gaussian Markov chains

3.1 Inverse Stieltjes covariance matrices give rise to color processes for $h = 0$

Definition 3.1. A Stieltjes matrix is a symmetric positive definite matrix with non-positive off-diagonal elements.

We will see later that the following result implies that for all discrete Gaussian free fields X (to be defined later), X^0 is a color process.

Theorem 3.2. *If $X \sim N(0, A)$ and A^{-1} is a Stieltjes matrix, then X^0 is a color process.*

The proof is based upon noting that the observation from [6] that the signs of a discrete Gaussian free field is an average of Ising Models is still true if one just assumes that one has a Stieltjes matrix.

Proof. Note first that as $(b_{ij}) := A^{-1}$ is a Stieltjes matrix, we have that $b_{ij} \leq 0$ whenever $i \neq j$. This implies in particular that if f_X is the probability density function of X, then

$$f_X(\mathbf{x}) \propto \exp\left(\frac{-\mathbf{x}^T A^{-1} \mathbf{x}}{2}\right) = \exp\left(\sum_{\{i,j\}} -b_{ij}\mathbf{x}_i\mathbf{x}_j - \frac{1}{2}\sum_i b_{ii}\mathbf{x}_i^2\right).$$

Now for each i, define $\sigma_i := \operatorname{sgn} X_i$ so that $X_i = |X_i|\sigma_i$. Then the conditional probability density function of (σ_i) given $|X_1| = y_1$, $|X_2| = y_2$, ..., $|X_n| = y_n$ satisfies

$$f(\sigma) \propto \exp\left(\sum_{\{i,j\}} -b_{ij}y_iy_j\sigma_i\sigma_j\right).$$

This is a ferromagnetic Ising model with parameters $\beta_{ij} = -b_{ij}y_iy_j \geq 0$ and no external field. It is well known that the (Fortuin Kastelyn) random cluster model yields a color representation for the Ising model after we identify -1 with 0. Since an average of color processes is a color process, we are done. $\square$

Remark 3.3. The proof of Theorem 3.2 does not apply to other threshold levels. With nonzero thresholds, this argument would lead to Ising model with a varying external field. The marginals of this (conditioned) process are not in general equal, which precludes it from being a color process, and even if the marginals were equal, there is no known color representation in this case in general.

We end this subsection by pointing out that there are fully supported Gaussian vectors whose threshold zero processes are color processes but whose inverse covariance matrix is not a Stieltjes matrix.

To see this, let $a \in (0, 1)$ and $\varepsilon \in (0, 1)$. Then the matrix

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a^2 - \varepsilon \\ a & a^2 - \varepsilon & 1 \end{pmatrix}$$

has eigenvalues $1 - a^2 + \varepsilon$ and $\frac{2 + a^2 - \varepsilon \pm \sqrt{8a^2 + (a^2 - \varepsilon)^2}}{2}$. Hence A is positive definite if $\varepsilon < 1 - a^2$. Moreover, we have

$$A^{-1} = \frac{1}{1 - a^2 - \varepsilon} \begin{pmatrix} 1 + a^2 - \varepsilon & -a & -a \\ -a & \frac{1-a^2}{1-a^2+\varepsilon} & \frac{\varepsilon}{1-a^2+\varepsilon} \\ -a & \frac{\varepsilon}{1-a^2+\varepsilon} & \frac{1-a^2}{1-a^2+\varepsilon} \end{pmatrix}$$

Hence, A is not an inverse Stieltjes matrix for any $\varepsilon > 0$, since for any $\varepsilon > 0$ we have that $A^{-1}(2,3) > 0$. Consequently, if $0 < \varepsilon < 1 - a^2$, then A is symmetric, positive and positive definite but not an inverse Stieltjes matrix. Finally, the fact that the threshold zero process is a color process follows from Proposition 2.12 in [12] which states that for $n = 3$, any $\{0,1\}$-symmetric process with nonnegative pairwise correlations is a color process.

3.2 Examples of Stieltjes matrices: discrete Gaussian free fields and embeddings into tree-indexed Gaussian Markov chains

In this subsection, we provide two (partially overlapping) examples of fully supported Gaussian vectors whose inverse covariance matrices are Stieltjes.

3.2.1 Discrete Gaussian free fields

An extremely important class of Gaussian processes are the so-called discrete Gaussian free fields which we now define. Let $n \in \mathbb{N}$. Given n, let $(c(\{x,y\}))_{\{x,y\} \subseteq [n], \, x \neq y}$ and $(\kappa(x))_{x \in [n]}$ be two sets of nonnegative real numbers, called *conductances* and *killing rates* respectively. Let $Z = (Z_t)_{t \in [0,\zeta]}$ be a continuous time sub-Markovian jump process on $[n]$, with transition rate between $x \in [n]$ and $y \in [n]$ given by $c(x,y)$, and $\kappa(x)$ being the transition rate from $x \in [n]$ to a cemetery point outside of $[n]$. When this latter transition occurs, we think of the process as being killed, and denote this random time by ζ. Let $(G(x,y))_{x,y \in [n]}$ be the Green's function for Z, i.e.

$$G(x,y) := \mathbb{E}_x \left[\int_0^\zeta I(Z_t = y) dt \right]. \tag{17}$$

We will restrict consideration to the case when G is finite for all x and y. Defining the graph $\mathcal{G} = (V, E)$ where V is indexed by $[n]$ and $\{x,y\} \in E$ if and only if $c(\{x,y\}) > 0$, we have $G(x,y) < \infty$ for all x and y if and only if each component of $\mathcal{G}$ contains an x with $\kappa(x) > 0$.

It is well known that G is a symmetric and positive definite matrix. The mean zero multivariate Gaussian vector X whose covariance matrix is G is called the *discrete Gaussian free field* (DGFF) associated to $\{c(x,y)\}$ and $\{\kappa(x)\}$.

Letting C denote the matrix

$$C(x,y) = \begin{cases} -c(x,y) & \text{if } x \neq y \\ \kappa(x) + \sum_{x' : \, x' \neq x} c(x,x') & \text{if } x = y. \end{cases}$$

It is well known and easy to check that $C = G^{-1}$. Consequently, the inverse covariance matrix of a DGFF $X \sim N(0, G)$ is a Stieltjes matrix, yielding the following corollary of Theorem 3.2.

Corollary 3.4. *The threshold zero process corresponding to a DGFF is a color process.*

We point out that if $X \sim N(0, G)$ with G invertible and if $B := G^{-1}$ is a Stieltjes matrix *with positive row sums*, then X is a DGFF. To see this, one simply lets $c(x,y) := -b(x,y)$ for $x \neq y$ and $\kappa(x) := \sum_y b(x,y)$ and then checks

that B is the matrix C given above using these rates. Hence $G = C^{-1}$ as desired.

Does the set of inverse Stieltjes matrices give us a larger class than the set of DGFFs? The answer is yes and no. If we look at the process, the answer is yes. For example

$$\begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1/2 \end{pmatrix}^{-1}$$

is a Stieltjes matrix but cannot correspond to a DGFF since the sum of the elements in the first row is negative.

However, if we look at the corresponding threshold zero processes, one obtains the same collection. Therefore Theorem 3.2 is equivalent to the result for DGFFs. To see the previous statement, let X^G be a Gaussian vector whose inverse covariance matrix $B = G^{-1}$ is a Stieltjes matrix. It is known (see e.g. K_{35} in [9]) that there exists a positive diagonal matrix D so that BD has strictly positive row sums. It follows that DBD has strictly positive row sums and it is easy to check that DBD is still a Stieltjes matrix. By the earlier discussion, the Gaussian vector Y with covariance matrix $(DBD)^{-1}$ is a DGFF. On the other hand, $Y = D^{-1}X$ and hence the sign processes are the same.

3.2.2 Embeddings into tree-indexed Gaussian Markov chains

We now consider a certain class of tree-indexed Markov chains as follows. Let (T, ρ) be a finite or infinite tree with a designated root ρ. Fix a state space S and assume that for each edge e, we have a Markov chain M_e on S. Assume that there is a probability measure μ on S which is a stationary distribution for each M_e. This yields a measure on S^T by choosing the state at ρ according to μ and then independently running the Markov chains going outward from ρ. Note that the final process is independent of ρ if and only if μ is reversible with respect to each M_e.

Now let $S = \mathbb{R}$ and let M_e be given by $s \mapsto b_e s + (1 - b_e^2)^{1/2}W$ where $b_e \in (0, 1)$ and W is a standard normal random variable. Clearly the standard Gaussian distribution is reversible with respect to each of these chains. The final tree-indexed process is clearly a mean zero, variance one Gaussian process whose covariance between two vertices is the product of the b_e values along the path between the two vertices. If the tree is finite with covariance matrix A, it is straightforward to show that the inverse covariance matrix is given by

$$A^{-1}(i, j) = \begin{cases} -b_e/(1 - b_e^2) & \text{if } \{i, j\} = e \\ 1 + \sum_{e \,:\, i \in e} b_e^2/(1 - b_e^2) & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since A is an inverse Stieltjes matrix and since every principal submatrix of an inverse Stieltjes matrix is known to be an inverse Stieltjes matrix (see e.g. [7]), we obtain the following corollary.

Corollary 3.5. *If X is embeddable into a tree-indexed Gaussian Markov chain (meaning that there is a tree-indexed Gaussian Markov chain and a subset U of the vertices of the tree such that the process restricted to U corresponds to X), then the covariance matrix for X is an inverse Stieltjes matrix and hence the corresponding threshold zero process is a color process.*

Of course, if we just wanted to know it is a color process, one does not need the result from [7] since the covariance matrix for the full tree is, as we saw, an inverse Stieljtes matrix and a subset of a color process is of course a color process.

3.2.3 Relationship between the two classes

Although certainly not disjoint, we now show that for the two set of processes in the previous subsubsections, neither class contains the other when $n \geq 4$ but they are the same when $n \leq 3$.

Theorem 3.6.

1. *There is a tree-indexed Gaussion Markov chain with four vertices which is not a DGFF.*

2. *There is a four-dimensional DGFF whose marginals have variance one which cannot be embedded into any tree-indexed Gaussian Markov chain.*

3. *Let A be a covariance matrix for a zero mean, variance one Gaussion vector with $n \leq 3$. Then the following are equivalent. A is an inverse Stieltjes matrix, X corresponds to a DGFF, and X is embeddable in a tree-indexed Gaussian Markov chain.*

Proof.

1. It is immediate from the discussion of the DGFF and the form of the inverse covariance matrix in the previous subsubsection, that a tree-indexed Gaussian Markov chain is a DGFF if and only if for all i,

$$1 - \sum_{e \, : \, i \in e} \frac{b_e}{1 + b_e} \geq 0$$

with strict inequality for some i and in this case, the DGFF would have conductances given by the b_e's and the killing rates $\kappa(i)$'s given by the expression above. It easily follows that for any tree-indexed Gaussian Markov chain which has a vertex v with three edges emanating from it all of which have b_e-value larger than $1/2$, the chain cannot be a DGFF. This clearly can be achieved with a tree on 4 vertices.

2. Consider a four-dimensional mean zero Gaussian vector X with covariance matrix given by
$$A = \begin{pmatrix} 1 & 0.2 & 0.2 & 0.1 \\ 0.2 & 1 & 0.1 & 0.1 \\ 0.2 & 0.1 & 1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 1 \end{pmatrix}$$
whose inverse covariance matrix is therefore given by

$$\frac{10}{8973} \begin{pmatrix} 972 & -171 & -171 & -63 \\ -171 & 944 & -53 & -72 \\ -171 & -53 & 944 & -72 \\ -63 & -72 & -72 & 918 \end{pmatrix}.$$

22

Since the latter is a Stieltjes matrix whose row sums are positive, we know that X^A corresponds to a DGGF. We will now show that X^A is not embeddable into a tree-indexed Gaussian Markov chain. Assume on the contrary that there exists a tree-indexed Gaussian Markov chain and vertices v_1, v_2, v_3, v_4 such that the distribution at these 4 points corresponds to X.

Given any three points x, y, z in a tree, there exists a unique *meeting point* w with the property that there are edge-disjoint paths from each of x, y and z to w. It is clear that if the covariance between z and x is the same as the covariance between z and y, then w must be on the *midpoint* between x and y in the sense that the covariance between w and x is the same as the covariance between w and y.

It follows that the meeting point m for v_1, v_2 and v_3 is the same as the meeting point for v_2, v_3 and v_4. Moreover, the correlations between v_1, v_2, v_3, v_4 and m must be given by $\frac{2}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}$ and $\frac{1}{\sqrt{10}}$. It follows that the covariance between v_1 and v_4 is at least 0.2; it would be exactly 0.2 if the paths from v_1 to m and from v_4 to m are disjoint and otherwise it would be larger. However, this contradicts the fact that this correlation is only 0.1.

3. We only do the case $n = 3$. We first show if a variance one Gaussian vector has a covariance matrix which is inverse Stieltjes, then it can be embedded into a tree-indexed Gaussian Markov chain on 4 vertices.

   Consider a three-dimensional mean zero, variance one Gaussian vector with covariances a_{12}, a_{13} and a_{23}, each in $(0, 1]$. (The case that some covariance is zero is easily handled separately.) Consider a tree on four vertices consisting of a root, labelled 4, with three edges coming out to the vertices labelled 1,2, and 3. For $i \in \{1, 2, 3\}$, let $b_i \in (0, 1]$ be assigned to the edge between i and 4 and consider the associated Gaussian Markov chain. If we can solve the equations

   $$a_{12} = b_1 b_2, \ a_{13} = b_1 b_3, \ a_{23} = b_2 b_3,$$

   then the distribution at the three leaves will correspond to our original Gaussian vector, demonstrating the embeddability. One checks that the unique solution is given by

   $$b_1 = \sqrt{\frac{a_{12} a_{13}}{a_{23}}}, \ b_2 = \sqrt{\frac{a_{12} a_{23}}{a_{13}}} \text{ and } b_3 = \sqrt{\frac{a_{13} a_{23}}{a_{12}}}.$$

   To insure these lie in $(0, 1]$, we need that for distinct i, j, k in $\{1, 2, 3\}$

   $$a_{ij} \geq a_{ik} a_{jk}.$$

   However, this is the so-called path product property, known to hold for all inverse Stieltjes matrices (see e.g. [4]).

   For the other two implications, we have seen in Subsubsection 3.2.1 that a DGFF necessarily has an inverse covariance matrix which is a Stieltjes matrix. Consequently, the desired conclusion follows if we can show that when $n = 3$, any Gaussian vector which is embeddable into a tree-indexed Gaussian Markov chain is a DGFF. To this end, suppose that

$x = (X_1, X_2, X_3)$ is embeddable into a tree-indexed Markov chain at vertices labelled by v_1, v_2 and v_3 respctively. Let m be the meeting point of v_1, v_2 an v_3, and further, for $i \in \{1, 2, 3\}$, let b_i be the product of the b_e's associated to the edges on the shortest path between m and v_i in the tree. Note that since our variables are always distinct, at most one of the b_i's is equal to 1. Then the covariance matrix of X is given by

$$\begin{pmatrix} 1 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & 1 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & 1 \end{pmatrix}$$

and it is easy to verify that the inverse covariance matrix is given by

$$\frac{1}{\det A} \begin{pmatrix} 1 - b_2^2 b_3^2 & -b_1 b_2 (1 - b_3^2) & -b_1 b_3 (1 - b_2^2) \\ -b_1 b_2 (1 - b_3^2) & 1 - b_1^2 b_3^2 & -b_2 b_3 (1 - b_1^2) \\ -b_1 b_3 (1 - b_2^2) & -b_2 b_3 (1 - b_1^2) & 1 - b_1^2 b_2^2 \end{pmatrix}.$$

This is clearly a Stieltjes matrix. To see that X is in fact a DGFF, note that the row sums of this matrix are given by

$$\frac{1}{\det A} \begin{cases} (1 - b_2 b_3)(1 - b_1 b_2 - b_1 b_3 + b_2 b_3) \\ (1 - b_1 b_3)(1 - b_1 b_2 - b_2 b_3 + b_1 b_3) \\ (1 - b_1 b_2)(1 - b_1 b_3 - b_2 b_3 + b_1 b_2) \end{cases}$$

and it is easy to check that since at most one b_i is 1, all these row sums are strictly positive, as desired.

$\square$

We end this subsection by discussing a simple Gaussian vector and show that different points of view can lead to very different color representations. To this end, consider the fully symmetric multivariate normal $X := (X_1, X_2, \ldots, X_n)$ with covariance matrix $A = (a_{ij})$ where $a_{ij} = a \in (0, 1)$ for $i \neq j$ and $a_{ii} = 1$ for all i. (While not needed, it is immediate to check that this is a DGFF on the complete graph with conductances $c(i, j) = \frac{a}{(1+(n-1)a)(1-a)}$ for all $i \neq j$ and constant killing rates $\kappa(i) = \frac{1}{1+(n-1)a}$.) It is easy to check that

$$A^{-1}(i, j) = \begin{cases} \frac{1+(n-2)a}{(1+(n-1)a)(1-a)} & \text{if } i = j \\ \frac{-a}{(1+(n-1)a)(1-a)} & \text{otherwise.} \end{cases}$$

Since this is a Stieltjes matrix, X^0 is a color process by Theorem 3.2 and moreover, by the proof, the resulting color representation has full support on all partitions. (The fact that this particular example is a color process is also covered by Section 3.5 in [12] using a different method.)

Now suppose we would add a variable X_0 with $a_{00} = 1$ and $a_{i0} = \sqrt{a}$ for all $i \in \{1, 2, \ldots, n\}$. One can check that this defines a Gaussian vector $(X_0, X_1, X_2, \ldots, X_n)$ and it is easy to check that this is a tree-indexed Gaussian Markov chain where the tree is a vertex with n edges coming out with each $b_e = \sqrt{a}$. If we let A_0 be the covariance matrix of $Y := (X_0, X_1, X_2, \ldots, X_n)$,

then its inverse is given by

$$A_0^{-1}(i,j) = \begin{cases} \frac{1+(n-1)a}{1-a} & \text{if } i = j = 0 \\ \frac{1}{1-a} & \text{if } i = j > 0 \\ \frac{-\sqrt{a}}{1-a} & \text{if } i > j = 0 \\ \frac{-\sqrt{a}}{1-a} & \text{if } j > i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This is also a Stieltjes matrix (which also follows from Corollary 3.5), and hence Y^0 has a color representation by Theorem 3.2. (This is a DGFF for some values of a and n and not for others but this will not concern us.) If we take the resulting color representation of Y^0 and restrict to $\{1, 2, \ldots, n\}$, it is clear that we have support only on partitions with at most one non-singleton cluster. In particular, this implies that when $n = 4$, these color representations will assign different probabilities to the partition $(12, 34)$, and hence the representations are distinct.

4   An alternative embedding proof for tree-indexed Gaussian Markov chains which extends to the stable case

The purpose of this section is twofold: first to give an alternative proof of the fact, contained in Corollary 3.5, that tree-indexed Gaussian Markov chains are color processes and then to use a variant of this alternative method to obtain a result in the context of stable random variables.

4.1   The Gaussian case

Alternative proof of Corollary 3.5. We give this proof only for a path where the correlations between successive variables are the same value a. The extension to the tree case and varying correlations is analogous.

To show that $X := (X_1, X_2, \ldots, X_n)$ has a color representation for any $n \geq 1$, we want to construct, on some probability space, a random partition π of $[n]$ and random variables $Y = (Y_1, Y_2, \ldots, Y_n)$ so that

(i) X and Y have the same distribution (which implies that their corresponding sign processes have the same distribution) and

(ii) (Y^0, π) is a color process (for $p = 1/2$) with its color representation.

To do this, let (Z_t) be the so-called Ornstein-Uhlenbeck (OU) process defined by

$$Z_t = e^{-t} W_{e^{2t}}$$

where $(W_t)_{t \geq 0}$ is a standard Brownian motion. It is well known and immediate to check that $Z_t \sim N(0, 1)$ for any $t \in \mathbb{R}$ and that $\text{Cov}(Z_s, Z_t) = e^{-|s-t|}$ for any $s, t \in \mathbb{R}$.

Now, given n, consider the random vector Y given by

$$Z_{\log(1/a)}, Z_{2\log(1/a)}, \ldots, Z_{n\log(1/a)}$$

and consider the random partition π of $\{1, 2, \ldots, n\}$ given by $i \sim j$ if Z_t does not hit zero between times $i \log(1/a)$ and $j \log(1/a)$.

It is immediate from the Markovian structure of both vectors and the co-variances in the OU process that (i) holds. Next, (ii) is clear using the reflection principle (which uses the strong Markov property) and the fact that the hitting time of 0 is a stopping time. $\qquad\square$

Remark 4.1. This argument (also) does not work for any threshold other than zero. For it to work, one would need that for $h > 0$ and any time $t \geq 0$, the probability that an OU process started at h is larger than h at time t is equal to the unconditioned probability. This however does not hold.

Remark 4.2. In [5], the author studies a similar construction as the construction above for discrete Gaussian free fields. More precisely, the author shows that one can obtain a color representation for a DGFF X as follows. Given X, for each pair of adjacent vertices he adds a Brownian bridge with length determined by their coupling constant. Two vertices are then put in the same partition element if the corresponding Brownian bridge does not hit zero. Since DGFF's have no stable analogue, this does not generalize to any class of stable distributions.

4.2 The stable case

We now obtain our first result for stable vectors. Given $\alpha \in (0, 2)$ and $a \in (0, 1)$, let U have distribution $S_\alpha(1, 0, 0)$ and consider the Markov chain on $\mathbb{R}$ given by $s \mapsto as + (1 - a^\alpha)^{1/\alpha} U$. It is straightforward to check that U is a stationary distribution for this Markov chain. Hence, given a tree T and a designated root, we obtain a tree-indexed α-stable Markov chain on T. Interestingly, unlike the Gaussian case, this process depends on the chosen root as this Markov Chain is not reversible. In particular, if (X_0, X_1) are two consecutive times for this Markov chain started in stationarity, then (X_0, X_1) and (X_1, X_0) have different distributions; one can see this by looking at the two spectral measures.

Proposition 4.3. *Fix $\alpha \in (0, 2)$, $a \in (0, 1)$, a tree T with designated root ρ and consider the corresponding tree-indexed α-stable Markov chain X on T. Then X^0 is a color process.*

Proof. We give the proof only for a path and with ρ being the start of the path. The extension to the tree case is analogous. As in the previous proof, we want to construct, on some probability space, a random partition π of $[n]$ and random variables $Y = (Y_1, Y_2, \ldots, Y_n)$ so that

  (i) $(X_1, \ldots, X_n)$ and Y have the same distribution, and

  (ii) (Y^0, π) is a color process (for $p = 1/2$) with its color representation.

We first recall (see Proposition 1.3.1 in [10], p.20) that if $(B_t)_{t \geq 0}$ is a standard Brownian motion and $S \sim S_{\alpha/2}(2 \cos(\pi\alpha/4)^{2/\alpha}, 1, 0)$ is independent, then $S^{1/2} B_1 \sim S_\alpha(1, 0, 0)$. S is an example of a so-called subordinator.

Now let $Y_1, S_2, \ldots, S_n, (B_t)_{t \geq 0}^2, \ldots, (B_t)_{t \geq 0}^n$ be independent with $Y_1 \sim S_\alpha(1, 0, 0)$, each $S_i \overset{\mathcal{D}}{=} S$, where S is as above and each $(B_t)_{t \geq 0}^i$ being a standard Brownian motion.

Define Y_i for $i \in \{2, \ldots, n\}$ inductively by

$$Y_{i+1} = aY_i + (1 - a^\alpha)^{1/\alpha} S_{i+1}^{1/2} B_1^{(i+1)}.$$

It is clear from the above discussion that (i) holds.

Now we extend this process to all times $t \in [1, n]$ as follows. Let, for $t \in (i, i+1)$,

$$Y_t = aY_i + (1 - a^\alpha)^{1/\alpha} S_{i+1}^{1/2} B_{t-i}^{(i+1)}.$$

Note that (Y_t) is left-continuous and has jumps exactly at the integers. Note also that this process never jumps over the x-axis.

Next, considering the random partition π of $\{1, 2, \ldots, n\}$ given by $i \sim j$ if Y_t does not hit zero between times i and j. Again using the reflection principle, properties of Brownian motion and the fact that (Y_t) never jumps over the x-axis it is clear that (ii) holds. $\qquad \square$

We apply this to a particular symmetric, fully symmetric stable n-dimensional vector. To this end, let S_0, S_1, $\ldots$, S_n be i.i.d. each having distribution $S_\alpha(1, 0, 0)$ and for $i = 1, 2, \ldots, n$ let

$$X_1 := aS_0 + (1 - a^\alpha)^{1/\alpha} S_i.$$

We claim that $(X_1^0, X_2^0, \ldots, X_n^0)$ is a color process. To see this, consider Proposition 4.3 with a homogeneous n-ary tree and α and a being as above. By that proposition, the threshold zero process for the corresponding tree-indexed Markov chain is a color process.

5 A geometric approach to Gaussian vectors

5.1 The geometric picture of a Gaussian vector

In this section we will switch to a more geometric perspective and view a mean zero Gaussian vector of length n as the values of a certain random function at a set of n points in $\mathbb{R}^k$ for some k. More precisely, let $k \geq 1$, $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$, and $W \sim N(0, I_k)$ be a standard normal distribution in $\mathbb{R}^k$. If we now let

$$X := (X_i)_{1 \leq i \leq n} := (\mathbf{x}_i \cdot W)_{1 \leq i \leq n},$$

then X is a Gaussian vector with mean zero and covariances $\operatorname{Cov}(X_i, X_j) = \mathbf{x}_i \cdot \mathbf{x}_j$. Note that X_i having variance one corresponds to $\mathbf{x}_i$ being on the unit sphere $\mathbb{S}^{k-1}$ in $\mathbb{R}^k$.

The above representation can always be achieved with $k = n$. To see this, let $X := (X_i)_{1 \leq i \leq n}$ be a Gaussian vector with mean zero and covariance matrix A. Then for $W \sim N(0, I_n)$ we have that

$$X \stackrel{d}{=} \sqrt{A} \, W,$$

where $\sqrt{A}$ is the nonnegative square root of A. Now for $i = 1, \ldots, n$, let $\mathbf{x}_i = \sqrt{A}(i, \cdot)$ in $\mathbb{R}^n$ be the ith row of $\sqrt{A}$. Then for any $i, j \in [n]$ we have that $\mathbf{x}_i \cdot \mathbf{x}_j = a_{ij}$ as desired. Such a representation can be achieved in $\mathbb{R}^k$ if and only if X lives on a k-dimensional subspace of $\mathbb{R}^n$. We say that X has dimension k

if k is the smallest integer where one has this representation. When we have $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$ as above, without loss of generality, we will always assume that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ spans $\mathbb{R}^k$ so that the dimension of X is k.

Now given a standard Gaussian vector $X \coloneqq (X_i)_{1 \le i \le n}$ (recall this means the marginals have mean zero and variance one) and $h \in \mathbb{R}$, let $(X_i^h)_{1 \le i \le n}$ be, as before, the threshold process defined by $X_i^h \coloneqq I(X_i > h)$. It will be useful to have a simple way to generate $(X_i^h)_{1 \le i \le n}$ which can be done as follows. Assume that X is k-dimensional with variances all being one. We take n points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ on $\mathbb{S}^{k-1}$ corresponding to $(X_i)_{1 \le i \le n}$ as described above. Let $Z \sim N(0, I_k)$. It is well known that when Z is written in polar coordinates (r, θ) with $r \ge 0$ and $\theta \in \mathbb{S}^{k-1}$, then r and θ are independent with θ uniform on $\mathbb{S}^{k-1}$ and r having the distribution of the square root of a χ-squared distribution with k degrees of freedom. We then have that $X_i^h = 1$ if and only if $\mathbf{x}_i \cdot Z > h$. Note that $\{\mathbf{x} : \mathbf{x} \cdot Z = h\}$ is a random hyperplane H_h in $\mathbb{R}^k$ perpendicular to $\theta(Z)$ and so X^h is equal to one for points on $\mathbb{S}^{k-1}$ which lie on one side of H_h and zero for points lying on the other side. Note that when $h = 0$, the hyperplane goes through the origin and it is the points on the same side as $\theta(Z)$ that get value one; in particular, when $h = 0$, the value of X_i^h only depends on $\theta(Z)$ and not on $r(Z)$. However, when $h > 0$, the hyperplane H_h can go through any point of the one-sided infinite line from the origin going through $\theta(Z)$. In particular, H_h might not intersect $\mathbb{S}^{k-1}$ at all; this would correspond exactly to $r(Z) < h$.

5.2 Gaussian vectors canonically indexed by the circle

Proposition 5.1. *Consider n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ on $\mathbb{S}^1$ satisfying $\mathbf{x}_i \cdot \mathbf{x}_j \ge 0$ for all i, j; this is equivalent to the correlations a_{ij} of the corresponding Gaussian process X being nonnegative. Then X^0 is a color process.*

Proof. Using the nonnegative correlations, it is easy to check that the n points $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subseteq \mathbb{S}^1$ must lie on an arc of length at most $\pi/2$. Since the distribution of a Gaussian process is invariant under rotations, we may assume that the n points lie on the arc $0 \le \theta \le \pi/2$. Hence we can assume that $\mathbf{x}_j = e^{i\theta_j}$ with $0 \le \theta_1 < \theta_2 < \ldots < \theta_n \le \pi/2$.

We will couple X^0 with a color process together with its color representation in such a way that X^0 and the color process match exactly. We first show how one uniform point U on $\mathbb{S}^1$ generates a color process together with its color representation. Let

$$I_1 = [0, \theta_1], I_2 = [\theta_1, \theta_2], \ldots, I_k = [\theta_{k-1}, \theta_k], \ldots, I_{n+1} = [\theta_n, \pi/2]$$

noting that the first and last arcs might be trivial. Letting I_k^θ be I_k rotated counterclockwise by θ, we note that

$$\{I_k^\theta : k \in \{1, \ldots, n+1\}, \theta \in \{0, \pi/2, \pi, 3\pi/2\}\}$$

partitions $\mathbb{S}^1$. Now for $k = 1, \ldots, n+1$, if U falls in $I_k^0 \cup I_k^{\frac{\pi}{2}} \cup I_k^\pi \cup I_k^{\frac{3\pi}{2}}$, we partition $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ into the two sets $J_1 \coloneqq \{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ and $J_2 \coloneqq \{\mathbf{x}_k, \ldots, \mathbf{x}_n\}$ with the obvious caveat when $k \in \{1, n+1\}$. Next we color J_1 and J_2 as follows. If U is in $[0, \pi/2]$, we color each cluster 1, if U is in $[\pi/2, \pi]$, we color J_1 0 and J_2 1, if U is in $[\pi, 3\pi/2]$, we color each cluster 0 and if U is in $[3\pi/2, 2\pi]$, we

color J_1 1 and J_2 0. This clearly yields a color process (with $p = 1/2$) together with its color representation. Finally observe that this color process is exactly X^0 if we use U for $\theta(Z)$. $\qquad\square$

Remark 5.2. It is clear from the construction that the probability that the color process is constant is at least $1/2$ (corresponding to when U falls in the top-left or bottom-right arcs) and so Theorem 2.2(i) could also be used here. In addition, the color representation given here corresponds to that given in Theorem 2.2(i). The above description will however be useful when dealing with the case $h \neq 0$ as in Proposition 5.5.

Remark 5.3. For any color process (Y_i) with $p = 1/2$, for any i and j it is clear that

$$\operatorname{Cov}(Y_i, Y_j) = \frac{q_{ij}}{4} \tag{18}$$

and that

$$P(Y_i = Y_j = 1) = \frac{1}{4} + \frac{q_{ij}}{4}.$$

In the case of Proposition 5.1, it is clear that

$$1 - q_{ij} = \frac{|\theta_j - \theta_i|}{\pi/2} \tag{19}$$

and hence that

$$P(X_i^0 = X_j^0 = 1) = \frac{1}{2} - \frac{|\theta_j - \theta_i|}{2\pi}. \tag{20}$$

Since $|\theta_j - \theta_i| = \arccos a_{ij}$ it follows that

$$P(X_i^0 = X_j^0 = 1) = \frac{1}{2} - \frac{\arccos a_{ij}}{2\pi}. \tag{21}$$

This is of course one of many ways to derive this last expression which is known as Sheppard's formula (see [11]).

This discussion also leads as well to the formula

$$q_{ij} = 1 - \frac{2\arccos a_{ij}}{\pi}. \tag{22}$$

The proof of the following elementary lemma, based on inclusion-exclusion, is left to the reader.

Lemma 5.4. *If $X := (X_1, X_2, X_3)$ is $\{0,1\}$-symmetric, then*

$$\nu_{000} = \frac{\nu_{00.} + \nu_{0.0} + \nu_{.00}}{2} - \frac{1}{4}. \tag{23}$$

In particular, using (20), *if X corresponds to threshold zero for a mean zero Gaussian vector, the above is equal to*

$$\frac{1}{2} - \frac{\theta_{12} + \theta_{13} + \theta_{23}}{4\pi}. \tag{24}$$

Proposition 5.5. *Consider n points $\mathbf{x}_1, \ldots, \mathbf{x}_n$ on $\mathbb{S}^1$ satisfying $\mathbf{x}_i \cdot \mathbf{x}_j \geq 0$ for all i, j. Then X^h does not have a color representation for any $h \neq 0$, $n \geq 3$.*

Proof. It suffices to prove this for $h > 0$ and $n = 3$. Since $h > 0$, it is clear from the construction of $X^{A,0}$ described above that $(0, 1, 0)$ has positive probability but that $(1, 0, 1)$ has probability zero. However, it is immediate that no color process can have this property. $\square$

Remark 5.6. By taking the three points very close together, we see that there is no analogue of Theorem 2.2(i) when we don't have $\{0, 1\}$-symmetry, even in the threshold Gaussian case (i.e., when $h \neq 0$).
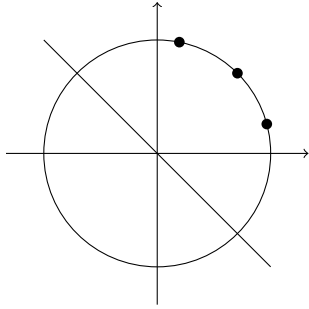

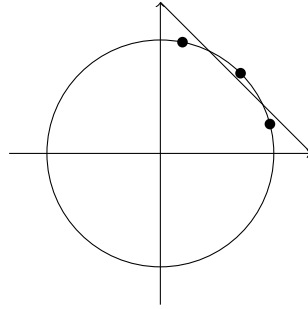Figure 1: The image above illustrates the situation when $h = 0$.

Figure 2: The image above illustrates the situation when $h > 0$.

5.3 A general obstruction for having a color representation for $h > 0$

The following is precisely a higher dimensional analogue of Proposition 5.5. The latter is the special case $n = 3$ together with the fact that any three points on the circle are in general position.

Theorem 5.7. *The standard Gaussian process X associated to n points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{S}^{n-2}$ in general position (equivalently not contained in an $(n - 2)$-dimensional hyperplane) is such that X^h is not a color process for any $h > 0$.*

More generally, if $X := (X_1, X_2, \ldots, X_n)$ is a random vector such that

- *$(X_1, X_2, \ldots, X_{n-1})$ is fully supported on $\mathbb{R}^{n-1}$*

- *there is $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \backslash \{0\}$ such that a.s.*

$$\sum_{i=1}^{n} a_i X_i = 0 \tag{25}$$

and

$$\sum_{i=1}^{n} a_i \neq 0, \tag{26}$$

then X^h is not a color process for any $h > 0$.

Remark 5.8. Any n-dimensional standard Gaussian vector which is not fully dimensional can be represented by points on $\mathbb{S}^{n-2}$. When the n points are *not* in general position, which can only happen if $n \geq 4$, in which case the above

30

result is not applicable, we will see in Corollary 7.3 that nonetheless X^h is not a color process for large h. Perhaps the simplest example of a four-dimensional Gaussian vector which is not fully dimensional but does not correspond to points on $\mathbb{S}^2$ in general position appears in Figure 3. In the next subsection, we will see in Theorem 5.9 that this case will lead us to an important example for which we will have a phase transition.

Proof of Theorem 5.7. We will first observe that the second statement implies the first. One can order the n points $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{S}^{n-2}$ in general position such that the first $n-1$ points are linearly independent. This implies that the corresponding Gaussian vector $X = (X_1, \ldots, X_n)$ satisfies the first condition. Next, since $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent (as they sit inside $\mathbb{R}^{n-1}$) there exists $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \backslash \{0\}$ such that

$$\sum_{i=1}^{n} a_i \mathbf{x}_i = 0$$

which implies (25). Finally (26) must hold since $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are in general position.

For the second statement, note first that we can assume that $|a_i| > 0$ for $i = 1, 2, \ldots, n$ since we can remove the X_i's for which $a_i = 0$. If $a_j > 0$ for all j (with a similar argument if $a_j < 0$ for all j), then for all $h > 0$, $\nu_{1^n}(h) = 0$ in which case there clearly cannot be any color representation. We hence assume that there are both positive and negative values among the a_j's. Furthermore since $\sum_{i=1}^{n} a_i X_i = 0$ and $(X_1, X_2, \ldots, X_{n-1})$ is fully supported, for any i, if we define $I_i = \{1, 2, \ldots, n\} \backslash \{i\}$, then the vector $(X_j)_{j \in I_i}$ is fully supported. This implies in particular that we, possibly after reordering the random variables and changing all the signs, can assume that

$$\sum_{i:\, a_i > 0} |a_i| < \sum_{j:\, a_j < 0} |a_j|$$

and that $a_n > 0$.

Fix now $h > 0$. Now define the binary string ρ by $\rho(i) = I(a_i < 0)$ and let $\mathcal{E}$ be the event that

$$\forall j < n \colon X_j > h \text{ if } a_j < 0 \text{ and } X_j \leq h \text{ if } a_j > 0.$$

Since $(X_1, X_2, \ldots, X_{n-1})$ is fully supported, the probability of the event $\mathcal{E}$ is strictly positive. Since

$$\sum_{i:\, a_i > 0} |a_i| X_i = \sum_{j:\, a_j < 0} |a_j| X_j,$$

this implies that on $\mathcal{E}$,

$$X_n = \frac{-\sum_{j<n} a_j X_j}{a_n} = \frac{\sum_{j<n:\, a_j<0} |a_j| X_j}{a_n} - \frac{\sum_{j<n:\, a_j>0} |a_j| X_j}{a_n}$$

$$\geq h \cdot \left(\frac{\sum_{j<n:\, a_j<0} |a_j|}{a_n} - \frac{\sum_{j<n:\, a_j>0} |a_j|}{a_n} \right) > h$$

which in particular implies that $\nu_\rho = 0$.

On the other hand, since $(X_1, X_2, \ldots, X_{n-1})$ is fully supported, the event

$$\forall j < n \colon X_j \in [\alpha h, h] \text{ if } a_j < 0 \text{ and } X_j \in (h, \beta h] \text{ if } a_j > 0$$

has strictly positive probability for any $\alpha \in (0,1)$ and $\beta \in (1, \infty)$. On this event we have that

$$X_n = \frac{\sum_{j<n \colon a_j<0} |a_j| X_j - \sum_{i<n \colon a_i>0} |a_i| X_i}{a_n}$$

$$\geq h \cdot \frac{\alpha \sum_{j<n \colon a_j<0} |a_j| - \beta \sum_{i<n \colon a_i>0} |a_i|}{a_n}.$$

Since

$$\sum_{j<n \colon a_j<0} |a_j| - \sum_{i<n \colon a_i>0} |a_i| > a_n$$

it follows that $X_n > h$ if α and β are both sufficiently close to one. In particular, this implies that $\nu_{-\rho} > 0$. Since $\nu_\rho = 0$ but $\nu_{-\rho} > 0$, it follows that X^h cannot have a color representation. $\qquad\square$
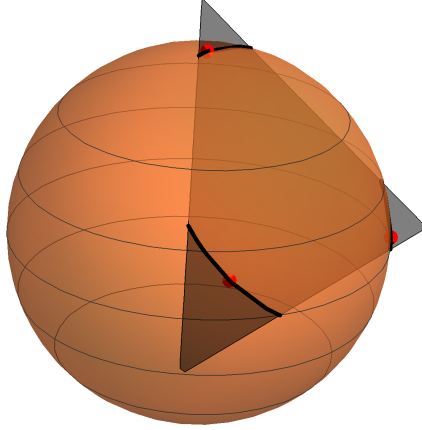

Figure 3: The picture above shows the three points $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ corresponding to a mean zero variance one Gaussian vector with $a_{12} = a_{23} = 0.2$ and $a_{13} = 0.2^2$. The black lines are the positions where we could add a fourth point $\mathbf{x}_4$ without the existence of a color representation for some $h > 0$ being ruled out by Theorem 5.7.

5.4 A four-dimensional Gaussian exhibiting a non-trivial phase transition

In this subsection we will study an example, corresponding to four points on $\mathbb{S}^2$, for which the existence of a color representation for positive h is not ruled out by Theorem 5.7. To this end, let $\theta \in (0, \pi/2]$ and define $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{S}^2$ by

$$\begin{cases} \mathbf{x}_1 = (\sin \theta, 0, \cos \theta) \\ \mathbf{x}_2 = (0, \sin \theta, \cos \theta) \\ \mathbf{x}_3 = (-\sin \theta, 0, \cos \theta) \\ \mathbf{x}_4 = (0, -\sin \theta, \cos \theta) \end{cases}$$

and for $i = 1, 2, 3, 4$, let $X_i = \mathbf{x}_i \cdot W$, where $W \sim N(0, I_3)$. Then $X \sim N(0, A)$ for

$$A = \begin{pmatrix} 1 & \cos^2 \theta & \cos^2 \theta - \sin^2 \theta & \cos^2 \theta \\ \cos^2 \theta & 1 & \cos^2 \theta & \cos^2 \theta - \sin^2 \theta \\ \cos^2 \theta - \sin^2 \theta & \cos^2 \theta & 1 & \cos^2 \theta \\ \cos^2 \theta & \cos^2 \theta - \sin^2 \theta & \cos^2 \theta & 1 \end{pmatrix}. \quad (27)$$

Geometrically, this corresponds to having four points in a square on a 2-sphere at the same latitude, and it follows easily that

$$X_1 + X_3 = X_2 + X_4. \quad (28)$$

Note that A has nonnegative entries if and only if $\theta \leq \pi/4$.

The following theorem implies Theorem 1.3.

Theorem 5.9. *Let X^θ be a Gaussian vector with covariance matrix given by (27). Then*

(i) $X^{\theta,0}$ is a color process for all $\theta \in (0, \pi/4]$,

(ii) there is $\theta_0 > 0$ such that for all $\theta < \theta_0$, there exists $h_\theta > 0$ such that $X^{\theta,h}$ is a color process for all $h \in (0, h_\theta)$.

(iii) for all $\theta \in (0, \pi/4)$, there is $h_\theta > 0$ such that $X^{\theta,h}$ has no color representation for any $h > h_\theta$.

Lemma 5.10. *Let X^θ be a Gaussian vector with covariance matrix given by (27). Then for all h, $X^{\theta,h}$ has a color representation if and only if there is a color representation of $(X_1^{\theta,h}, X_2^{\theta,h}, X_3^{\theta,h})$ which satisfies*

$$\begin{cases} q_{123} \geq q_{13,2} \geq 0 \\ 2q_{12,3} - 2q_{13,2} \geq q_{1,2,3} \geq 0. \end{cases} \quad (29)$$

Proof. Fix $h \geq 0$ and assume first that there is a color representation (q_σ) (the dependence on h will be suppressed) of (ν_ρ^h). Since the distribution of X^θ is invariant under the action of the dihedral group, we can assume that (q_σ) also is. Note that it follows from (28) that $\nu_{0101} = 0$, and hence $\nu_{010\cdot} = \nu_{0100}$. In particular, this implies that

$$q_{1,2,3,4} = q_{13,2,4} = q_{1,24,3} = q_{13,24} = 0, \quad (30)$$

and using this, we obtain (using the assumed symmetry)

$$
\begin{cases}
q_{1234} = q_{123} - q_{123,4} = q_{123} - q_{13,4} = q_{123} - q_{13,2} \\
q_{123,4} = q_{13,4} = q_{13,2} \\
q_{12,3,4} = q_{2,3,4} - q_{14,2,3} = q_{1,2,3} - q_{12,3,4} = q_{1,2,3}/2 \\
q_{12,34} = q_{12,3} - q_{124,3} - q_{12,3,4} = q_{12,3} - q_{13,2} - q_{1,2,3}/2.
\end{cases}
\tag{31}
$$

Since this is a color representation by assumption, $q_\sigma \geq 0$ for all σ, which is equivalent to (29). This proves the necessity in the first part of the lemma.

To see that we also have sufficiency, let $q = (q_{123}, q_{12,3}, q_{13,2}, q_{1,23}, q_{1,2,3})$ be a color representation of $(X_1^{\theta,h}, X_2^{\theta,h}, X_3^{\theta,h})$ which satisfies the inequalities in (29). Define q_σ for $\sigma \in \mathcal{B}_4$ by (30) and (31) and extend to all partitions by making it invariant under the dihedral group. Since (29) holds, $q_\sigma \geq 0$ for all $\sigma \in \mathcal{P}_4$. Also, one checks that they add to one and the projection onto $\{1,2,3\}$ is q above. Using the fact that $\nu_{010\cdot} = \nu_{0100}$, one can check that the probability of any configuration is determined by the three–dimensional marginals. From here, one verifies that this yields a color representation of $X^{\theta,h}$, as desired. $\square$

Proof of Theorem 5.9. To see that (i) holds, let $h = 0$. We will apply Lemma 5.10. By (11), the process $(X_1^{\theta,0}, X_2^{\theta,0}, X_3^{\theta,0})$ has a *signed* color representation given by

$$
\begin{cases}
q_{123} & = 1 - 4(\nu_{001} + \nu_{010} + \nu_{100}) + t \\
q_{12,3} & = 4\nu_{001} - t \\
q_{1,23} & = 4\nu_{100} - t \\
q_{13,2} & = 4\nu_{010} - t \\
q_{1,2,3} & = 2t
\end{cases}
$$

for some free variable $t \in \mathbb{R}$. This will give a color representation for all t which is such that $q_\sigma \geq 0$ for all $\sigma \in \mathcal{B}_3$. Using (20) and (24), one easily verifies that in a Gaussian setting, the set of equations above can equivalently be written as

$$
\begin{cases}
q_{123} & = 1 - \frac{\theta_{12} + \theta_{13} + \theta_{23}}{\pi} + t \\
q_{12,3} & = \frac{(\theta_{12} + \theta_{13} + \theta_{23}) - 2\theta_{12}}{\pi} - t \\
q_{1,23} & = \frac{(\theta_{12} + \theta_{13} + \theta_{23}) - 2\theta_{23}}{\pi} - t \\
q_{13,2} & = \frac{(\theta_{12} + \theta_{13} + \theta_{23}) - 2\theta_{13}}{\pi} - t \\
q_{1,2,3} & = 2t.
\end{cases}
$$

Rearranging, we see that these are all nonnegative if and only if

$$
0 \vee \left(\frac{\sum_{i \neq j} \theta_{ij}}{\pi} - 1 \right) \leq t \leq \frac{\sum_{i \neq j} \theta_{ij} - 2(\theta_{12} \vee \theta_{13} \vee \theta_{23})}{\pi}.
\tag{32}
$$

In our specific example, we have that

$$
\begin{cases}
\theta_{12} = \theta_{23} = \arccos \cos^2 \theta \\
\theta_{13} = 2\theta \geq \theta_{12}
\end{cases}
$$

and hence (32) simplifies to

$$
0 \vee \left(\frac{2 \arccos \cos^2 \theta + 2\theta}{\pi} - 1 \right) \leq t \leq \frac{2 \arccos \cos^2 \theta - 2\theta}{\pi}.
$$

Similarly, we can rewrite (29) as

$$
\begin{cases}
t \geq \frac{\sum_{i \neq j} \theta_{ij} - \theta_{13} - \pi/2}{\pi} = \frac{2 \arccos \cos^2 \theta - \pi/2}{\pi} \\
t \leq \frac{2(\theta_{13} - \theta_{12})}{\pi} = \frac{2(2\theta - \arccos \cos^2 \theta)}{\pi}.
\end{cases}
$$

If we put these sets of inequalities together, and use that $\theta \in (0, \pi/4]$, we obtain the following necessary and sufficient condition for the existence of such a t:

$$
0 \vee \frac{2 \arccos \cos^2 \theta - \pi/2}{\pi} \leq \frac{2 \arccos \cos^2 \theta - 2\theta}{\pi} \wedge \frac{2(2\theta - \arccos \cos^2 \theta)}{\pi}.
$$

Here it is easy to verify that

$$
0 \vee \frac{2 \arccos \cos^2 \theta - \pi/2}{\pi} \leq \frac{2 \arccos \cos^2 \theta - 2\theta}{\pi}
$$

and that

$$
0 \leq \frac{2(2\theta - \arccos \cos^2 \theta)}{\pi}
$$

and hence to see that we can always pick t so that the above inequalities hold it suffices to show that

$$
\frac{2 \arccos \cos^2 \theta - \pi/2}{\pi} \leq \frac{2(2\theta - \arccos \cos^2 \theta)}{\pi}
$$

for all $\theta \in (0, \pi/4]$. To this end, note first that we can rewrite the inequality above as

$$
\arccos \cos^2 \theta - \theta \leq \pi/8.
$$

This can be verified to hold for all $\theta \in (0, \pi/4]$ by verifying that the left hand side is increasing in θ for $\theta \in (0, \pi/4]$ and noting that

$$
\arccos \cos^2(\pi/4) - \pi/4 = \arccos(1/2) - \pi/4 = \pi/3 - \pi/4 = \pi/12 \leq \pi/8.
$$

The desired conclusion now follows.

To see that (ii) holds, note first that by Theorem 8.1 and a computation, the value of the free parameter t corresponding to the limit of $h \to 0$ is given by

$$
t = 1 - \frac{\arccos \left(\frac{2 \sin^4 \theta}{(1 + \cos^2 \theta)^2} - 1 \right)}{\pi}.
$$

Using the proof of (i), it follows that it suffices to show that

$$
\frac{2 \arccos \cos^2 \theta - \pi/2}{\pi} < 1 - \frac{\arccos \left(\frac{2 \sin^4 \theta}{(1 + \cos^2 \theta)^2} - 1 \right)}{\pi} < \frac{2(2\theta - \arccos \cos^2 \theta)}{\pi}
$$

for all sufficiently small θ. To this end, note first that at $\theta = 0$ the first expression is equal to $-1/2$ while the second and third expression are both equal to zero, and hence the first inequality is strict for all sufficiently small θ. To compare the last two expressions, one verifies that the derivatives of these two expressions at $\theta = 0$ are given by 0 and $4 - 2\sqrt{2}$ respectvely, and hence (ii) is established.

Finally, (iii) follows from Corollary 7.3. $\qquad \square$

5.5 A four-dimensional Gaussian with nonnegative correlations whose 0 threshold has no color representation

In this subsection, we study a particular example which will in particular yield a proof of Theorem 1.5; see (ii) and (iii) below.

Theorem 5.11. *Let $(X_1, X_2, \ldots, X_{n-1})$ be a fully symmetric multivariate mean zero variance one Gaussian random vector with pairwise correlation $a \in [0, 1)$, and let*

$$X_n = (X_1 + X_2 + \ldots + X_{n-1})/\sqrt{a(n-1)^2 + (1-a)(n-1)}.$$

insuring that X_n has mean zero and variance one. In addition, nonnegative pairwise correlations is immediate to check. If $X^a := (X_1, X_2, \ldots, X_n)$, then the following hold.

   (i) *When $n = 3$, $X^{a,0}$ is a color process for any $a \in [0, 1)$.*

   (ii) *When $n \geq 4$ and a is sufficiently close to zero (or zero), $X^{a,0}$ is not a color process.*

   (iii) *For $n \geq 4$, there exists a fully supported multivariate mean zero variance one Gaussian random variable X with nonnegative correlations for which X^0 is not a color process.*

   (iv) *When $n \geq 4$ and a is sufficiently close to one, $X^{a,0}$ is a color process.*

   (v) *For any $n \geq 3$, $a \in [0, 1)$ and $h > 0$, $X^{a,h}$ is not a color process.*

Proof. (i). The claim for $n = 3$ follows immediately from Proposition 5.1 or Proposition 2.12 in [12].

   (ii). We first consider $n \geq 4$ and $a = 0$ and obtain the result in this case. If X^0 is a color process, then it must be the case that the color representation gives weight $1/(n-1)$ to each of the $n-1$ partitions which consist of all singletons except n is in a block of size 2. This is because (1) since $X_1, X_2, \ldots, X_{n-1}$ are independent, none of $1, 2, \ldots, n-1$ can ever be in the same cluster, (2) if n is in its own cluster with positive probability, then $\nu_{0^{n-1}1} > 0$ which contradicts the fact that $X_1, X_2, \ldots, X_{n-1}$ all negative and X_n positive is impossible and (3) symmetry. On the other hand, by (22), each of the above partition elements must have value $1 - \frac{2\arccos\frac{1}{\sqrt{n-1}}}{\pi}$. The conclusion is that if it is a color process, then

$$\frac{1}{n-1} = 1 - \frac{2\arccos\frac{1}{\sqrt{n-1}}}{\pi}.$$

This is true for $n = 3$ (as it must be) but we show this is false for all $n \geq 4$. Rearranging, this is equivalent to

$$\frac{\pi}{2} \cdot \frac{n-2}{n-1} = \arcsin\sqrt{\frac{n-2}{n-1}}. \tag{33}$$

Now consider the two functions $f(x) = \pi x^2/2$ and $g(x) = \arcsin x$ for $x \in [0, 1]$. Then we clearly have $f(0) = g(0)$ and $f(1) = g(1)$. Moreover, one can easily check that both functions are continuously differentiable, that their

first derivatives agree only at $x = \sqrt{\frac{1}{2} \pm \frac{\sqrt{\pi^2-4}}{2\pi}}$ (i.e. at $x \approx 0.338247$ and $x \approx 0.941057$) and that $f'(0) = 0 < 1 = g'(0)$ and $f'(1) = \pi < \infty = g'(1)$. This easily implies that $\{x : f(x) > g(x)\}$ is of the form $(b, 1)$. Hence we need only check that (33) fails for $n = 4$ with the left side being larger. However, this is immediate to check. Finally, to obtain the result for small a depending on n, one just uses the fact that the set of color processes is closed.

(iii). Fix $n \geq 4$, take $a = 0$ and replace X_n by $X'_n := \epsilon Z + (1 - \epsilon^2)^{1/2} X_n$ where Z is another standard Gaussian independent of everything else. Then for every $\epsilon > 0$, the resulting vector X is fully supported with nonnegative correlations. However, for small ϵ, X^0 cannot be a color process since the color processes are closed and the limit as $\epsilon \to 0$ is not a color process by (ii).

For (iv), note that by Theorem 2.2(i), a sufficient condition for being a color process is that $\nu_{0^n} \geq 1/4$. In our case, we clearly have that for any n, $\nu_{0^n} \to 1/2$ as $a \to 1$, and hence the desired conclusion follows.

Finally for (v), with $n \geq 3$, $a \in [0, 1)$ and $h > 0$, this follows immediately from Theorem 5.7. $\qquad\square$

6 An extension to the stable case and two interesting integrals

In this section, we explain to which extent the results in the last subsection of the previous section can be carried out for the stable case. We assume now that $X_1, X_2, \ldots, X_{n-1}$ are i.i.d. each with distribution $S_\alpha(1, 0, 0)$ for some $\alpha \in (0, 2)$ and we let
$$X_n = (X_1 + X_2 + \ldots + X_{n-1})/(n-1)^{1/\alpha}$$
and $X := (X_1, X_2, \ldots, X_n)$.

Proposition 2.12 in [12] implies, as before, that when $n = 3$, X^0 is a color process (the $\{0, 1\}$-symmetry is obvious and the nonnegative correlations being an easy consequence of Harris' inequality).

Concerning whether X^0 can be a color process for some $n \geq 3$ and $h > 0$, Theorem 5.7 implies that it cannot be *except* perhaps when $\alpha = 1$.

We now look at the case $n \geq 4$ and $h = 0$. The same argument as in the previous case allows us to conclude that if X^0 is a color process, then it must be the case that the color representation gives weight $1/(n-1)$ to each of the $n-1$ partitions which consist of all singletons except n is in a block of size 2. Next (18) easily yields that the probability that 1 and n are in the same block is $\mathbb{E}[\text{sgn}(X_1)\,\text{sgn}(X_n)]$ and hence it must be the case that

$$\mathbb{E}[\text{sgn}(X_1)\,\text{sgn}(X_n)] = 1/(n-1).$$

We conclude that for any α where the left hand side above is not $1/(n-1)$, X^0 cannot be a color process. Next, Corollary 6.12 in [8] implies, after some work, that the above expectation is given by $2/\pi^2$ times the following integral to be taken in the Cauchy principal value sense.

$$\lim_{\varepsilon \to 0} \int_{(\varepsilon, \pi/2-\varepsilon) \cup (\pi/2+\varepsilon, \pi-\varepsilon)} \frac{\log\left(\frac{n-2}{n-1}|\sin\theta|^\alpha + |\cos\theta + (\frac{1}{n-1})^{\frac{1}{\alpha}}\sin\theta|^\alpha\right)}{\alpha \cos\theta \sin\theta}\, d\theta.$$
(34)

Restricting now to $n = 4$, using Mathematica, we draw this integral as function of α in Figure 4 and it appears to be strictly increasing. (We will see later it is constant if $n = 3$!) If we knew that to be the case, we could conclude that there is at most one value of α for which X^0 is a color process.
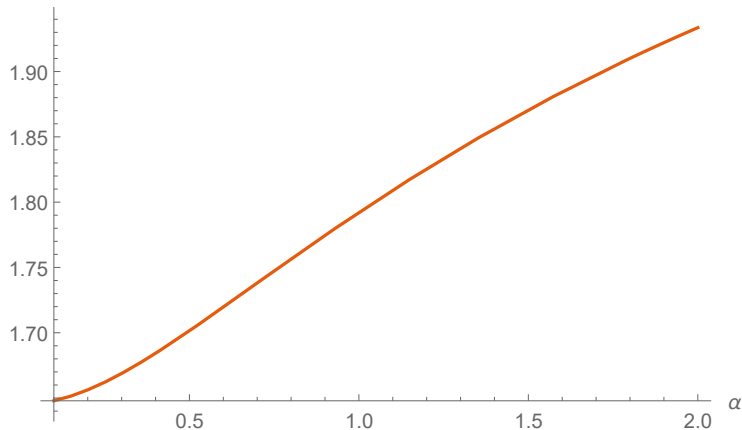

Figure 4: The figure above shows the integral in (34) as a function of α.

Exactly as in the previous subsection, using the fact that the set of color processes is closed, once we know that there is a least one α for which it is not a color process, we can easily construct fully supported symmetric stable distributions with $n = 4$, where the necessary pairwise nonnegative correlations of the signs holds, whose zero thresholds are not color processes.

In [12], a very special case of Corollary 3.4 was obtained, namely the case where we have exchangability (i.e., full permutation invariance). As described below, a consequence of that analysis, the argument in the proof of Theorem 5.11(ii) and a result of I. Molchanov yields a proof of Theorem 1.6 from Section 1.

Proof of Theorem 1.6. The independence in α of the first integral and its value is established as follows. Let S_1, S_2, S_3 be i.i.d. each having distribution $S_\alpha(1, 0, 0)$ and let

$$X_1 := \frac{S_1 + S_2}{2^{1/\alpha}}$$

and

$$X_2 := \frac{S_1 + S_3}{2^{1/\alpha}}.$$

Considering $\mathbb{E}[\mathrm{sgn}(X_1)\,\mathrm{sgn}(X_2)]$, it follows easily from the discussion in Section 3.5 in [12] that this expectation is $1/3$ for each α. On the other hand, Corollary 6.12 in [8] implies, after some work, that the above expectation is, for a given α, the first integral above times $2/\pi^2$. Hence this integral is independent of α with value $\pi^2/6$.

The independence in α of the second integral and its value is established as follows. Let S_1, S_2 be i.i.d. each having distribution $S_\alpha(1, 0, 0)$ and let

$$X := \frac{S_1 + S_2}{2^{1/\alpha}}$$

Consider (S_1^0, S_2^0, X^0). It is easy to see that these random variables have pairwise nonnegative correlations (e.g., using Harris' inequality) and is $\{0, 1\}$-symmetric. It follows from Proposition 2.12 in [12] that this is a color process. It is clear, using e.g. the argument in the proof of Theorem 5.11(ii) that the random partition must satisfy $q_{13,2} = q_{23,1} = 1/2$. Next (18) easily yields $q_{13,2} = \mathbb{E}[\mathrm{sgn}(S_1)\,\mathrm{sgn}(X)]$. On the other hand, Corollary 6.12 in [8] implies, after some work, that the above expectation is, for a given α, the second integral above times $2/\pi^2$. Hence the second integral is independent of α with value $\pi^2/4$. □

Remark 6.1. (i) Without this theory, one can show directly, although it is nontrivial, that the first integral is independent of α and equal to $\pi^2/6$ (as shown by Jack D'Aurizio, see `https://math.stackexchange.com/questions/2698982/why-does-this-integral-not-depend-on-the-parameter`).

(ii) Given the analysis in [12], we knew that any formula for $\mathbb{E}[\mathrm{sgn}(X_1)\,\mathrm{sgn}(X_2)]$ would have to be independent of α, in particular the formula given in Corollary 6.12 in [8] which is the above integral. However we would have guessed that the independence in α of such a formula would have appeared in a more transparent way; surprisingly this was not the case.

(iii) Extending the analysis of the first integral, letting

$$X_k := \frac{S_1 + S_{k+1}}{2^{1/\alpha}}, \qquad k = 1, 2, \ldots, n,$$

one can consider $\mathbb{E}[\mathrm{sgn}(X_1 X_2 \cdots X_n)]$, the analogue of $\mathbb{E}[\mathrm{sgn}(X_1)\,\mathrm{sgn}(X_2)]$. Using Section 3.5 in [12], one can show that the former integral is $1/(n+1)$ for n even and zero for n odd. One can then combine this and Corollary 6.12 in [8] to obtain an infinite number of higher-dimensional integrals which also suprisingly do not depend upon α.

7   Results for large thresholds and the discrete Gaussian free field

In the first subsection of this section, we show that non-fully supported Gaussian vectors do not have color representations for large h. On the other hand, in the second subsection, we give the proof of Theorem 1.7 that discrete Gaussian free fields (DGFF) have color representations for large h.

7.1   An obstruction for large h

We first deal with the case $n = 2$, where we have the following easy result.

Proposition 7.1. *Let* $X := (X_1, X_2)$ *be a standard Gaussian vector with* $\mathrm{Cov}(X_1, X_2) \in [0, 1)$. *Then* X^h *has a (unique) color representation* $(q_\sigma)_{\sigma \in \mathcal{B}_2}$ *for all* $h \in \mathbb{R}$ *and* $\lim_{h \to \infty} q_{12}(h) = 0$.

This result essentially follows from Theorem 2.1 in [2] (see also Lemma 7.10 here) but we include a proof sketch here.

Proof of Proposition 7.1. Note first that since $n = 2$, the nonnegative correlation immediately implies that X^h has a color representation for all $h \in \mathbb{R}$, and hence we need only show that $\lim_{h \to \infty} q_{12}(h) = 0$. Since it can be easily checked that

$$q_{12}(h) = \frac{\nu_{11}(h) - \nu_1(h)^2}{\nu_0(h)\nu_1(h)}$$

we need to show that

$$\lim_{h \to \infty} \nu_{11}(h)/\nu_1(h) = 0; \tag{35}$$

this however is straighforward. $\qquad \square$

The previous result immediately implies the following.

Corollary 7.2. *If $X := (X_1, X_2, \ldots, X_n)$ is a standard Gaussian vector with $\mathrm{Cov}(X_i, X_j) \in [0, 1)$ for all $i < j$ and X^h has a color representation $(q_\sigma)_{\sigma \in \mathcal{B}_n}$ for arbitrarily large h, then*

$$\lim_{h \to \infty} q_{1,2,3,\ldots,n}(h) = 1.$$

Interestingly, this gives the following negative result when X is not fully dimensional.

Corollary 7.3. *Let $(X_1, X_2, \ldots, X_n)$ be a standard Gaussian vector with $\mathrm{Cov}(X_i, X_j) \in [0, 1)$ for all $i < j$. If X is not fully supported, then for all sufficiently large h, X^h is not a color process.*

Proof. Since X is not fully dimensional, there must exist a linear relationship between the variables. As a result, there must exist $\rho \in \{0, 1\}^n$ so that for all $h > 0$, $\nu_\rho(h) = 0$. Hence, if there is a color representation $(q_\sigma(h))$ for some h, it must satisfy $q_{1,2,\ldots,n}(h) = 0$. The desired conclusion now follows from Corollary 7.2. $\qquad \square$

7.2 Discrete Gaussian free fields and large thresholds

In this section, our main goal will be to prove Theorem 1.7. Note that all our random vectors in this section will be fully supported which we know is anyway necessary in view of Corollary 7.3.

We first note the following corollaries of Theorem 1.7.

Corollary 7.4. *Let $a \in (0, 1)$ and let $X := (X_1, X_2, \ldots, X_n)$ be a standard Gaussian vector with $\mathrm{Cov}(X_i, X_j) = a$ for all $i < j$. Then X^h is a color process for all sufficiently large h.*

Proof. Let A be the covariance matrix of X. Then one verifies that for $i, j \in [n]$ we have

$$A^{-1}(i, j) = \begin{cases} \frac{1 + (n-2)a}{(1-a)(1+(n-1)a)} & \text{if } i = j \\ \frac{-a}{(1-a)(1+(n-1)a)} & \text{if } i \neq j. \end{cases}$$

Consequently, A is an inverse Stieltjes matrix. Moreover, for all $j \in [n]$ we have that

$$\mathbf{1}^T A^{-1}(j) = \frac{1}{1 + (n-1)a}.$$

and hence $\mathbf{1}^T A^{-1} > \mathbf{0}$. Applying Theorem 1.7, the desired conclusion follows. $\qquad \square$

Corollary 7.5. *Let $a \in (0,1)$ and let $X := (X_1, X_2, \ldots, X_n)$ be a standard Gaussian vector with $\mathrm{Cov}(X_i, X_j) = a^{|i-j|}$ for all $i, j \in [n]$, yielding a Markov chain. Then X^h is a color process for all sufficiently large h.*

Proof. Let A be the covariance matrix of X. Then one verifies that for $i, j \in [n]$ we have

$$A^{-1}(i,j) = \begin{cases} \frac{1}{1-a^2} & \text{if } i = j \in \{1, n\} \\ \frac{1+a^2}{1-a^2} & \text{if } i = j \notin \{1, n\} \\ \frac{-a}{1-a^2} & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, A is an inverse Stieltjes matrix. Moreover, for all $j \in [n]$ we have that

$$\mathbf{1}^T A^{-1}(j) = \begin{cases} \frac{1}{1+a} & \text{if } j \in \{1, n\} \\ \frac{1-a}{1+a} & \text{if } j \notin \{1, n\} \end{cases}$$

and hence $\mathbf{1}^T A^{-1} > \mathbf{0}$. Applying Theorem 1.7, the desired conclusion follows. $\square$

We now state and prove a few lemmas that will be needed in the proof of Theorem 1.7. The first of these will give sufficient conditions for X^h to be a color process for large h in terms of the decay of the tails of $\nu(1^S)$ for sets S.

Lemma 7.6. *Let $(\nu_p)_{p \in (0,1)}$ be a family of probability measures on $\{0,1\}^n$. Assume that ν_p has marginals $p\delta_1 + (1-p)\delta_0$ and that for all $S \subseteq [n]$ with $|S| \geq 2$ and all $k \in S$, as $p \to 0$, we have that*

$$p\nu_p(1^{S \setminus \{k\}}) \ll \nu_p(1^S) \asymp \nu_p(1^S 0^{S^c}) \tag{36}$$

and

$$\lim_{p \to 0} \sum_{S \subseteq [n]\,:\, |S| \geq 2} \frac{\nu_p(1^S 0^{S^c})}{p} < 1.$$

Then $X_p \sim \nu_p$ is a color process for all sufficiently small $p > 0$.

Proof. We will show that given the assumptions of the lemma, for $p > 0$ sufficiently small there is a color representation $(q_\sigma) = (q_\sigma(p))$ of $X_p \sim \nu_p$ which is such that $q_\sigma = 0$ for all $\sigma \in \mathcal{B}_n$ with more than one non-singleton partition element.

To this end, fix $p \in (0, 1/2)$. We now refer to the proof of Theorem 2.4. By Step 1 in that proof we have that a color representation $(q_\sigma(p))$ with the desired properties exists if and only if the unique solution $(q_{\sigma^S}(p))_{|S| \neq 1}$ to

$$\nu_p(1^S) = \sum_{T \subseteq [n]\,:\, |T| \neq 1} p^{\|\sigma_S^T\|} q_{\sigma^T}(p), \qquad S \subseteq [n]\colon |S| \neq 1 \tag{37}$$

is non-negative. As in the proof of Theorem 2.4, let A'' be the $2^n \times (2^n - n)$ matrix corresponding to (37) and define $B = (B(S, S'))_{S, S' \subseteq [n]}$ by

$$B(S, S') := (-p)^{|S| - |S'|} I(S' \subseteq S).$$

In the proof of Step 1 of Theorem 2.4, we saw that for $S, T \subseteq [n]$ with $|T| \neq 1$,

$$(BA'')(S,T) = \begin{cases} p(1-p)^{|S|} + (-p)^{|S|}(1-p) & \text{if } S \subseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Note that since $p \in (0, 1/2)$, if $S \subseteq T$ and $|S| \neq 1$, we have that $(BA'')(S,T) > 0$. Next, let $D = (D(S, S'))_{S, S' \subseteq [n]}$ be the diagonal matrix with $D(S,S) = (p(1-p)^{|S|} + (-p)^{|S|}(1-p))^{-1}I(|S| \neq 1)$. Then for $S, T \subseteq [n]$ with $|T| \neq 1$,

$$(DBA'')(S,T) = I(S \subseteq T) \cdot I(|S| \neq 1).$$

Furthermore, one verifies that if we define the matrix $C := (C(S, S'))_{S, S' \subseteq [n]}$ by

$$C(S, S') := \begin{cases} (-1)^{|S'|-|S|}I(S \subseteq S') & \text{if } |S| \geq 2 \text{ or } S = S' = \emptyset \\ (-1)^{|S'|-|S|}I(S \subseteq S') \cdot (1 - |S'|) & \text{if } S' \neq S = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

then (since $p \neq \{0, 1/2, 1\}$)

$$(CDBA'')(S,T) = I(S = T) \cdot I(|S| \neq 1).$$

Since, by Step 1 in the proof of Theorem 2.4, the rank of A'' is exactly $2^n - n$, it follows that if we think of ν_p as a column vector, then (37) is equivalent to

$$q_\sigma(p) = \begin{cases} e_S^T CDB\nu_p & \text{if } \sigma = \sigma^S, |S| \neq 1 \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

(with T here meaning transpose and e_S denoting the vector $(I(S' = S))_{S' \subseteq [n]}$). Now note that $DB\nu_p(1^\emptyset) = \nu_p(1^\emptyset)$ and that if $S \subseteq [n]$ has size $|S| \geq 2$, we have that

$$DB\nu_p(1^S) = e_S^T DB\nu_p = \frac{\sum_{S' \subseteq S}(-p)^{|S|-|S'|}\nu_p(1^{S'})}{p(1-p)^{|S|} + (-p)^{|S|}(1-p)}.$$

Since $|S| \geq 2$, the denominator is asymptotic to p, and by (36) the numerator is asymptotic to $\nu_p(1^S)$. It follows that $DB\nu_p(1^S) \sim p^{-I(|S|\geq 2)}\nu_p(1^S)$ for any $S \subseteq [n]$ with $|S| \neq 1$. If we apply C to the vector $(p^{-I(|S|\geq 2)}\nu_p(1^S))_{S \subseteq [n], |S| \neq 1}$, a computation shows that we get the vector $(p^{-I(|S|\geq 2)}\nu_p(1^S 0^{[n]\setminus S}))_{S \subseteq [n], |S| \neq 1}$. Since $\nu_p(1^S 0^{[n]\setminus S}) \asymp \nu_p(1^S)$ by assumption, it follows that for $S \subseteq [n]$ with $|S| \geq 2$ we have

$$q_{\sigma^S} \asymp p^{-1}\nu_p(1^S)$$

and hence $q_{\sigma^\emptyset} \sim 1 - \sum_{S \subseteq [n]: |S| \geq 2} p^{-1}\nu_p(1^S 0^{S^c})$. Since by assumption, this expression is positive for p close to zero, it follows that $q_\sigma(p) \geq 0$ for all sufficiently small $p > 0$ and all $\sigma \in \mathcal{B}_n$. This concludes the proof. $\qquad \square$

Lemma 7.7. *Let $X := (X_1, X_2, \ldots, X_n)$ be a standard Gaussian vector with strictly positive, positive definite covariance matrix A. Assume further that A is an inverse Stieltjes matrix and that $\mathbf{1}^T A^{-1} \geq \mathbf{0}$. Then for each $S \subseteq [n]$, the covariance matrix A_S of $X_S := (X_i)_{i \in S}$ is a strictly positive, positive definite inverse Stieltjes matrix with $\mathbf{1}^T A_S^{-1} \geq \mathbf{0}$.*

Remark 7.8. The main part of the proof of this lemma consists of showing that if the weak Savage condition holds for a matrix A which is an inverse Stieltjes matrix, then the weak Savage condition will also hold for any principal submatrix. Without the additional assumption that A is an inverse Stieltjes matrix, this will not be true. To see this, take e.g.

$$
A = \begin{pmatrix} 1 & 0.81 & 0.51 & 0.4 \\ 0.81 & 1 & 0.3 & 0.5 \\ 0.51 & 0.3 & 1 & 0.5 \\ 0.4 & 0.5 & 0.5 & 1 \end{pmatrix}.
$$

One can verify that A is a positive definite matrix for which the Savage condition holds, but that the Savage condition does not hold for the principal submatrix corresponding to the first three rows and columns.

Remark 7.9. Lemma 7.7 essentially proves that if X is a DGFF, then for any $S \subseteq [n]$, $X_S := (X_i)_{i \in S}$ is also a DGFF.

Proof of Lemma 7.7. By induction, it suffices to show that the conclusion of the lemma holds for S of the form $[n] \backslash \{k\}$ for some $k \in [n]$. To this end, fix $k \in [n]$. Clearly, $A_{[n] \backslash \{k\}}$ is a positive and positive definite matrix. By a lemma on page 328 in [7], $A_{[n] \backslash \{k\}}$ is also an inverse Stieltjes matrix. Next, let $(b_{ij}) := A^{-1}$. Since A is positive definite, so is A^{-1}, and hence $b_{kk} = e_k^T A^{-1} e_k > 0$. Next, since $b_{kk} > 0$, for $i, j \in [n] \backslash \{k\}$, it is well known that

$$
A_{[n] \backslash \{k\}}^{-1}(i, j) = b_{ij} - \frac{b_{ik} b_{jk}}{b_{kk}}
$$

and hence for $j \neq k$

$$
\begin{aligned}
\mathbf{1}^T A_{[n] \backslash \{k\}}^{-1}(j) &= \sum_{i \in [n] \backslash \{k\}} \left(b_{ij} - \frac{b_{ik} b_{jk}}{b_{kk}} \right) = \sum_{i \in [n]} \left(b_{ij} - \frac{b_{ik} b_{jk}}{b_{kk}} \right) \\
&= \frac{\left(\sum_{i \in [n]} b_{ij} \right) b_{kk} - \left(\sum_{i \in [n]} b_{ik} \right) b_{jk}}{b_{kk}} \\
&= \frac{\mathbf{1}^T A^{-1}(j) b_{kk} - \mathbf{1}^T A^{-1}(k) b_{jk}}{b_{kk}} \\
&= \mathbf{1}^T A^{-1}(j) - \frac{\mathbf{1}^T A^{-1}(k) b_{jk}}{b_{kk}}.
\end{aligned} \tag{39}
$$

Since $b_{jk} \leq 0$, $b_{kk} > 0$ and $\mathbf{1}^T A^{-1}(k) \geq 0$, we obtain the inequality

$$
\mathbf{1}^T A_{[n] \backslash \{k\}}^{-1}(j) \geq \mathbf{1}^T A^{-1}(j).
$$

Since this holds for all $j \neq k$, the desired conclusion follows. $\qquad \square$

The following lemma, which collects special cases of Theorems 2.1 and 2.2 in [2] and Theorem 3.1 in [3], will be needed here and in addition in the proof of some lemmas used in the proof of Theorem 1.8

Lemma 7.10. *Let X be a fully supported n-dimensional standard Gaussian vector with positive definite covariance matrix $A = (a_{ij})$. If the vector $\alpha := \mathbf{1}^T A^{-1}$ has no zero component, then as $h \to \infty$ one has that*

$$\nu_{(I(\alpha(i)>0))_i}(h) \sim \frac{1}{(2\pi)^{n/2}\sqrt{\det A} \cdot (\prod_{i=1}^n |\alpha(i)|) \cdot h^n} \cdot \exp\left(-\frac{h^2}{2} \cdot \mathbf{1}^T A^{-1}\mathbf{1}\right).$$

Furthermore if $\mathbf{1}^T A^{-1}(1) = 0$, then

$$\lim_{h \to \infty} \frac{\nu_{1^n}(h)}{\nu_{\cdot 1^{n-1}}(h)} = \frac{1}{2}.$$

We note that if $n = 3$, then assuming $\alpha(1) \leq \alpha(2) \leq \alpha(3)$, then it is immediate to check that $\alpha(2)$ and $\alpha(3)$ are strictly positive, while $\alpha(1)$ can be negative, zero or positive.

Lemma 7.11. *Let $X := (X_1, X_2, \ldots, X_n)$ be a standard Gaussian vector with positive, positive definite covariance matrix A which is an inverse Stieltjes matrix and satisfies $\mathbf{1}^T A^{-1} \geq \mathbf{0}$. Then for any $S \subseteq [n]$ with $|S| \geq 2$ and $k \in S$, as $h \to \infty$, we have*

$$p\nu_h(1^{S\setminus\{k\}}) \ll \nu_h(1^S) \asymp \nu_h(1^S 0^{[n]\setminus S}). \tag{40}$$

Proof. Let $S \subseteq [n]$ and define $X_S := (X_i)_{i \in S}$. Let A_S be the covariance matrix of X_S. By Lemma 7.7, the matrix A_S is a strictly positive, positive definite inverse Stieltjes matrix which satisfies $\mathbf{1}^T A_S^{-1} \geq \mathbf{0}$. To simplify notation, let $(a_{ij}^{(S)}) := A_S$ and $(b_{ij}^{(S)}) := A_S^{-1}$. The rest of the proof of this lemma will be divided into several steps

Step 1. Fix $S \subseteq [n]$ with $|S| \geq 2$ and $k \in S$. In this step, we will prove the inequality

$$b_{kk}^{(S)} > \left(\sum_{i \in S} b_{ki}^{(S)}\right)^2 \tag{41}$$

or equivalently

$$\sum_{i \in S} b_{ki}^{(S)} > \left(\sum_{i \in S} b_{ki}^{(S)}\right)^2 + \sum_{i \in S\setminus\{k\}} b_{ki}^{(S)}. \tag{42}$$

To this end, note first that since $(b_{ij}^{(S)})$ is the inverse of $(a_{ij}^{(S)})$, we have that

$$1 = \sum_{i \in S} a_{ik}^{(S)} b_{ki}^{(S)}.$$

Since X is a standard Gaussian vector, we have that $a_{kk}^{(S)} = 1$ and that $a_{ki}^{(S)} < 1$ if $i \in S\setminus\{k\}$. Moreover, since A_S is a positive definite inverse Stieltjes matrix by Lemma 7.7, we have that $b_{kk}^{(S)} > 0$ and that $b_{ji}^{(S)} \leq 0$ for $i \neq j$. In addition, since $a_{ij}^{(S)} > 0$ for all $i, j \in S$, we also obtain that

$$\sum_{i \in S\setminus\{k\}} b_{ki}^{(S)} < 0. \tag{43}$$

Combining these observations, we have

$$1 = \sum_{i \in S} a_{ik}^{(S)} b_{ki}^{(S)} = b_{kk}^{(S)} + \sum_{i \in S \setminus \{k\}} a_{ik}^{(S)} b_{ki}^{(S)} > b_{kk}^{(S)} + \sum_{i \in S \setminus \{k\}} b_{ki}^{(S)} \cdot 1 = \sum_{i \in S} b_{ki}^{(S)}.$$

Since $\sum_{i \in S} b_{ki}^{(S)} = \mathbf{1}^T A_S^{-1}(k) \geq 0$, it follows that

$$1 > \sum_{i \in S} b_{ki}^{(S)} \geq 0.$$

This implies in particular that

$$\sum_{i \in S} b_{ki}^{(S)} \geq \left(\sum_{i \in S} b_{ki}^{(S)} \right)^2$$

with equality if and only if $\mathbf{1}^T A_S^{-1}(k) = 0$. In view of (43), (41) follows.

Step 2. In this step, we will prove that for all $S \subseteq [n]$ with $|S| \geq 2$ and $k \in S$, we have

$$\mathbf{1}^T A_{S \setminus \{k\}}^{-1} \mathbf{1} \leq \mathbf{1}^T A_S^{-1} \mathbf{1} < 1 + \mathbf{1}^T A_{S \setminus \{k\}}^{-1} \mathbf{1} \tag{44}$$

with the first inequality being strict if and only if $\mathbf{1}^T A_S^{-1}(k) > 0$. To this end, note first that since A is positive definite, so is A_S and $A_{S \setminus \{k\}}$. So, as before, $b_{kk}^{(S)} = e_k^T A_S^{-1} e_k > 0$ and if $i, j \in S \setminus \{k\}$ then

$$A_{S \setminus \{k\}}^{-1}(i, j) = b_{ij}^{(S)} - \frac{b_{ik}^{(S)} b_{jk}^{(S)}}{b_{kk}^{(S)}}.$$

Using this, we obtain

$$\mathbf{1}^T A_{S \setminus \{k\}}^{-1} \mathbf{1} = \sum_{i,j \in S \setminus \{k\}} \left(b_{ij}^{(S)} - \frac{b_{ik}^{(S)} b_{jk}^{(S)}}{b_{kk}^{(S)}} \right)$$

and

$$\mathbf{1}^T A_S^{-1} \mathbf{1} = \sum_{i,j \in S} b_{ij}^{(S)}$$

$$= \mathbf{1}^T A_{S \setminus \{k\}}^{-1} \mathbf{1} + \left(\sum_{i,j \in S \setminus \{k\}} \frac{b_{ik}^{(S)} b_{jk}^{(S)}}{b_{kk}^{(S)}} \right) + 2 \left(\sum_{i \in S \setminus \{k\}} b_{ik}^{(S)} \right) + b_{kk}^{(S)} \tag{45}$$

$$= \mathbf{1}^T A_{S \setminus \{k\}}^{-1} \mathbf{1} + \frac{\left(\sum_{i \in S} b_{ik}^{(S)} \right)^2}{b_{kk}^{(S)}}.$$

Recalling that $b_{kk}^{(S)} > 0$ and using the conclusion of Step 1, the desired conclusion follows.

Step 3. For $S \subseteq [n]$ with $|S| \geq 2$, define $J_S := \{j \in S \colon \mathbf{1}^T A_S^{-1}(j) = 0\}$. Note that since A_S is positive definite, we have that $\mathbf{1}^T A_S^{-1} \mathbf{1} > 0$ and hence $J_S \neq S$. In this step, we claim that the following hold for any sets $S' \subseteq S \subseteq [n]$.

(i) If $i \in J_S$, then $J_{S \setminus \{i\}} = J_S \setminus \{i\}$

(ii) $|S \setminus J_S| \geq 2$

(iii) $J_{S'} \subseteq J_S$

(iv) $\mathbf{1}^T A_{(S \setminus J_S) \setminus S'}^{-1} > \mathbf{0}$

(v) The set $\{T \subseteq [n] \setminus S \colon T \subseteq J_{S \cup T}\}$ is a power set of some set

To see this, note first that by (39), for any set $S \subseteq [n]$ and any distinct $i, j \in S$ we have that

$$\mathbf{1}^T A_{S \setminus \{i\}}^{-1}(j) = \mathbf{1}^T A_S^{-1}(j) - \mathbf{1}^T A_S^{-1}(i) \cdot \frac{b_{ji}^{(S)}}{b_{ii}^{(S)}}. \tag{46}$$

From this (i) immediately follows.

For (ii), one first checks that if S has 2 elements, then $J_S = \emptyset$. For larger S, we argue by induction. Take $i \in J_S$. By induction, $(S \setminus \{i\}) \setminus (J_{S \setminus \{i\}}) \geq 2$ which by (i) implies $(S \setminus \{i\}) \setminus (J_S \setminus \{i\}) \geq 2$, which yields the result for S..

Next, by Lemma 7.7, A_S is an inverse Stieltjes matrix which satisfies $\mathbf{1}^T A_S^{-1} \geq \mathbf{0}$. In particular, this implies that $b_{ji}^{(S)} \leq 0$ and $\mathbf{1}^T A_S^{-1}(i) \geq 0$, and hence it follows from (46) that

$$\mathbf{1}^T A_{S \setminus \{i\}}^{-1}(j) \geq \mathbf{1}^T A_S^{-1}(j) \geq 0; \tag{47}$$

(iii) follows.

Next, (iv) follows easily from (iii).

We will now show that (v) holds. To simplify notation, let $\mathcal{Z}_S := \{T \subseteq [n] \setminus S \colon T \subseteq J_{S \cup T}\}$. It suffices to show that if $T_1, T_2 \in \mathcal{Z}_S$ and $i \in T_1$, then

(a) $T_1 \setminus \{i\} \in \mathcal{Z}_S$

(b) $T_2 \cup \{i\} \in \mathcal{Z}_S$.

To this end, fix $S \subseteq [n]$, $T_1, T_2 \in \mathcal{Z}_S$ and $i \in T_1$. Since $T_1 \in \mathcal{Z}_S$, we have that $T_1 \subseteq [n] \setminus S$ and $T_1 \subseteq J_{S \cup T_1}$, or equivalently, that $\mathbf{1}^T A_{S \cup T_1}^{-1}(j) = 0$ for all $j \in T_1$. Since this in particular implies that $\mathbf{1}^T A_{S \cup T_1}^{-1}(i) = 0$, using (46) it follows that $\mathbf{1}^T A_{S \cup T_1 \setminus \{i\}}^{-1}(j) = 0$ for all $j \in T_1 \setminus \{i\}$. Since $T_1 \setminus \{i\} \subseteq T_1 \subseteq [n] \setminus S$, it follows that $T_1 \setminus \{i\} \in \mathcal{Z}_S$, and hence (a) holds. Next, since $\{i\} \in T_1 \subseteq [n] \setminus S$ and $T_2 \subseteq [n] \setminus S$, we clearly have that $T_2 \cup \{i\} \subseteq [n] \setminus S$, and hence to prove that (b) holds we need to show exactly that $\mathbf{1}^T A_{S \cup T_2 \cup \{i\}}^{-1}(j) = 0$ for all $j \in T_2 \cup \{i\}$. For this, fix $j \in T_2 \cup \{i\}$ and note that by (a), $\{j\} \in \mathcal{Z}_S$, and hence $\mathbf{1}^T A_{S \cup \{j\}}^{-1}(j) = 0$. However, since $\{j\} \subseteq T_2 \cup \{i\}$, by repeated application of (47), it follows that

$$0 = \mathbf{1}^T A_{S \cup \{j\}}^{-1}(j) \geq \mathbf{1}^T A_{S \cup T_2 \cup \{i\}}^{-1}(j) \geq 0$$

and hence $\mathbf{1}^T A_{S \cup T_2 \cup \{i\}}^{-1}(j) = 0$, implying in particular that $T_2 \cup \{i\} \in \mathcal{Z}_S$. Ths concludes the proof of (b).

Step 4. In this step, we will now show that for any $S \subseteq [n]$ with $|S| \geq 2$ and $k \in S$, as $h \to \infty$, we have that

$$p\nu_h(1^{S\setminus\{k\}}) \ll \nu_h(1^S).$$

To this end, fix $S \subseteq [n]$ and let J_S be as in Step 3. By Step 2, for any $k \in S\setminus J_S$, we have that

$$\mathbf{1}^T A_{S\setminus J_S}^{-1} \mathbf{1} < 1 + \mathbf{1}^T A_{S\setminus(J_S\cup\{k\})}^{-1} \mathbf{1}. \tag{48}$$

Since this trivially holds for $k \in J_S$, it follows that these inequalities in fact hold for all $k \in S$. Now fix $k \in S$. By Step 3 (iv) we have that $\mathbf{1}^T A_{S\setminus J_S}^{-1} > \mathbf{0}$ and $\mathbf{1}^T A_{S\setminus(J_S\cup\{k\})}^{-1} > \mathbf{0}$, and hence by applying the first part of Lemma 7.10 and using (48), it follows that as $h \to \infty$, we have

$$p\nu_h(1^{S\setminus(J_S\cup\{k\})}) \ll \nu_h(1^{S\setminus J_S}).$$

Applying the second part of Lemma 7.10 several times together with Step 3 (iii), we see that

$$\nu_h(1^{S\setminus J_S}) \sim 2^{|J_S|}\nu_h(1^S) \tag{49}$$

Using this, it follows that as $h \to \infty$,

$$p\nu_h(1^{S\setminus\{k\}}) \leq p\nu_h(1^{S\setminus(J_S\cup\{k\})}) \ll \nu_h(1^{S\setminus J_S}) \asymp \nu_h(1^S)$$

and hence the desired conclusion holds.

Step 5. In this step, we show that for each $S \subseteq [n]$ with $|S| \geq 2$, as $h \to \infty$, we have that

$$\nu_h(1^S) \asymp \nu_h(1^S 0^{[n]\setminus S}). \tag{50}$$

To this end, fix $S \subseteq [n]$. By an inclusion-exclusion argument, we see that

$$\nu_h(1^S 0^{[n]\setminus S}) = \sum_{T \subseteq [n]\setminus S} \nu_h(1^{S\cup T})(-1)^{|T|}.$$

For each $T \subseteq [n]\setminus S$, let $J_{S\cup T}$ be as in Step 3. By (49) applied to $S \cup T$, it follows that

$$\nu_h(1^{S\cup T}) \sim 2^{-|J_{S\cup T}|}\nu_h(1^{(S\cup T)\setminus J_{S\cup T}}).$$

Now note that by (45) and Step 3 (iii), we have that

$$\mathbf{1}^T A_{(S\cup T)\setminus J_{S\cup T}}^{-1} \mathbf{1} = \mathbf{1}^T A_{S\cup T}^{-1} \mathbf{1}.$$

(44) and induction now implies that

$$\mathbf{1}^T A_{(S\cup T)\setminus J_{S\cup T}}^{-1} \mathbf{1} = \mathbf{1}^T A_{S\cup T}^{-1} \mathbf{1} \geq \mathbf{1}^T A_S^{-1} \mathbf{1} = \mathbf{1}^T A_{S\setminus J_S}^{-1} \mathbf{1}$$

with equality if and only if $T \subseteq J_{S\cup T}$. Since by Step 3 (iv) we have that $\mathbf{1}^T A_{(S\cup T)\setminus J_{S\cup T}}^{-1} > \mathbf{0}$, if we combine these observations and apply Lemma 7.10, it follows that

$$\nu_h(1^S 0^{S^c}) \sim \sum_{T \subseteq S^c \,:\, T \subseteq J_{S\cup T}} \nu_h(1^{S\cup T})(-1)^{|T|} \sim \nu_h(1^S) \sum_{T \subseteq S^c \,:\, T \subseteq J_{S\cup T}} 2^{-|T|}(-1)^{|T|}.$$

By Step 3 (v), the set $\{T \subseteq [n]\backslash S \colon T \subseteq J_{S \cup T}\}$ is a power set of some set S_0. Using this, it follows that

$$\sum_{T \subseteq S^c \colon T \subseteq J_{S \cup T}} 2^{-|T|}(-1)^{|T|} = \sum_{T \subseteq S_0} 2^{-|T|}(-1)^{|T|} = (1 - 2^{-1})^{|S_0|} = 2^{-|S_0|}$$

and hence (50) holds.

Since Step 4 and Step 5 together give the conclusions of the lemma, this concludes the proof. $\qquad \square$

Remark 7.12. If we assumed Savage instead of weak Savage, the proof could be somewhat shortened.

We are now ready to give the proof of Theorem 1.7.

Proof of Theorem 1.7. The covariance matrix for a DGFF is a block matrix with each block satisfying the assumptions of Lemma 7.11. Hence, restricting to a box, we have that for all S within this box with $|S| \geq 2$ and for $k \in S$, we have that

$$p\nu_h(1^{S \backslash \{k\}}) \ll \nu_h(1^S) \asymp \nu_h(1^S 0^{S^c}).$$

The second condition in Lemma 7.6 trivially holds and hence applying this lemma, we obtain conclude that for large h, the threshold Gaussian corresponding to this fixed box is a color process. Since the full process is independent over the different boxes, we easily obtain the desired result for the full process. $\qquad \square$

8 General results for small and large thresholds for $n = 3$ in the Gaussian case

8.1 h small

Theorem 8.1. *Let X be a three-dimensional random vector with a continuous and strictly positive probability density function. Assume further that $X \overset{\mathcal{D}}{=} -X$ and that X has equal marginal distributions. Then*

$$\begin{cases} \lim_{h \to 0} q_{1,2,3}(h) = 4 - \frac{4\nu_{000}'(0)}{\nu_0'(0)} \\ \lim_{h \to 0} q_{12,3}(h) = 4\nu_{001}(0) - 2 + \frac{2\nu_{000}'(0)}{\nu_0'(0)} \\ \lim_{h \to 0} q_{13,2}(h) = 4\nu_{010}(0) - 2 + \frac{2\nu_{000}'(0)}{\nu_0'(0)} \\ \lim_{h \to 0} q_{1,23}(h) = 4\nu_{100}(0) - 2 + \frac{2\nu_{000}'(0)}{\nu_0'(0)} \\ \lim_{h \to 0} q_{123}(h) = 4\nu_{000}(0) + 1 - \frac{2\nu_{000}'(0)}{\nu_0'(0)}. \end{cases}$$

(Note that $\nu_0'(0)$ is the one-dimensional marginal density at zero). As a consequence, if X is a three-dimensional standard Gaussian vector with covariance

matrix $A = (a_{ij})$ and $\theta_{ij} := \arccos a_{ij}$, then

$$
\begin{cases}
\lim_{h\to 0} q_{1,2,3}(h) = 2 - \dfrac{2\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{\pi} \\[2ex]
\lim_{h\to 0} q_{12,3}(h) = \dfrac{\theta_{13}+\theta_{23}-\theta_{12}}{\pi} - 1 + \dfrac{\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{\pi} \\[2ex]
\lim_{h\to 0} q_{13,2}(h) = \dfrac{\theta_{12}+\theta_{23}-\theta_{13}}{\pi} - 1 + \dfrac{\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{\pi} \\[2ex]
\lim_{h\to 0} q_{1,23}(h) = \dfrac{\theta_{12}+\theta_{13}-\theta_{23}}{\pi} - 1 + \dfrac{\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{\pi} \\[2ex]
\lim_{h\to 0} q_{123}(h) = 2 - \dfrac{\theta_{12}+\theta_{13}+\theta_{23}}{\pi} - \dfrac{\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{\pi}.
\end{cases}
$$

Proof. Since X has equal marginals, it follows from Theorem 2.4 that X^h has a unique signed color representation for each $h > 0$, and by (10) we have

$$
q_{1,2,3}(h) = \frac{\nu_{100}(h) - \nu_{011}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))}.
$$

Since ν_ρ is differentiable at zero, it follows that

$$
\begin{aligned}
\lim_{h\to 0} q_{1,2,3}(h) &= \lim_{h\to 0} \frac{\nu_{100}(h) - \nu_{011}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))} \\
&= 4\lim_{h\to 0} \frac{\nu_{100}(h) - \nu_{100}(-h)}{2h} \cdot \frac{2h}{\nu_0(h) - \nu_0(-h)} = \frac{4\nu'_{100}(0)}{\nu'_0(0)}.
\end{aligned}
$$

Similarly, again using (10), one has that

$$
\begin{aligned}
\lim_{h\to 0} q_{12,3}(h) &= \lim_{h\to 0} \frac{\nu_0(h)\nu_{110}(h) - \nu_1(h)\nu_{001}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))} \\
&= 4\lim_{h\to 0} \left(\frac{\nu_0(h)\nu_{110}(h) - \nu_0(-h)\nu_{110}(-h)}{2h} \right) \cdot \frac{2h}{\nu_0(h) - \nu_0(-h)} \\
&= 4 \cdot \frac{\nu'_0(0)\nu_{110}(0) + \nu_0(0)\nu'_{110}(0)}{\nu'_0(0)} = 4\nu_{110}(0) + \frac{2\nu'_{110}(0)}{\nu'_0(0)} \\
&= 4\nu_{001}(0) - \frac{2\nu'_{001}(0)}{\nu'_0(0)}.
\end{aligned}
$$

If we can show that

$$
\nu'_{.00}(0) = \nu'_{0\cdot 0}(0) = \nu'_{00\cdot}(0) = \nu'_0(0) \tag{51}
$$

the first part of the theorem will follow using symmetry and the fact that $\sum q_\sigma = 1$. To see that (51) holds, let f be the probability density function of (X_1, X_2) and note that $\nu'_0(x)$ is the marginal density of both X_1 and X_2. Then for any $h_1, h_2 \in \mathbb{R}$ we have that

$$
\begin{aligned}
\frac{d}{dh_2} P(X_1 \leq h_1, X_2 \leq h_2) &= \frac{d}{dh_2} \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} f(x_1, x_2)\, dx_2\, dx_1 \\
&= \int_{-\infty}^{h_1} f(x_1, h_2)\, dx_1 = P(X_1 \leq h_1 \mid X_2 = h_2) \cdot \nu'_0(h_2).
\end{aligned}
$$

Differentiating with respect to h_1 in the same way and then setting $h_1 = h_2 = 0$, it follows that

$$\nu'_{00.}(0) = \nu'_0(0)\left(P(X_1 \leq 0 \mid X_2 = 0) + P(X_2 \leq 0 \mid X_1 = 0)\right).$$

By symmetry, the two summands are each equal to $1/2$, and hence $\nu'_{00.}(0) = \nu'_0(0)$ as desired. The other equalities follow by an analogous argument.

For the second part of the theorem, note first that by an analogous argument as above, one obtains in general that

$$\frac{\nu'_{000}(0)}{\nu'_0(0)} = P(X_2, X_3 \leq 0 \mid X_1 = 0) + P(X_1, X_3 \leq 0 \mid X_2 = 0) + P(X_1, X_2 \leq 0 \mid X_3 = 0).$$

Using basic facts about Gaussian vectors, one has that $(X_2, X_3) \mid X_1 = 0$ is a Gaussian vector with correlation

$$\alpha = \frac{a_{23} - a_{12}a_{13}}{\sqrt{(1 - a_{12}^2)(1 - a_{13}^2)}}.$$

Using (20), it follows that

$$P(X_2 \leq 0, X_3 \leq 0 \mid X_1 = 0) = \frac{1}{2} - \frac{\arccos\left(\frac{a_{23} - a_{12}a_{13}}{\sqrt{(1 - a_{12}^2)(1 - a_{13}^2)}}\right)}{2\pi}$$

and hence, by symmetry, we obtain

$$\frac{\nu'_{000}(0)}{\nu'_0(0)} = \frac{3}{2} - \frac{\arccos\left(\frac{a_{23} - a_{12}a_{13}}{\sqrt{(1 - a_{12}^2)(1 - a_{13}^2)}}\right) + \arccos\left(\frac{a_{13} - a_{12}a_{23}}{\sqrt{(1 - a_{12}^2)(1 - a_{23}^2)}}\right) + \arccos\left(\frac{a_{12} - a_{13}a_{23}}{\sqrt{(1 - a_{13}^2)(1 - a_{23}^2)}}\right)}{2\pi}.$$

Now recall that for any $\alpha, \beta \in [-1, 1]$ we have that

$$\arccos\alpha + \arccos\beta = \begin{cases} \arccos(\alpha\beta - \sqrt{(1 - \alpha^2)(1 - \beta^2)}) & \text{if } \alpha + \beta \geq 0 \\ 2\pi - \arccos(\alpha\beta - \sqrt{(1 - \alpha^2)(1 - \beta^2)}) & \text{if } \alpha + \beta \leq 0. \end{cases}$$

and hence if $\alpha, \beta \in [-1, 1]$ satisfies $\alpha + \beta \geq 0$ and $\alpha\beta - \sqrt{1 - \alpha^2}\sqrt{1 - \beta^2} + \gamma \leq 0$, then

$$\arccos\alpha + \arccos\beta + \arccos\gamma$$
$$= 2\pi - \arccos\left(\alpha\beta\gamma - \alpha\sqrt{(1 - \beta^2)(1 - \gamma^2)} - \beta\sqrt{(1 - \alpha^2)(1 - \gamma^2)} - \gamma\sqrt{(1 - \alpha^2)(1 - \beta^2)}\right).$$

Now let

$$\begin{cases} \alpha = \frac{a_{23} - a_{12}a_{13}}{\sqrt{(1 - a_{12}^2)(1 - a_{13}^2)}} \\ \beta = \frac{a_{13} - a_{12}a_{23}}{\sqrt{(1 - a_{12}^2)(1 - a_{23}^2)}} \\ \gamma = \frac{a_{12} - a_{13}a_{23}}{\sqrt{(1 - a_{13}^2)(1 - a_{23}^2)}}. \end{cases}$$

Using that as A is positive definite, then $a_{12} \leq a_{13}a_{23} + \sqrt{(1 - a_{13}^2)(1 - a_{23}^2)}$, it follows that we indeed have that $\alpha + \beta \geq 0$. Moreover, with some work, one verifies that

$$\alpha\beta - \sqrt{1 - \alpha^2}\sqrt{1 - \beta^2} + \gamma = \frac{-\det A}{(1 + a_{12})\sqrt{1 - a_{13}^2}\sqrt{1 - a_{23}^2}} \leq 0$$

50

and that

$$\alpha\beta\gamma - \alpha\sqrt{(1-\beta^2)(1-\gamma^2)} - \beta\sqrt{(1-\alpha^2)(1-\gamma^2)} - \gamma\sqrt{(1-\alpha^2)(1-\beta^2)}$$
$$= \frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1.$$

This implies in particular that

$$\frac{\nu'_{000}(0)}{\nu'_0(0)} = \frac{3}{2} - \frac{\arccos\alpha + \arccos\beta + \arccos\gamma}{2\pi}$$

$$= \frac{3}{2} - \frac{2\pi - \arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{2\pi}$$

$$= \frac{1}{2} + \frac{\arccos\left(\frac{\det A}{\prod_{i<j}(1+a_{ij})} - 1\right)}{2\pi}.$$

Combining this with (24) and the first part of the theorem, the desired conclusion follows. □

We now apply Theorem 8.1 to a few examples.

Corollary 8.2. *Let $a \in (0,1)$ and let $X := (X_1, X_2, X_3)$ be a standard Gaussian vector with $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = a$. Then X^h is a color process for all sufficiently small h.*

Proof. Note first that by using Theorem 8.1, after a computation, we obtain

$$\begin{cases} \lim_{h\to 0} q_{1,2,3}(h) = 2 - \dfrac{2\arccos\left(\frac{a(a^2-6a-3)}{(1+a)^3}\right)}{\pi} \\ \lim_{h\to 0} q_{12,3}(h) = \dfrac{\arccos a}{\pi} - 1 + \dfrac{\arccos\left(\frac{a(a^2-6a-3)}{(1+a)^3}\right)}{\pi} \\ \lim_{h\to 0} q_{123}(h) = 2 - \dfrac{3\arccos a}{\pi} - \dfrac{\arccos\left(\frac{a(a^2-6a-3)}{(1+a)^3}\right)}{\pi}. \end{cases}$$

It suffices to show that the above limits are positive. Since $\arccos x \in (0, \pi)$ for all $x \in (-1, 1)$ and $\arccos x$ is strictly decreasing in x, it follows that the first of these is strictly positive whenever

$$\frac{a(a^2 - 6a - 3)}{(1+a)^3} > -1.$$

By rearranging, one easily sees this to be true whenever $a \in (0, 1)$. Next, since $\pi - \arccos x = \arccos(-x)$ for all $x \in (0, 1)$ it follows that the second limit is strictly positive whenever

$$a + \frac{a(a^2 - 6a - 3)}{(1+a)^3} = \frac{-a(1-a)(2 + 5a + a^2)}{(1+a)^3} < 0.$$

which clearly holds for all $a \in (0, 1)$. To see that X^h has a color representation for all sufficiently small $h > 0$, it thus only remains to show that $\lim_{h\to 0} q_{123}(h) > 0$. To this end, first note that this is equivalent to that

$$3\arccos a + \arccos\left(\frac{a(a^2 - 6a - 3)}{(1+a)^3}\right) < 2\pi.$$

It is easy to verify that we get equality when $a = 0$, and hence it would be enough to show that the left hand side is strictly decreasing in a. If we differentiate the left hand side one we obtain, after a detailed computation, that

$$\frac{3}{(1+a)\sqrt{1+2a}} - \frac{3}{\sqrt{1-a^2}}$$

which is clearly negative for all $a \in (0,1)$. From this the desired conclusion follows. $\qquad\square$

Corollary 8.3. *Let $a \in (0,1)$ and let $X := (X_1, X_2, X_3)$ be a standard Gaussian vector with $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_2, X_3) = a$ and $\mathrm{Cov}(X_1, X_3) = a^2$. Then X^h is a color process for all sufficiently small h.*

Remark 8.4. With $X = (X_1, X_2, X_3)$ defined as a above, X is a Markov chain.

Proof of Corollary 8.3. Note first that by using Theorem 8.1, after a computation, we obtain

$$\begin{cases} \lim_{h \to 0} q_{1,2,3}(h) = 2 - \frac{2 \arccos\left(\frac{-2a}{1+a^2}\right)}{\pi} \\ \lim_{h \to 0} q_{12,3}(h) = \frac{\arccos a^2}{\pi} - 1 + \frac{\arccos\left(\frac{-2a}{1+a^2}\right)}{\pi} \\ \lim_{h \to 0} q_{13,2}(h) = \frac{2 \arccos a - \arccos a^2}{\pi} - 1 + \frac{\arccos\left(\frac{-2a}{1+a^2}\right)}{\pi} \\ \lim_{h \to 0} q_{1,23}(h) = \frac{\arccos a^2}{\pi} - 1 + \frac{\arccos\left(\frac{-2a}{1+a^2}\right)}{\pi} \\ \lim_{h \to 0} q_{123}(h) = 2 - \frac{2 \arccos a + \arccos a^2}{\pi} - \frac{\arccos\left(\frac{-2a}{1+a^2}\right)}{\pi}. \end{cases}$$

It suffices to show that the above limits are positive. By using the fact that $\pi - \arccos x = \arccos(-x)$ for all $x \in (-1,1)$ and the fact that arccosine is a strictly decreasing function, one easily verifies that the first, second and fourth of these are strictly positive for all $a \in (0,1)$. To see that the third limit is strictly positive for $a \in (0,1)$, we differentiate this limit with respect to a to obtain

$$(a\sqrt{1+a^2} + \sqrt{1-a^2} - (1+a^2)) \cdot \frac{2}{(1+a^2)\sqrt{1-a^2}}.$$

This expression can be equal to zero if and only if

$$a\sqrt{1+a^2} + \sqrt{1-a^2} = 1 + a^2.$$

Squaring both sides and simplifying, we see that this is equivalent to that

$$\sqrt{1-a^4} = a$$

which in turn is equivalent to that

$$1 - a^2 - a^4 = 0.$$

This equation clearly has exactly one solution in $(0,1)$. Hence in particular, there can be only one maxima or minima in $(0,1)$. Since $\lim_{h \to 0} q_{13,2}(h)(a)$ is continuous in a for all $a \in [0,1]$, $\lim_{h \to 0} q_{13,2}(h)(0) = \lim_{h \to 0} q_{13,2}(h)(1) = 0$ and one easily verifies that $\lim_{h \to 0} q_{13,2}(h)(0.5) > 0$ it follows that $\lim_{h \to 0} q_{13,2}(h)(a) > 0$ for all $a \in (0,1)$.

Finally, one easily verifies that the derivative of $\lim_{h\to 0} q_{123}(h)(a)$ with respect to a is given by

$$(a\sqrt{1+a^2} + (1+a^2) - \sqrt{1-a^2}) \cdot \frac{2}{(1+a^2)\sqrt{1-a^2}}$$

which has no zeros in $(0,1)$. Since $\lim_{h\to 0} q_{123}(h)(0) = 0$, $\lim_{h\to 0} q_{123}(h)(1) = 1$ and $\lim_{h\to 0} q_{123}(h)(a)$ is continuous in a, it must be strictly increasing in a in $(0,1)$, and hence it follows that $\lim_{h\to 0} q_{123}(h)(a) > 0$ for all $a \in (0,1)$. $\qquad\square$

8.2 h large

Before proving Theorem 1.8, we start off by giving some interesting applications of it.

Corollary 8.5. *For each case below, there is at least one Gaussian vector X with non-negative correlations which satisfies it.*

 (i) *X^h has a color representation for all sufficiently large h and for all sufficiently small $h > 0$.*

 (ii) *X^h has no color representation for any sufficiently large h nor for any sufficiently small $h > 0$.*

 (iii) *X^h has a color representation for all sufficiently large h but not for any sufficiently small $h > 0$.*

 (iv) *X^h has a color representation for all sufficiently small h but not for any sufficiently large h.*

In particular, the property of X^h being a color process for a fixed X is not monotone in h (in either direction) for $h > 0$.

Proof.

 (i) Of course one can take an i.i.d. process here. A more interesting example is as follows. Let X be a three-dimensional standard Gaussian vector with $\mathrm{Cov}(X_1, X_2) = \mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = a \in (0,1)$. By combining Corollary 8.2 and Theorem 1.8(i), it follows that X^h has a color representation for both sufficiently small and sufficiently large $h > 0$.

 (ii) Let X be a three-dimensional Gaussian vector with $\mathrm{Cov}(X_1, X_2) = 0.05$, $\mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = 0.6825$. One can verify that this corresponds to a positive definite covariance matrix. Using Theorem 8.1, one verifies that $\lim_{h\to 0} q_{12,3}(h) \approx -0.05$ and hence X^h does not have a color representation for any sufficiently small h. Using Theorem 1.8, it follows that X^h does not either have a color representation for large h.

 (iii) Let X be a three-dimensional standard Gaussian vector with $\mathrm{Cov}(X_1, X_2) = 0.1$, $\mathrm{Cov}(X_1, X_3) = \mathrm{Cov}(X_2, X_3) = 0.5$. One can verify that this corresponds to a positive definite covariance matrix. Now by Theorem 8.1, the limit $\lim_{h\to 0} q_{12,3}(h) \approx -0.016$ and hence X^h does not have a color representation for any sufficiently small $h > 0$. Next, since the Savage condition (2) holds, we have that X^h has a color representation for all sufficiently large h.

(iv) This follows immediately from Theorem 5.9.

$\square$

Example 8.6. It is illuminating to look at the subset of the set of three-dimensional standard Gaussians for which at least two of the covariances are equal. So, we let $X_{a,b} = (X_1, X_2, X_3)$ be a standard Gaussian vector with covariance matrix

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & b \\ a & b & 1 \end{pmatrix}$$

for some $a, b \in (0, 1)$. One can verify that A is positive definite exactly when $2a^2 < 1 + b$. Applying Theorem 1.8, one can check that $X_{a,b}^h$ is a color process for all sufficiently large h if and only if either $2a - 1 \le b$ or $(2a - 1)^2 < b$ (note both of these inequalities imply that $2a^2 < 1 + b$). Cases (i) and (ii) correspond to the first inequality holding and Case (iii) corresponds to the first inequality failing and the second inequality holding. For a fixed h, the set of parameters which yield a color process for threshold h is a closed set. However the set of parameters which yield a color process for sufficiently large h is not a closed set; for example, $a = .1$ and $b = \epsilon$ belongs to this set for every $\epsilon > 0$ but not for $\epsilon = 0$.

In Figure 5, we first draw the regions corresponding to the various cases in Theorem 1.8 and the region corresponding to having a positive definite covariance matrix. In the second picture, we superimpose the region corresponding to all choices of a and b for which $X_{a,b}^h$ has a color representation for all h which are sufficiently close to zero. Interestingly, this figure suggests that if $X_{a,b}^h$ is a color process for h close to zero, then $X_{a,b}^h$ is also a color process for h sufficiently large. Moreover, the region corresponding to the set of a and b for which $X_{a,b}^h$ has a color representation for h close to zero intersects both the regions corresponding to Cases (i) and (iii).

We now proceed with the proof of Theorem 1.8.

Lemma 8.7. *Let $X := (X_1, X_2)$ be a fully supported standard Gaussian vector with covariance matrix $A = (a_{ij})$. Then $\nu_{11}(h) \ll \nu_1(h)$ and if $a_{12} > 0$, then $\nu_1(h)^2 \ll \nu_{11}(h)$.*

Proof. We have that $\mathbf{1}^T A^{-1} = \left((1 + a_{12})^{-1}, (1 + a_{12})^{-1}\right) > \mathbf{0}$ and hence Lemma 7.10 implies that

$$\nu_{11}(h) \asymp h^{-2} \cdot \exp\left(-\frac{h^2}{2} \cdot \frac{2}{1 + a_{12}}\right).$$

Since $p_1(h) \asymp h^{-1} \cdot \exp\left(-h^2/2\right)$, $p_1(h)^2 \asymp h^{-2} \cdot \exp\left(-h^2\right)$ and $a_{12} < 1$ by the fully supported assumption, the result easily follows. $\square$

Lemma 8.8. *Let X be a fully supported 3-dimensional standard Gaussian vector with covariance matrix $A = (a_{ij})$. If $a_{ij} \in [0, 1)$ for all $i < j$, then*

$$\nu_1(h) \max(\{\nu_{11\cdot}(h), \nu_{1\cdot1}(h), \nu_{\cdot11}(h)\})$$
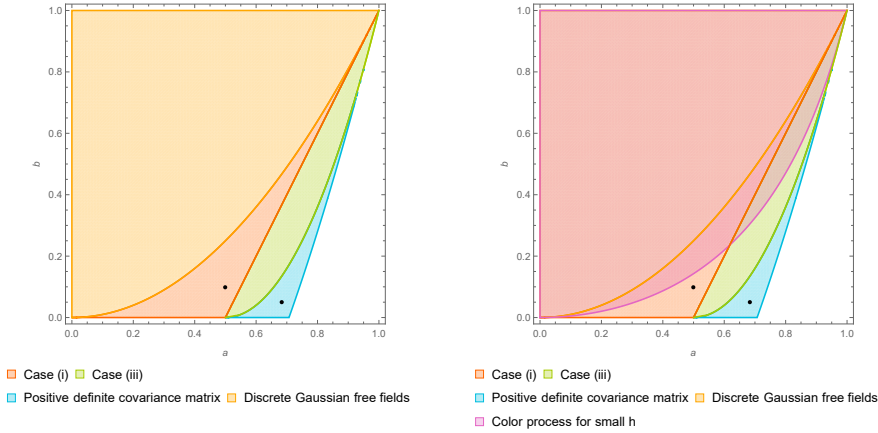$$\ll \min(\{\nu_{11\cdot}(h), \nu_{1\cdot1}(h), \nu_{\cdot11}(h)\}).$$

Figure 5: The figure to the left shows, for Example 8.6, the different cases in Theorem 1.8. A is positive definite in the blue region and to its left, Case (iii) is the green region, Case (i) is the red region and to its left and the set of DGFFs is the orange region. Case (ii) corresponds to the straight line $b = 2a - 1$. The boundary of the orange region, which is the line $b = a^2$, corresponds to the family of standard Gaussian Markov chains. The boundary between the green and blue regions is the right half of the parabola $b = (2a - 1)^2$. Finally the two black points correspond to the two examples given in the proof of (ii) and (iii) of Corollary 8.5. The picture to the right is the same except with the region where there is a color representation for h sufficiently close to zero being superimposed.

Proof. For $i < j$, let A_{ij} be the covariance matrix of (X_i, X_j). Then $\mathbf{1}^T A_{ij}^{-1} = \left((1 + a_{ij})^{-1}, (1 + a_{ij})^{-1} \right) > \mathbf{0}$ and hence Lemma 7.10 implies that

$$\nu_{1\{i,j\}}(h) \asymp h^{-2} \cdot \exp\left(-\frac{h^2}{2} \cdot \frac{2}{1 + a_{ij}} \right) \tag{52}$$

and so

$$\nu_1(h)\nu_{1\{i,j\}}(h) \asymp h^{-3} \cdot \exp\left(-\frac{h^2}{2} \cdot \left(1 + \frac{2}{1 + a_{ij}} \right) \right). \tag{53}$$

In particular, this implies that the desired conclusion follows if we can show that

$$\max_{i<j} \frac{2}{1 + a_{ij}} < 1 + \min_{i<j} \frac{2}{1 + a_{ij}}.$$

However, since $a_{ij} \in [0, 1)$ for all $i < j$ we have that

$$\max_{i<j} \frac{2}{1 + a_{ij}} \leq 2 < 1 + \min_{i<j} \frac{2}{1 + a_{ij}}.$$

$\square$

Lemma 8.9. *Let X be a fully supported 3-dimensional standard Gaussian vector with covariance matrix $A = (a_{ij})$. If $\mathbf{1}^T A^{-1} > \mathbf{0}$ and at most one of the covariances a_{ij} is equal to zero, then*

$$\nu_1(h) \max(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\}) \ll \nu_{111}(h)$$
$$\ll \min(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\}). \tag{54}$$

55

Proof. We first show that the second of the two inequalities holds. First, since $\mathbf{1}^T A^{-1} > \mathbf{0}$ by assumption, we have that

$$\nu_{111}(h) \asymp h^{-3} \cdot \exp\left(-\frac{h^2}{2} \cdot \mathbf{1}^T A^{-1} \mathbf{1}\right). \tag{55}$$

Since

$$\nu_{111}(h) \leq \min(\{\nu_{11\cdot}(h), \nu_{1\cdot1}(h), \nu_{\cdot11}(h)\}),$$

(52) implies

$$\mathbf{1}A^{-1}\mathbf{1} \geq \frac{2}{1 + a_{ij}}$$

for all $i < j$. However since $h^{-3} \ll h^{-2}$, it follows that we then must have that

$$\nu_{111}(h) \ll \min(\{\nu_{11\cdot}(h), \nu_{1\cdot1}(h), \nu_{\cdot11}(h)\}).$$

This shows that the second inequality in (54) holds.

Next, to show that the first of the two inequalities in (54) holds, we will show that

$$\mathbf{1}^T A^{-1}\mathbf{1} < 1 + \frac{2}{1 + a_{ij}} \tag{56}$$

for all $i < j$, since if this holds, then (53) and (55) immediately imply the desired conclusion. To this end, using (1), one first verifies that $\mathbf{1}^T A^{-1} > \mathbf{0}$ is equivalent to

$$1 + 2\min_{i<j} a_{ij} > \sum_{i<j} a_{ij}. \tag{57}$$

Similarly, (56) can be shown to be equivalent to

$$(1 + \max_{i<j}\{a_{ij}\})\prod_{i<j}(1 - a_{ij}) < 1 - \sum_{i<j} a_{ij}^2 + 2\prod_{i<j} a_{ij}. \tag{58}$$

If $a_{ij} = 0$ for exactly one of the covariances, then one easily verifies that (58) holds when (57) holds. Now instead assume that $a_{ij} > 0$ for all $i > j$. If we think of $a_{12} > 0$ as being fixed, then (58) holds for all a_{13} and a_{23} in the interior of the ellipse E given by

$$(1 - a_{12}^2)(1 - x)(1 - y) = 1 - a_{12}^2 - x^2 - y^2 + 2a_{12}xy, \quad x, y \in \mathbb{R}.$$

One verifies that the boundary of E passes through the origin and the points $(0, 1 - a_{12}^2)$, $(1 - a_{12}^2, 0)$, $(a_{12}, 1)$ and $(1, a_{12})$. Since we are assuming the Savage condition (2), any possible a_{13} and a_{23}) under consideration necessarily lies in the region R given by

$$1 + 2\min(\{a_{12}, x, y\}) > a_{12} + x + y, \quad x, y > 0.$$

Hence we need only show that $R \subseteq E$. (See Figure 6.) To see this containment, note that R is a polygon with vertices given by $(0,0)$, $(0, 1 - a_{12})$, $(1 - a_{12}, 0)$, $(1, a_{12})$ and $(a_{12}, 1)$. We already know that the first, fourth and fifth of these vertices lie on the boundary of E while one easily checks that the other two lie inside E. Since E is convex, and R is a polygon, it follows that $R \subseteq E$. $\qquad\square$
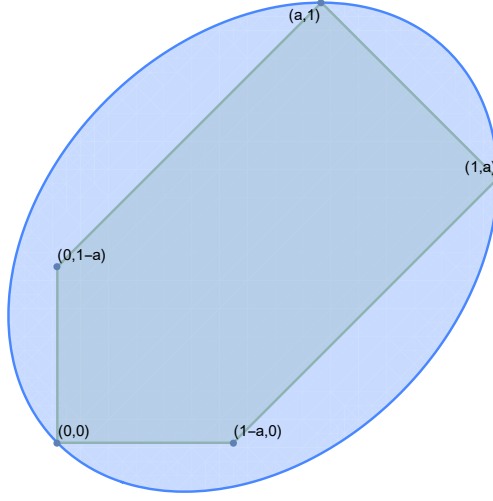
Figure 6: The image above shows the situation in the proof of Lemma 8.9, where we are interested in whether a region R is contained inside a given ellipse E.

We are now ready to give the proof of Theorem 1.8. We remark that in the proof, Case 1 and Case 2 can alternatively be proven, using the lemmas in this section, by appealing to Lemma 7.6.

Proof of Theorem 1.8. Let $(q_\sigma)_{\sigma \in \mathcal{B}_3}$ be the unique solution to (5) guaranteed to exist by Theorem 2.4. By (10), using inclusion-exclusion, we see that for any $h > 0$ we have that

$$q_{1,2,3}(h) = \frac{\nu_1(h) - (\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h)) + 2\nu_{111}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))},$$

$$q_{12,3}(h) = \frac{(1 - 2\nu_1(h))\nu_{11.}(h) + \nu_1(h)(\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h)) - \nu_1(h)^2 - \nu_{111}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))}$$

and

$$q_{123}(h) = \frac{2\nu_1(h)^3 + \nu_{111}(h) - \nu_1(h)(\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h))}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))}.$$

This implies that there is a color representation for large h if and only if for all large h we have that

$$\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h) \le \nu_1(h) + 2\nu_{111}(h), \tag{59}$$

$$\begin{aligned} \nu_{111}(h) + \nu_1(h)^2 &\le \nu_1(h)(\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h)) \\ &\quad + (1 - 2\nu_1(h))\min(\{\nu_{11.}(h), \nu_{1.1}(h), \nu_{.11}(h)\}) \end{aligned} \tag{60}$$

and

$$\nu_1(h)(\nu_{.11}(h) + \nu_{1.1}(h) + \nu_{11.}(h)) \le \nu_{111}(h) + 2\nu_1(h)^3. \tag{61}$$

We will check when (59), (60) and (61) hold for large h by comparing the decay rate of the various tails. Before we do this, note that by (1), one has

57

that $\mathbf{1}^T A^{-1}(1) \leq 0$ exactly when $1 + a_{23} \leq a_{12} + a_{13}$. If this holds, then clearly $a_{23} = \min_{i<j}(a_{ij})$ and hence $\nu_{\cdot 11}(h) = \min(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\})$.

Without loss of generality, we assume that $0 \leq a_{23} \leq a_{13} \leq a_{12}$ and that $a_{12} > 0$, since the case $a_{12} = a_{13} = a_{23} = 0$ is trivial. Note that this assumption implies by (1) that

$$\mathbf{1}A^{-1}(1) \leq \mathbf{1}A^{-1}(2) \leq \mathbf{1}A^{-1}(3)$$

with the largest two terms being positive.

We now claim that (59) holds for all sufficiently large h, without making any additional assumptions on A. To see this, note that Lemma 8.7 implies that

$$\nu_{\cdot 11}(h) + \nu_{1\cdot 1}(h) + \nu_{11\cdot}(h) \leq 3\nu_{11\cdot}(h) \ll \nu_1(h) \leq \nu_1(h) + 2\nu_{111}(h)$$

and hence (59) holds for all large h.

We now divide into three cases.

Case 1. Assume that $\mathbf{1}^T A^{-1}(1) > \mathbf{0}$ and $a_{23} > 0$. We will show that both (60) and (61) hold in this case without any further assumptions. To this end, note first that since $a_{23} > 0$, Lemma 8.7 implies that $\nu_1(h)^2 \ll \nu_{\cdot 11}(h)$. Moreover, since $\mathbf{1}^T A^{-1} > \mathbf{0}$, Lemma 8.9 implies that

$$\nu_1(h) \max(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\}) \ll \nu_{111}(h)$$
$$\ll \min(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\}).$$

Combining these observations, we obtain

$$\nu_{111}(h) + \nu_1(h)^2 \ll \nu_{\cdot 11}(h) \sim (1 - 2\nu_1(h))\nu_{\cdot 11}(h)$$
$$\leq \nu_1(h)(\nu_{\cdot 11}(h) + \nu_{1\cdot 1}(h) + \nu_{11\cdot}(h))$$
$$+ (1 - 2\nu_1(h)) \min(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\})$$

and hence (60) holds. Similarly, we obtain

$$\nu_1(h)(\nu_{\cdot 11}(h) + \nu_{1\cdot 1}(h) + \nu_{11\cdot}(h)) \ll \nu_{111}(h) \leq \nu_{111}(h) + 2\nu_1(h)^3.$$

establishing (61). This concludes the proof of (i).

Case 2. Assume that $\mathbf{1}^T A^{-1}(1) = 0$ and $a_{23} > 0$. We will show that (60) and (61) both hold in this case. To this end, note first that since $\mathbf{1}^T A^{-1}(1) = 0$, Lemma 7.10 implies that $\nu_{111}(h) \sim \nu_{\cdot 11}(h)/2$ and, since $a_{23} > 0$, Lemma 8.7 implies that $\nu_{\cdot 11}(h) \gg \nu_1(h)^2$. This implies in particular that

$$\nu_{111}(h) + \nu_1(h)^2 \sim \nu_{\cdot 11}(h)/2 + \nu_1(h)^2 \sim \nu_{\cdot 11}(h)/2$$
$$\sim \frac{1}{2}(1 - 2\nu_1(h))\nu_{\cdot 11}(h)$$
$$= \frac{1}{2}(1 - 2\nu_1(h)) \min(\{\nu_{11\cdot}(h), \nu_{1\cdot 1}(h), \nu_{\cdot 11}(h)\})$$

and hence (60) holds for all sufficiently large h. Next, using Lemma 8.8, we obtain

$$\nu_1(h)(\nu_{\cdot 11}(h) + \nu_{1\cdot 1}(h) + \nu_{11\cdot}(h)) \ll \nu_{\cdot 11}(h)/2 \sim \nu_{111}(h)$$

and hence (61) holds for all sufficiently large h in this case. This finishes the proof of (ii).

Case 3. Assume that $\mathbf{1}^T A^{-1}(1) < 0$. By Lemma 7.10, we have that $\nu_{111}(h) \sim \nu_{\cdot 11}(h)$. Using this, one easily checks that (61) holds by the same argument as in Case 2, and hence it remains only to check when (60) holds. To this end, note first that if we use the assumption that $a_{23} \le a_{13} \le a_{12}$, then (60) is equivalent to

$$\nu_1(h)^2 + \nu_1(h)(\nu_{\cdot 11}(h) - \nu_{1 \cdot 1}(h) - \nu_{11 \cdot}(h)) \le \nu_{011}(h). \tag{62}$$

Since $a_{ij} < 1$ for all $i < j$, Lemma 8.7 implies that

$$\nu_1(h)^2 + \nu_1(h)(\nu_{\cdot 11}(h) - \nu_{1 \cdot 1}(h) - \nu_{11 \cdot}(h)) \sim \nu_1(h)^2.$$

Therefore, by Lemma 7.10, we see that if $\mathbf{1}^T A^{-1} \mathbf{1} < 2$ holds, then $\nu_{011}(h) \gg \nu_1(h)^2$ yielding (62). On the other hand, if $\mathbf{1}^T A^{-1} \mathbf{1} \ge 2$ holds, then $\nu_{011}(h) \ll \nu_1(h)^2$ in which case (62) fails.

Case 4 Assume now that $a_{23} = 0$, i.e. that X_2 and X_3 are independent. Note that if $a_{13} = a_{23} = 0$, then there is a color representation by Proposition 7.1, and hence we can assume that $a_{13} > 0$. Now note that since X_2 and X_3 are independent by assumption, if X^h has a color representation $(q_\sigma(h))$ for some h, it must satisfy $q_{1,23}(h) = q_{123}(h) = 0$. Using the general formula for these expressions, we obtain that

$$\nu_{111}(h) + \nu_1(h)^2 = \nu_1(h)(\nu_{\cdot 11}(h) + \nu_{1 \cdot 1}(h) + \nu_{11 \cdot}(h))$$
$$+ (1 - 2\nu_1(h))\nu_{\cdot 11}(h)$$

and

$$\nu_1(h)(\nu_{\cdot 11}(h) + \nu_{1 \cdot 1}(h) + \nu_{11 \cdot}(h)) = \nu_{111}(h) + 2\nu_1(h)^3.$$

Using that $\nu_{\cdot 11}(h) = \nu_1(h)^2$ by assumption, we see that these equations are both equivalent to that

$$\nu_{111}(h) + \nu_1(h)^3 = \nu_1(h)(\nu_{1 \cdot 1}(h) + \nu_{11 \cdot}(h)). \tag{63}$$

We will show that (63) does not hold for any large h. To this end, note first that if $\mathbf{1}^T A^{-1} > \mathbf{0}$ and $a_{12}, a_{13} > 0$, then by Lemma 8.9 we have that

$$\nu_{111}(h) + \nu_1(h)^3 \sim \nu_{111}(h) \gg \nu_1(h)(\nu_{1 \cdot 1}(h) + \nu_{11 \cdot}(h))$$

and hence (63) cannot hold, implying that there can be no color representation for any large h in this case.

Next, if $\mathbf{1}^T A^{-1}(1) = 0$, then using Lemma 7.10 we get that

$$\nu_{111}(h) + \nu_1(h)^3 \sim \nu_{\cdot 11}(h)/2 + \nu_1(h)^3 = \nu_1(h)^2/2 + \nu_1(h)^3 \sim \nu_1(h)^2/2.$$

Using Lemma 8.7 and the assumption that $a_{12}, a_{13} < 1$, it follows that

$$\nu_1(h)^2 \gg \nu_1(h)(\nu_{1 \cdot 1}(h) + \nu_{11 \cdot}(h))$$

and hence (63) cannot hold, implying that there can be no color representation for any large h in this case.

Finally, if $\mathbf{1}^T A^{-1}(1) < 0$ then we can use Case 3. Observing that if $a_{23} = 0$, then $\det A > 0$ implies that $a_{12}^2 + a_{13}^2 < 1$, we have that

$$\mathbf{1}^T A^{-1} \mathbf{1} < 2 \Leftrightarrow 1 + \frac{2(1 - a_{12})(1 - a_{13})}{1 - a_{12}^2 - a_{13}^2} < 2$$
$$\Leftrightarrow (1 - a_{12} - a_{13})^2 < 0$$

implying in particular that there can be no color representation. $\qquad \square$

9 Large threshold results for stable vectors with emphasis on the $n = 3$ case

9.1 Two-dimensional stable vectors and nonnegative correlations

In this subsection, we give a proof of Proposition 1.10.

Proof of Proposition 1.10. We may stick to $h \geq 0$ throughout. It is elementary to check that X_1^h and X_2^h have nonnegative correlation if and only if

$$P((1 - a^\alpha)^{1/\alpha} S_2 \geq a|S_1| + h) \geq P(S_1 \geq h)^2. \tag{64}$$

When $h = 0$ and $a = 2^{-1/\alpha}$, we have that

$$P((1 - a^\alpha)^{1/\alpha} S_2 \geq a|S_1| + h) = P(S_2 \geq |S_1|) = 1/4$$

and

$$P(S_1 \geq h)^2 = 1/4.$$

Hence we get equality in (64) in this case. Now note that the left hand side of (64) is strictly decreasing in a. This implies that when $h = 0$, we get nonnegative correlations if and only if $a \leq 2^{-1/\alpha}$, establishing the equivalence of (i) and (ii).

We now show that (ii) implies (iii). To see this, note first that since the left hand side of (64) is strictly decreasing in a, it suffices to show that (64) holds for all $h \geq 0$ when $a = 2^{-1/\alpha}$. To this end, note first that in this case, we have that

$$P((1 - a^\alpha)^{1/\alpha} S_2 \geq a|S_1| + h) = P(S_2 \geq |S_1| + h2^{1/\alpha}).$$

Now observe that

$$2P(S_2 \geq S_1 + h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha})$$
$$= 2P(S_2 \geq S_1 + h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha}, \, S_1 < 0)$$
$$\qquad + 2P(S_2 \geq S_1 + h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha}, \, S_1 \geq 0)$$
$$= 2P(S_2 \geq h2^{1/\alpha}, \, S_1 < 0)$$
$$\qquad + 2P(S_2 \geq |S_1| + h2^{1/\alpha}, \, S_1 \geq 0)$$
$$= 2P(S_2 \geq h2^{1/\alpha})P(S_1 < 0)$$
$$\qquad + 2P(S_2 \geq |S_1| + h2^{1/\alpha})P(S_1 \geq 0)$$
$$= P(S_2 \geq h2^{1/\alpha}) + P(S_2 \geq |S_1| + h2^{1/\alpha})$$

and that

$$P(S_1 \geq h) = P(S_2 + S_1 \geq h2^{1/\alpha})$$
$$= P(S_2 + S_1 \geq h2^{1/\alpha}, \, S_1 \geq h2^{1/\alpha})$$
$$\qquad + P(S_2 + S_1 \geq h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha})$$
$$\qquad - P(S_2 + S_1 \geq h2^{1/\alpha}, \, S_1 \geq h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha})$$
$$= 2P(S_2 + S_1 \geq h2^{1/\alpha}, \, S_2 \geq h2^{1/\alpha}) - P(S_1 \geq h2^{1/\alpha})^2.$$

Putting these observations together, we obtain

$$P(S_2 \geq |S_1| + h2^{1/\alpha})$$
$$= 2P(S_2 \geq S_1 + h2^{1/\alpha},\ S_2 \geq h2^{1/\alpha}) - P(S_2 \geq h2^{1/\alpha})$$
$$= P(S_1 \geq h) + P(S_1 \geq h2^{1/\alpha})^2 - P(S_2 \geq h2^{1/\alpha}).$$

In particular, we get nonnegative correlations if and only if

$$P(S_1 \geq h) + P(S_1 \geq h2^{1/\alpha})^2 - P(S_2 \geq h2^{1/\alpha}) \geq P(S_1 \geq h)^2.$$

Rearranging, we see that this is equivalent to

$$P(S_1 \geq h) - P(S_1 \geq h)^2 \geq P(S_2 \geq h2^{1/\alpha}) - P(S_1 \geq h2^{1/\alpha})^2$$

which will hold for all $h \geq 0$ since $P(S_1 \geq h) - P(S_1 \geq h)^2$ is decreasing in h for all $h \geq 0$. This establishes (iii). $\qquad\square$

9.2 h large and a phase transition in the stability exponent

In this subsection we will look at what happens when X is a symmetric multivariate stable random variable with index $\alpha < 2$ and marginals $S_\alpha(1,0,0)$, and the threshold $h > 0$ is large. The fact that stable distributions have fat tails for $\alpha < 2$ will result in behavior that is radically different from the Gaussian case. We will obtain various results, perhaps the most interesting being a phase transition in α at $\alpha = 1/2$; this is Theorem 1.12.

Our main tool when working in this setting will be the following theorem.

Theorem 9.1. *Let $\alpha \in (0,2)$ and let X be a symmetric n-dimensional α-stable random vector with marginals $S_\alpha(1,0,0)$ and with spectral measure Λ and let $\rho \in \{0,1\}^n$. Then for any $k \geq 1$ we have that the limit of $\nu_\rho(h)/\nu_1^k(h)$ as $h \to \infty$ is equal to*

$$\frac{2^k}{k!} \int_0^\infty \cdots \int_0^\infty \Lambda^k \left(\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{S}_{n-1} \colon \forall j \in [n] \colon \sum_{i=1}^k s_i \mathbf{x}_i(j) > 1 \Leftrightarrow \rho(j) = 1 \right)$$
$$\cdot \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i.$$

$$(65)$$

Remark 9.2. The case $k = 1$ and $\rho = 1^n$ of this theorem is essentially a special case of Theorem 4.4.1 in [10] (see their equation (4.4.2)).

Remark 9.3. If Λ is in addition finitely supported, a change of variables shows that the expression in (65) is equivalent to

$$\frac{1}{k!} \sum_{\mathbf{x}_1,\ldots,\mathbf{x}_k \in \mathrm{supp}(\Lambda)} \int_0^\infty \cdots \int_0^\infty I\left(\forall j \in [n] \colon \sum_{i=1}^k s_i (2\Lambda(\mathbf{x}_i))^{1/\alpha} \mathbf{x}_i(j) > 1 \Leftrightarrow \rho(j) = 1 \right)$$
$$\cdot \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i.$$

Remark 9.4. Theorem 9.1 can easily be extended quite a lot. First, the same statement but with different thresholds $h_i = b_i h$, $b_i > 0$, for each i follows analogously if we replace $\mathbf{1}$ with $(b_i)_{i=1,2,\dots,n}$ but keep $\nu_1(h)$. Furthermore, we could drop the assumption that the b_i is positive by applying the theorem to the random vector $(\mathrm{sgn}(b_i)X_i)$. Finally, it in fact already follows from the proof that we do not need the assumptions on the marginals if we replace the term $\nu_1(h)$ in the limit with $P(Y \geq h)$ for $Y \sim S_\alpha(1,0,0)$.

Proof sketch of Theorem 1.11. We show that the assumptions of Lemma 7.6 hold. First Theorem 9.1 implies that (36) holds. Next, a computation using Theorem 9.1 shows that the last condition in Lemma 7.6 holds if (3) holds. □

We will now apply Theorem 9.1 to a simple example.

Corollary 9.5. *Let $\alpha \in (0,2)$ and let S_1, S_2 and S_3 be i.i.d. with $S_1 \sim S_\alpha(1,0,0)$. Furthermore, let $a \in (0,1)$ and define $X_1 := S_1$ and X_2 and X_3 by*

$$X_i := aX_{i-1} + (1-a^\alpha)^{1/\alpha}S_i, \quad i = 2,3.$$

Then X^h is a color process for all sufficiently large h.

Remark 9.6. The random vector X defined by this corollary is a stable Markov chain. We have already seen a Gaussian analogue of this result.

Proof of Corollary 9.5. Clearly (X_1, X_2, X_3) is a three-dimensional symmetric α-stable random vector whose marginals are $S_\alpha(1,0,0)$. If we let A be given by

$$\begin{pmatrix} 1 & 0 & 0 \\ a & (1-a^\alpha)^{1/\alpha} & 0 \\ a^2 & a(1-a^\alpha)^{1/\alpha} & (1-a^\alpha)^{1/\alpha} \end{pmatrix}$$

then

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}.$$

It follows that for each $\mathbf{x} \in \mathrm{supp}(\Lambda)$, exactly one of $\pm(2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$ is a column in A. Moreover, each column of A corresponds to a pair of points in the support of Λ in this way. To simplify notation, for $\mathbf{x} \in \mathrm{supp}(\Lambda)$ we write $\hat{\mathbf{x}} := (2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$. Using Theorem 9.1 and Remark 9.3 with $n = 3$ and $k = 1$, one easily verifies that this implies that

$$\lim_{h\to\infty} \frac{\nu_{111}(h)}{\nu_1(h)} = \sum_{\mathbf{x}_1 \in \mathrm{supp}(\Lambda)} \int_0^\infty I(s_1\hat{\mathbf{x}}_1 > \mathbf{1}) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1$$

$$= \int_{a^{-2}}^\infty \alpha s_1^{-(1+\alpha)} \, ds_1 = a^{2\alpha}$$

and similarly that

$$
\begin{cases}
\lim\limits_{h\to\infty} \dfrac{\nu_{110}(h)}{\nu_1(h)} = a^\alpha(1-a^\alpha) \\[2mm]
\lim\limits_{h\to\infty} \dfrac{\nu_{100}(h)}{\nu_1(h)} = 1-a^\alpha \\[2mm]
\lim\limits_{h\to\infty} \dfrac{\nu_{011}(h)}{\nu_1(h)} = a^\alpha(1-a^\alpha) \\[2mm]
\lim\limits_{h\to\infty} \dfrac{\nu_{010}(h)}{\nu_1(h)} = (1-a^\alpha)^2 \\[2mm]
\lim\limits_{h\to\infty} \dfrac{\nu_{001}(h)}{\nu_1(h)} = 1-a^\alpha \\[2mm]
\lim\limits_{h\to\infty} \dfrac{\nu_{101}(h)}{\nu_1(h)} = 0.
\end{cases}
\tag{66}
$$

Combining this with (10) we obtain

$$
\begin{cases}
\lim_{h\to\infty} q_{1,2,3}(h) &= (1-a^\alpha)^2 \\
\lim_{h\to\infty} q_{12,3}(h) &= a^\alpha(1-a^\alpha) \\
\lim_{h\to\infty} q_{13,2}(h) &= 0 \\
\lim_{h\to\infty} q_{1,23}(h) &= a^\alpha(1-a^\alpha) \\
\lim_{h\to\infty} q_{123}(h) &= a^{2\alpha}.
\end{cases}
$$

From this it follows that X^h has a color representation for all sufficiently large h if $q_{13,2}(h)$ is non-negative for large h. By (10), $q_{13,2}(h)$ is given by

$$
q_{13,2}(h) = \frac{\nu_0(h)\nu_{101}(h) - \nu_1(h)\nu_{010}(h)}{\nu_1(h)\nu_0(h)(\nu_0(h) - \nu_1(h))}.
$$

Here the denominator is strictly positive for all $h > 0$, and we know from (66) that $\nu_{010}(h) = (1-a^\alpha)^2\nu_1(h) + o(\nu_1(h))$. Hence it is sufficient to show that

$$
\lim_{h\to\infty} \frac{\nu_{101}(h)}{\nu_1(h)^2} > (1-a^\alpha)^2.
$$

To see this, we again apply Theorem 9.1 and Remark 9.3 to obtain

$$
\begin{aligned}
&\lim_{h\to\infty} \frac{\nu_{101}(h)}{\nu_1(h)^2} \\
&= \frac{1}{2} \sum_{\mathbf{x}_1,\mathbf{x}_2\in\mathrm{supp}(\Lambda)} \int_0^\infty \int_0^\infty I\big(s_1\hat{\mathbf{x}}_1(1) + s_2\hat{\mathbf{x}}_2(1) > 1, \\
&\qquad\qquad\qquad\qquad s_1\hat{\mathbf{x}}_1(2) + s_2\hat{\mathbf{x}}_2(2) \le 1, \\
&\qquad\qquad\qquad\qquad s_1\hat{\mathbf{x}}_1(3) + s_2\hat{\mathbf{x}}_2(3) > 1\big)\, \alpha^2 s_1^{-(1+\alpha)} s_2^{-(1+\alpha)}\, ds_2\, ds_1 \\
&= \int_0^\infty \int_0^\infty I\left(a^{-1} > s_1 > 1,\, s_2 > \frac{1-a^2 s_1}{(1-a^\alpha)^{1/\alpha}}\right) \alpha^2 s_1^{-(1+\alpha)} s_2^{-(1+\alpha)}\, ds_2\, ds_1 \\
&= (1-a^\alpha) \int_1^{a^{-1}} \left(1-a^2 s_1\right)^{-\alpha} \alpha s_1^{-(1+\alpha)}\, ds_1 \\
&> (1-a^\alpha) \int_1^{a^{-1}} \alpha s_1^{-(1+\alpha)}\, ds_1 = (1-a^\alpha)^2
\end{aligned}
$$

which is the desired conclusion. □

We can now prove Theorem 1.12 which is a stable version of the example in the proof of (i) of Corollary 8.5.

Proof of Theorem 1.12. We start a little more generally. Let $\alpha \in (0, 2)$ and let $S_0, S_1, \ldots, S_n$ be i.i.d. with $S_0 \sim S_\alpha(1, 0, 0)$. Furthermore, let $a \in (0, 1)$ and for $i = 1, 2, \ldots, n$, define

$$X_i = aS_0 + (1 - a^\alpha)^{1/\alpha} S_i.$$

Note first that for any $n \geq 1$, $(X_1, X_2, \ldots, X_n)$ is clearly an n-dimensional symmetric α-stable random vector whose marginals have distribution $S_\alpha(1, 0, 0)$. Moreover, for any $n \geq 2$, if we let A be the $n \times (n + 1)$ matrix defined by

$$A(i, j) = \begin{cases} a & \text{if } j = 1 \\ (1 - a^\alpha)^{1/\alpha} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\left(X_1, \ldots, X_n\right)^T = A \cdot \left(S_0, S_1, \ldots, S_n\right)^T.$$

It follows that for each $\mathbf{x} \in \text{supp}(\Lambda)$, exactly one of $\pm(2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$ is a column in A. Moreover, each column of A corresponds to a pair of points in the support of Λ in this way. To simplify notation, for $\mathbf{x} \in \text{supp}(\Lambda)$ we write $\hat{\mathbf{x}} := (2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$. Using Theorem 9.1 and Remark 9.3 one easily verifies that it follows that

$$\lim_{h \to \infty} \frac{\nu_{1^n}(h)}{\nu_1(h)} = \sum_{\mathbf{x}_1 \in \text{supp}(\Lambda)} \int_0^\infty I\left(s_1 \hat{\mathbf{x}}_1 > \mathbf{1}\right) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1 \tag{67}$$
$$= \int_0^\infty I\left(as_1 > 1\right) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1 = a^\alpha.$$

Returning to the case $n = 3$, let, for $h > 0$, $(q_{123}(h), q_{12,3}(h), q_{13,2}(h), q_{1,23}(h), q_{1,2,3}(h))$ be the unique formal solution to (5). Then by (10), symmetry and inclusion-exclusion, we have that

$$q_{1,2,3}(h) = \frac{\nu_{100}(h) - \nu_{011}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))} = \frac{\nu_1(h) - 3\nu_{11.}(h) + 2\nu_{111}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))}$$

and hence $\lim_{h \to \infty} q_{1,2,3}(h) = 1 - a^\alpha$. Similarly, one sees that $\lim_{h \to \infty} q_{12,3}(h) = \lim_{h \to \infty} q_{13,2}(h) = \lim_{h \to \infty} q_{1,23}(h) = 0$ and hence $\lim_{h \to \infty} q_{123}(h) = a^\alpha$. Since the solution is permutation invariant, it follows that we have a color representation for all sufficiently large h if and only if $q_{12,3}(h) \geq 0$ for all sufficiently large h. To see when this happens, note first that by symmetry, $\nu_{101} + \nu_{010} = \nu_{011} + \nu_{010} = \nu_{01.}$ and hence, using (10), it follows that

$$q_{13,2}(h) = \frac{\nu_0(h)\nu_{101}(h) - \nu_1(h)\nu_{010}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))} = \frac{\nu_{101}(h) - \nu_1(h)\nu_{01.}(h)}{\nu_0(h)\nu_1(h)(\nu_0(h) - \nu_1(h))}.$$

The denominator is strictly positive for all large h and by (67) we have that

$$\nu_{01.}(h) = \nu_1(h) - \nu_{11.}(h) = \nu_1(h)(1 - a^\alpha) + o(\nu_1(h)).$$

64

The question is now how $\lim_{h\to\infty} \nu_{101}(h)/\nu_1(h)^2$ compares with $1 - a^\alpha$. Applying Theorem 9.1 and Remark 9.3 with $\rho = (1, 0, 1)$ and $k = 2$, we obtain

$$
\lim_{h\to\infty} \frac{\nu_{101}(h)}{\nu_1(h)^2}
$$
$$
= \frac{1}{2} \sum_{\mathbf{x}_1, \mathbf{x}_2 \in \mathrm{supp}(\Lambda)} \int_0^\infty \int_0^\infty I\big(s_1\hat{\mathbf{x}}_1(1) + s_2\hat{\mathbf{x}}_2(1) > 1,
$$
$$
s_1\hat{\mathbf{x}}_1(2) + s_2\hat{\mathbf{x}}_2(2) \leq 1,
$$
$$
s_1\hat{\mathbf{x}}_1(3) + s_2\hat{\mathbf{x}}_2(3) > 1\big) \, \alpha^2 s_1^{-(1+\alpha)} s_2^{-(1+\alpha)} \, ds_2 \, ds_1
$$
$$
= \int_0^\infty \int_0^\infty I\big((1 - a^\alpha)^{1/\alpha} s_1 > 1, \, (1 - a^\alpha)^{1/\alpha} s_2 > 1\big) \, \alpha^2 s_1^{-(1+\alpha)} s_2^{-(1+\alpha)} \, ds_2 \, ds_1
$$
$$
+ \int_0^\infty \int_0^\infty I\big(as_1 > 1, \, as_1 - (1 - a^\alpha)^{1/\alpha} s_2 < 1\big) \, \alpha^2 s_1^{-(1+\alpha)} s_2^{-(1+\alpha)} \, ds_2 \, ds_1
$$
$$
= (1 - a^\alpha)^2 + a^\alpha(1 - a^\alpha) \int_1^\infty \alpha w^{-(1+\alpha)}(w - 1)^{-\alpha} \, dw.
$$

Hence X^h has a color representation for all sufficiently large h if

$$
\int_1^\infty \frac{\alpha}{w^{1+\alpha}(w - 1)^\alpha} \, dw > 1
$$

and has no color representation for any large h if

$$
\int_1^\infty \frac{\alpha}{w^{1+\alpha}(w - 1)^\alpha} \, dw < 1.
$$

To check when we are in these two cases, note first that by making the change of variables $w = x^{-1}$, we seethat the integral on the left hand side is equal to

$$
\int_0^1 \alpha x^{2\alpha - 1}(1 - x)^{-\alpha} \, dx.
$$

This integral is easily verified to be infinite when $\alpha \geq 1$, strictly positive for all $\alpha \in (0, 1)$ and equal to 1 if $\alpha = 1/2$. Furthermore, if $\alpha \in (0, 1)$, recognizing this as the Beta distribution we see that it is equal to

$$
\frac{\alpha\Gamma(2\alpha)\Gamma(1 - \alpha)}{\Gamma(1 + \alpha)} = \frac{\Gamma(2\alpha)\Gamma(1 - \alpha)}{\Gamma(\alpha)} = 2^{2\alpha - 1}\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma(1 - \alpha) \cdot \frac{1}{\sqrt{\pi}}
$$

where the last equality follows by using the Legendre Duplication Formula (see [1], 6.1.18, p. 256). We claim that this expression is strictly increasing in α. If we can show this, the conclusion of the theorem will follow since we get the value 1 at $\alpha = 1/2$. To see this, recall first that $\Gamma'(\alpha) = \Gamma(\alpha)\psi(\alpha)$, where ψ is the so-called digamma function. It follows that the derivative of the expression above is equal to

$$
2^{2\alpha - 1}\Gamma\left(\alpha + \frac{1}{2}\right)\Gamma(1 - \alpha) \cdot \frac{1}{\sqrt{\pi}} \cdot \left(2\log 2 + \psi\left(\alpha + \frac{1}{2}\right) - \psi(1 - \alpha)\right).
$$

Since the first term is equal to our original integral, it is clearly strictly larger than zero. Moreover, an integral representation of ψ given in [1] (see 6.3.21, p.

259) implies that $\psi(x)$ is strictly increasing in x for $x > 0$. It follows that the second term is strictly larger than

$$2\log 2 + \psi\left(1/2\right) - \psi\left(1\right).$$

Using the values of the digamma function at $1/2$ and 1 (see [1], 6.3.2 and 6.3.3, p. 258), this last expression is 0. This finishes the proof. $\qquad\square$

We next give the proof of Theorem 1.13.

Proof of Theorem 1.13. Clearly (X_1, X_2, X_3) is a three-dimensional symmetric α-stable random vector whose marginals are $S_\alpha(1, 0, 0)$.

If we define $c = c(\alpha, a, b) := (1 - 2a^\alpha - 2b^\alpha)^{1/\alpha}$ and let A be given by

$$\begin{pmatrix} a & b & 0 & b & a & 0 & c \\ 0 & a & b & 0 & b & a & c \\ b & 0 & a & a & 0 & b & c \end{pmatrix}$$

then

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = A \cdot (S_1,\, S_2,\, S_3,\, S_4,\, S_5,\, S_6,\, S_7)^T$$

It follows that for each $\mathbf{x} \in \text{supp}(\Lambda)$, exactly one of $\pm(2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$ is a column in A. Moreover, each column of A corresponds to a pair of points in the support of Λ in this way. To simplify notation, for $\mathbf{x} \in \text{supp}(\Lambda)$ we write $\hat{\mathbf{x}} := (2\Lambda(\mathbf{x}))^{1/\alpha}\mathbf{x}$. Using Theorem 9.1 and Remark 9.3, we get that

$$\lim_{h\to\infty} \frac{\nu_{111}(h)}{\nu_1(h)} = \sum_{\mathbf{x}_1 \in \text{supp}(\Lambda)} \int_0^\infty I\left(s_1\hat{\mathbf{x}}_1 > \mathbf{1}\right) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1$$

$$= \int_0^\infty I(cs_1 > 1) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1 = c^\alpha = 1 - 2a^\alpha - 2b^\alpha. \tag{68}$$

Similarly, we obtain

$$\lim_{h\to\infty} \frac{\nu_{110}(h)}{\nu_1(h)} = 2 \int_0^\infty I(s_1 \cdot \min(\{a, b\}) > 1) \cdot \alpha s_1^{-(1+\alpha)} \, ds_1$$

$$= 2\min(\{a, b\})^\alpha. \tag{69}$$

Using (10), it follows that

$$\begin{cases} \lim_{h\to\infty} q_{123}(h) = 1 - 2a^\alpha - 2b^\alpha \\ \lim_{h\to\infty} q_{12,3}(h) = 2\min(\{a, b\})^\alpha \\ \lim_{h\to\infty} q_{13,2}(h) = 2\min(\{a, b\})^\alpha \\ \lim_{h\to\infty} q_{1,23}(h) = 2\min(\{a, b\})^\alpha \end{cases} \tag{70}$$

and as $q_{1,2,3}(h) = 1 - q_{123}(h) - q_{12,3}(h) - q_{13,2}(h) - q_{1,23}(h)$ for $h \in \mathbb{R}$, we also obtain

$$\lim_{h\to\infty} q_{1,2,3}(h) = 1 - (1 - 2a^\alpha - 2b^\alpha) - 6\min(\{a, b\})^\alpha = 2\left(\max(\{a, b\})^\alpha - 2\min(\{a, b\})^\alpha\right).$$

Since $a, b \in (0, 1)$ and $2a^\alpha + 2b^\alpha < 1$ (as $\alpha > c_1$), it follows that all of the limits in (70) lie in $(0, 1)$ for any $\alpha \in (0, 1)$.

Let $g(\alpha) = \max(\{a, b\})^\alpha - 2\min(\{a, b\})^\alpha$ for $\alpha \in (0, \infty)$. If $a = b$, then $c_2 = \infty$ and $g(\alpha) = \max(\{a, b\})^\alpha - 2\min(\{a, b\})^\alpha$ is negative for all α and the claim holds. If $a \neq b$, then it is easy to check that c_2 is the unique zero of $g(\alpha)$ on $(0, \infty)$ and that g is negative (positive) to the left (right) of c_2. This immediately leads to all of the claims. $\square$

To be able to give the proof of Theorem 9.1 we will need the following lemma. The case $k = 2$ was stated in [10] (see equation (1.4.8) on p. 27), but no proof is given in the book. A sketch of the proof in this case was provided in private correspondence with one of the authors.

Lemma 9.7. *Let $\alpha \in (0, 2)$, $k \geq 2$ and let $\varepsilon \in (0, \min(\{\alpha, (k-1)(2-\alpha)\}))$. Furthermore, let $(W_i)_{i \geq k}$ be a sequence of i.i.d. random variables with $0 \leq W_i \leq 1$, $(\varepsilon_i)_{i \geq k}$ be a sequence of i.i.d. random variables with $\varepsilon_i \sim \mathrm{unif}(\{-1, 1\})$ and for $i \geq k$ let Γ_i be the i:th arrival time of a Poisson process with rate one. Assume that these three sequences are independent of each other. Then*

$$\mathbb{E}\left[\left|\sum_{i=k}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{(k-1)\alpha+\varepsilon}\right] < \infty.$$

Remark 9.8. An almost analogous proof works if we replace the assumption that $0 \leq W_i \leq 1$ by $\mathbb{E}\left[|W_i|^{(k-1)\alpha+\varepsilon}\right] < \infty$.

Proof of Lemma 9.7. To simplify notation, write $\beta := (k-1)\alpha + \varepsilon$. We then need to show that

$$\mathbb{E}\left[\left|\sum_{i=k}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{\beta}\right] < \infty.$$

To this end, note first that for any fixed $m \geq k$ we have that

$$\mathbb{E}\left[\left|\sum_{i=k}^{m} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{\beta}\right] \leq \mathbb{E}\left[\left(\sum_{i=k}^{m} \Gamma_i^{-1/\alpha} W_i\right)^{\beta}\right] \leq \mathbb{E}\left[\left(\sum_{i=k}^{m} \Gamma_i^{-1/\alpha}\right)^{\beta}\right]$$

$$\leq \mathbb{E}\left[\left((m-k+1)\Gamma_k^{-1/\alpha}\right)^{\beta}\right] = (m-k+1)^{\beta}\mathbb{E}\left[\Gamma_k^{-\beta/\alpha}\right].$$

Since $k > \beta/\alpha$, we have that $\mathbb{E}\left[\Gamma_k^{-\beta/\alpha}\right] < \infty$, and hence

$$\mathbb{E}\left[\left|\sum_{i=k}^{m} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{\beta}\right] < \infty \tag{71}$$

for any fixed $m \geq k$.

Now recall that for any real-valued random variables X and Y with $\mathbb{E}\left[|X|^{\beta}\right] < \infty$ and $\mathbb{E}\left[|Y|^{\beta}\right] < \infty$ we have that $\mathbb{E}[|X + Y|^{\beta}] < \infty$. Using (71), the conclusion of the lemma will thus follow if we can prove that

$$\mathbb{E}\left[\left|\sum_{i=m}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{\beta}\right] < \infty$$

for some $m \geq k$. To this end, fix $m > \beta/\alpha \cdot k/(k-1)$. Then

$$
\mathbb{E}\left[\left|\sum_{i=m}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right|^{\beta}\right] = \mathbb{E}\left[\left(\left(\sum_{i=m}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right)^{2(k-1)}\right)^{\beta/(2(k-1))}\right]
$$

$$
= \mathbb{E}\left[\mathbb{E}_{(\varepsilon_i)}\left[\left(\left(\sum_{i=m}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right)^{2(k-1)}\right)^{\beta/(2(k-1))} \mid \sigma\left((W_i),(\Gamma_i)\right)\right]\right].
$$

Since $\beta/(2(k-1)) = ((k-1)\alpha + \varepsilon)/(2(k-1)) < 1$ by the assumption on ε, we can apply Jensen's inequality to bound this expression from above by

$$
\mathbb{E}\left[\mathbb{E}_{(\varepsilon_i)}\left[\left(\sum_{i=m}^{\infty} \varepsilon_i \Gamma_i^{-1/\alpha} W_i\right)^{2(k-1)} \mid \sigma\left((W_i),(\Gamma_i)\right)\right]^{\beta/(2(k-1))}\right]
$$

$$
\leq \mathbb{E}\left[\left((2(k-1))! \sum_{m \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}} \prod_{j=1}^{k-1} \Gamma_{i_j}^{-2/\alpha} W_{i_j}^2\right)^{\beta/(2(k-1))}\right].
$$

Now we can again use the fact that $\beta/(2(k-1)) < 1$ and the so-called c_r-inequality to move this exponent into the summands to bound the previous expression from above by

$$
((2(k-1))!)^{\beta/(2(k-1))} \mathbb{E}\left[\sum_{m \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}} \prod_{j=1}^{k-1} \Gamma_{i_j}^{-\beta/(\alpha(k-1))} W_{i_j}^{\beta/(k-1)}\right]
$$

$$
= ((2(k-1))!)^{\beta/(2(k-1))} \sum_{m \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}} \mathbb{E}\left[\prod_{j=1}^{k-1} \Gamma_{i_j}^{-\beta/(\alpha(k-1))} W_{i_j}^{\beta/(k-1)}\right]
$$

$$
\leq ((2(k-1))!)^{\beta/(2(k-1))} \sum_{m \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}} \mathbb{E}\left[\prod_{j=1}^{k-1} \Gamma_{i_j}^{-\beta/(\alpha(k-1))}\right].
$$

In particular, this implies that it now only remains to show that

$$
\sum_{m \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1}} \mathbb{E}\left[\prod_{j=1}^{k-1} \Gamma_{i_j}^{-\beta/(\alpha(k-1))}\right] < \infty. \tag{72}
$$

To do this, first fix $\gamma \in \mathbb{R}_+$. If $i \in \mathbb{Z}_+$, then $\mathbb{E}[\Gamma_i^{-\gamma}] < \infty$ if and only if $i > \gamma$. Moreover, for such i and γ we easily have that $\mathbb{E}\left[\Gamma_i^{-\gamma}\right] = \frac{\Gamma(i-\gamma)}{\Gamma(i)}$. By Stirling's formula, it follows that for a fixed γ we have that $\mathbb{E}\left[\Gamma_i^{-\gamma}\right] \sim i^{-\gamma}$ and hence $\mathbb{E}\left[\Gamma_i^{-\gamma}\right] < C_\gamma i^{-\gamma}$ for some constant C_γ and all $i > \gamma$.

Now assume that $1 \leq i_1 \leq \ldots \leq i_{k-1}$ is a sequence of integers. Then for $j = 2, 3, \ldots, k-1$ we have that $\Gamma_{i_j} \geq \Gamma_{i_1}$, and $\Gamma_{i_j} \geq \Gamma_{i_j} - \Gamma_{i_{j-1}}$. The random variables $\Gamma_{i_j} - \Gamma_{i_{j-1}}$ are independent and equal in distribution to $\Gamma_{i_j-i_{j-1}}$ if $i_j \neq i_{j-1}$ and equal to zero else. Using this, it follows that for any fixed $M > 0$

we have that

$$\mathbb{E}\left[\prod_{j=1}^{k-1}\Gamma_{i_j}^{-\gamma}\right] \leq \mathbb{E}\left[\Gamma_{i_1}^{-\gamma-\gamma\sum_{j\geq2}^{k-1}\mathbf{1}(i_j-i_{j-1}<M)}\right]\prod_{j\geq2:\,i_j-i_{j-1}\geq M}^{k-1}\mathbb{E}\left[\Gamma_{i_j-i_{j-1}}^{-\gamma}\right]$$

$$\leq \mathbb{E}\left[\Gamma_{i_1}^{-\gamma}\cdot I(\Gamma_{i_1}\geq1)+\Gamma_{i_1}^{-(k-1)\gamma}\cdot I(\Gamma_{i_1}<1)\right]\prod_{j\geq2:\,i_j-i_{j-1}\geq M}^{k-1}\mathbb{E}\left[\Gamma_{i_j-i_{j-1}}^{-\gamma}\right]$$

$$\leq \mathbb{E}\left[\Gamma_{i_1}^{-\gamma}+\Gamma_{i_1}^{-(k-1)\gamma}\right]\prod_{j\geq2:\,i_j-i_{j-1}\geq M}^{k-1}\mathbb{E}\left[\Gamma_{i_j-i_{j-1}}^{-\gamma}\right].$$

If $i_1 > (k-1)\gamma$ and $M > \gamma$, then, using the above, this is bounded from above by

$$C_\gamma^{k-1}\cdot(C_\gamma+C_{(k-1)\gamma})\cdot i_1^{-\gamma}\prod_{j\geq2:\,i_j-i_{j-1}\geq M}^{k}(i_j-i_{j-1})^{-\gamma}.$$

In particular, if we let $\gamma = \beta/(\alpha(k-1)) = (\alpha(k-1)+\varepsilon)/(\alpha(k-1)) > 1$ and $M = m$, then for $i_1 \geq m$ we have that

$$i_1 \geq m > \beta/\alpha\cdot k/(k-1) = k\gamma > (k-1)\gamma$$

and therefore, since $k \geq 2$,
$$M = m > \gamma.$$

Hence it follows that (72) is bounded from above by

$$C_{\beta/(\alpha(k-1))}^{k-1}\big(C_{\beta/(\alpha(k-1))}+C_{\beta/\alpha}\big)$$

$$\cdot\sum_{m\leq i_1\leq i_2\leq\ldots\leq i_{k-1}}i_1^{-\beta/(\alpha(k-1))}\prod_{\substack{j\in\{2,3,\ldots,k-1\}:\\i_j-i_{j-1}\geq m}}(i_j-i_{j-1})^{-\beta/(\alpha(k-1))}$$

This implies in particular that it only remains to show that

$$\sum_{m\leq i_1\leq i_2\leq\ldots\leq i_{k-1}}i_1^{-\beta/(\alpha(k-1))}\prod_{\substack{j\in\{2,3,\ldots,k-1\}:\\i_j-i_{j-1}\geq m}}(i_j-i_{j-1})^{-\beta/(\alpha(k-1))} < \infty.$$

To see this, we first change the order of summation as follows. First, we will sum over all possible choices of i_1. Then we sum over the number G of terms in the product, which will range between 0 and $k-2$. Finally, we sum also over the possible choices of $\ell_j := i_j - i_{j-1}$ in the product, which will range from m to infinity. To sum over all possible sequences $m \leq i_1 \leq \ldots \leq i_{k-1}$, we find an upper bound on the number of ways to choose the differences $i_j - i_{j-1}$ which are smaller than m and also, on the number of ways to choose which of the differences are larger than or equal to m. The former of these quantities is clearly bounded from above by m^{k-2}, and the latter is equal to $\binom{k-2}{G} < 2^{k-2}$.

Putting these observations together, we get

$$\sum_{m \le i_1 \le i_2 \le \dots \le i_{k-1}} i_1^{-\beta/(\alpha(k-1))} \prod_{j \in \{2,3,\dots,k-1\}: \, i_j - i_{j-1} \ge m} (i_j - i_{j-1})^{-\beta/(\alpha(k-1))}$$

$$\le (2m)^{k-2} \sum_{i_1=m}^{\infty} i_1^{-\beta/(\alpha(k-1))} \sum_{G=0}^{k-2} \sum_{\ell_1,\dots,\ell_G=m}^{\infty} \prod_{j=1}^{G} \ell_j^{-\beta/(\alpha(k-1))}$$

$$= (2m)^{k-2} \sum_{i_1=m}^{\infty} i_1^{-\beta/(\alpha(k-1))} \sum_{G=0}^{k-2} \left(\sum_{\ell=m}^{\infty} \ell^{-\beta/(\alpha(k-1))} \right)^G$$

$$= (2m)^{k-2} \sum_{G=1}^{k-1} \left(\sum_{\ell=m}^{\infty} \ell^{-\beta/(\alpha(k-1))} \right)^G.$$

Since $\beta/(\alpha(k-1)) = (\alpha(k-1) + \varepsilon)/(\alpha(k-1)) > 1$, the desired conclusion now follows.

$\square$

We now state the following lemma which will be used in the proof of Theorem 9.1. For a proof of this lemma we refer the reader to [10].

Lemma 9.9 (Theorem 3.10.1 in [10])**.** *Let Λ be a symmetric spectral measure on $\mathbb{S}^{n-1}$. Furthermore, let C_α be defined by $P(Y \ge h) \sim C_\alpha h^{-\alpha}/2$ for $Y \sim S_\alpha(1, 0, 0)$, let $(\Gamma_i)_{i \ge 1}$ be the arrival times of a rate one Poisson process and let $(\mathbf{W}_i)_{i \ge 1}$ be i.i.d., each with distribution $\bar{\Lambda} := \Lambda/\Lambda(\mathbb{S}^{n-1})$ (the normalized spectral measure), independent of the Poisson process. Then*

$$C_\alpha^{1/\alpha} \Lambda(\mathbb{S}_{n-1})^{1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{W}_i$$

converges almost surely to a random vector with distribution $S_\alpha(\Lambda)$.

We now give a proof of Theorem 9.1 using Lemmas 9.7 and 9.9.

Proof of Theorem 9.1. Let C_α, (Γ_i) and $(\mathbf{W}_i)$ be as in Lemma 9.9 and define

$$X = (X_1, X_2, \dots, X_n) := C_\alpha^{1/\alpha} \Lambda(\mathbb{S}_{n-1})^{1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{W}_i.$$

Then Lemma 9.9 implies that X has distribution $S_\alpha(\Lambda)$. By Markov's inequality, for any $j = 1, 2, \dots, n$ and all $h > 0$ and $\varepsilon > 0$ we have that

$$P \left((C_\alpha \Lambda(\mathbb{S}_{n-1}))^{1/\alpha} \left| \sum_{i=k+1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{W}_i(j) \right| > h \right) \le \frac{\mathbb{E} \left[(C_\alpha \Lambda(\mathbb{S}_{n-1}))^{1/\alpha} \left| \sum_{i=k+1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{W}_i(j) \right|^{k\alpha + \varepsilon} \right]}{h^{k\alpha + \varepsilon}}.$$

By picking $\varepsilon > 0$ sufficiently small and applying Lemma 9.7 (noting that by symmetry, $W_i(j)$ has the same distribution as $\epsilon_i |W_i(j)|$ with the two factors independent), it follows that

$$P \left((C_\alpha \Lambda(\mathbb{S}_{n-1}))^{1/\alpha} \left| \sum_{i=k+1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{W}_i(j) \right| > h \right) \le o(h^{-k\alpha})$$

70

and hence

$$P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty > h\right) \le o(h^{-k\alpha}).$$

This implies in particular that for any $\varepsilon' > 0$

$$P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty > \varepsilon'h\right) \le o(h^{-k\alpha})$$

and hence, if we let $\mathbf{h} := h \cdot \mathbf{1}$, we have that

$$\nu_{1^n}(h) = P\left((X_1, X_2, \dots, X_n) \ge \mathbf{h}\right)$$

$$= P\left((X_1, X_2, \dots, X_n) \ge \mathbf{h}, (C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty < \varepsilon'h\right) + o(h^{-k\alpha})$$

$$\ge P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\left(\Gamma_1^{-1/\alpha}\mathbf{W}_1 + \dots + \Gamma_k^{-1/\alpha}\mathbf{W}_k\right) \ge \mathbf{h}(1 + \varepsilon') \text{ and}\right.$$

$$\left.(C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty < \varepsilon'h\right) + o(h^{-k\alpha})$$

$$= P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\left(\Gamma_1^{-1/\alpha}\mathbf{W}_1 + \dots + \Gamma_k^{-1/\alpha}\mathbf{W}_k\right) \ge \mathbf{h}(1 + \varepsilon')\right) + o(h^{-k\alpha}).$$
$$(73)$$

Similarly, we have that

$$\nu_{1^n}(h) = P\left((X_1, X_2, \dots, X_n) \ge \mathbf{h}\right)$$

$$= P\left((X_1, X_2, \dots, X_n) \ge \mathbf{h}, (C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty < \varepsilon'h\right) + o(h^{-k\alpha})$$

$$\le P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\left(\Gamma_1^{-1/\alpha}\mathbf{W}_1 + \dots + \Gamma_k^{-1/\alpha}\mathbf{W}_k\right) \ge \mathbf{h}(1 - \varepsilon'),\right.$$

$$\left.(C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\Big\|\sum_{i=k+1}^\infty \Gamma_i^{-1/\alpha}\mathbf{W}_i\Big\|_\infty < \varepsilon'h\right) + o(h^{-k\alpha})$$

$$\le P\left((C_\alpha\Lambda(\mathbb{S}_{n-1}))^{1/\alpha}\left(\Gamma_1^{-1/\alpha}\mathbf{W}_1 + \dots + \Gamma_k^{-1/\alpha}\mathbf{W}_k\right) \ge \mathbf{h}(1 - \varepsilon')\right) + o(h^{-k\alpha}).$$
$$(74)$$

To be able to simplify these expressions, first recall that if $(\Gamma_1, \Gamma_2, \dots, \Gamma_{k+1})$ are the first $k+1$ arrivals of a mean one Poisson process and $U_1, U_2, \dots, U_k \sim$ unif$(0, 1)$ are independent, then conditioned on Γ_{k+1},

$$\{\Gamma_1/\Gamma_{k+1}, \dots, \Gamma_k/\Gamma_{k+1}\} \overset{d}{=} \{U_1, \dots, U_k\}.$$

Using this and now letting $U_1, U_2, \dots, U_k$ be i.i.d. uniforms defined on the same probability space as everything else but independent of them, and also letting

$h' = h(1 \pm \varepsilon)$ and $\mathbf{h}' = h' \cdot \mathbf{1}$, we see that

$$P\left((C_\alpha \Lambda(\mathbb{S}_{n-1}))^{1/\alpha} \sum_{i=1}^k \Gamma_i^{-1/\alpha} \mathbf{W}_i \geq \mathbf{h}' \right)$$

$$= P\left((C_\alpha \Lambda(\mathbb{S}_{n-1}))^{1/\alpha} \Gamma_{k+1}^{-1/\alpha} \sum_{i=1}^k U_i^{-1/\alpha} \mathbf{W}_i \geq \mathbf{h}' \right)$$

$$= \int_0^\infty \frac{x^k e^{-x}}{\Gamma(k+1)} \int_0^1 \cdots \int_0^1 \bar{\Lambda}^k \left(\left\{ (w_i) \colon x^{-1/\alpha} \sum_{i=1}^k \left(\frac{C_\alpha \Lambda(\mathbf{S}_{n-1})}{u_i} \right)^{1/\alpha} w_i \geq \mathbf{h}' \right\} \right) \prod_{i=1}^k du_i \, dx.$$

If, for each fixed x, we make the change of variables

$$s_i = x^{-1/\alpha} \left(\frac{C_\alpha \Lambda(\mathbb{S}_{n-1})}{h'^\alpha u_i} \right)^{1/\alpha},$$

then this simplifies to

$$\int_0^\infty \frac{e^{-x}}{\Gamma(k+1)} \left(C_\alpha \Lambda(\mathbb{S}_{n-1}) h'^{-\alpha} \right)^k \int \cdots \int_{\left(\frac{C_\alpha \Lambda(\mathbb{S}_{n-1})}{h'^\alpha x} \right)^{1/\alpha} < s_i < \infty} \bar{\Lambda}^k \left(\left\{ (w_i) \colon \sum_{i=1}^k s_i w_i \geq \mathbf{1} \right\} \right) \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i \, dx$$

$$= \int_0^\infty \frac{e^{-x}}{\Gamma(k+1)} \left(C_\alpha h'^{-\alpha} \right)^k \int \cdots \int_{\left(\frac{C_\alpha \Lambda(\mathbb{S}_{n-1})}{h'^\alpha x} \right)^{1/\alpha} < s_i < \infty} \Lambda^k \left(\left\{ (w_i) \colon \sum_{i=1}^k s_i w_i \geq \mathbf{1} \right\} \right) \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i \, dx.$$

Note here that $\Gamma(k+1) = k!$ and that as $h' \to \infty$ we have that $C_\alpha h'^{-\alpha}/2 \sim \nu_1(h')$. Furthermore,, note that $\lim_{\varepsilon' \to 0} \lim_{h \to \infty} \nu_1(h)/\nu_1(h(1 \pm \varepsilon')) = 1$. Together with (73) and (74) this implies that

$$\lim_{h \to \infty} \frac{\nu_{1^n}(h)}{\nu_1(h)^k}$$

$$= \frac{2^k}{k!} \int_0^\infty e^{-x} \int_0^\infty \cdots \int_0^\infty \Lambda^k \left(\left\{ (w_i) \colon \sum_{i=1}^k s_i w_i \geq \mathbf{1} \right\} \right) \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i \, dx$$

$$= \frac{2^k}{k!} \int_0^\infty \cdots \int_0^\infty \Lambda^k \left(\left\{ (w_i) \colon \sum_{i=1}^k s_i w_i \geq \mathbf{1} \right\} \right) \prod_{i=1}^k \alpha s_i^{-(1+\alpha)} \, ds_i.$$

The conclusion for general binary strings ρ follows completely analogously.

$\square$

10 Ubiquitousness of zero-threshold stables and another phase transition in the stability exponent

In this section we prove Theorems 1.15 and 1.16.

Proof of Theorem 1.15. We create a probability measure Λ on $\mathbb{S}^{n-1}$ with at most 2^n point masses as follows. By identifying 0 and -1, we have a natural

bijection between $\{0,1\}^n$ and the subset $J = \{-1/\sqrt{n}, 1/\sqrt{n}\}^n$ of $\mathbb{S}^{n-1}$. Using this identification, place a point mass at $x \in J$ of weight $\mu(x)$. Let X_α denote the n-dimensional stable vector with index α and spectral measure Λ. It suffices to show that X_α^0 converges in distribution to μ as $\alpha \to 0$.

Letting $\alpha < 1$, it follows from p. 69 in [10] that

$$X_\alpha \overset{\mathcal{D}}{=} \sum_{y \in J} \Lambda(y)^{1/\alpha} \mathbf{y} S_y^{(\alpha)}$$

where the $S_y^{(\alpha)}$'s are i.i.d. each with distribution $S_\alpha(1,1,0)$. Since $\alpha < 1$, each $S_y^{(\alpha)}$ has support only on $\mathbb{R}_+$.

For $y \in J$, let $\mathcal{E}_{y,\alpha}$ be the event

$$\{\Lambda(y)^{1/\alpha} S_y^{(\alpha)} > \sum_{y' \in J, y' \neq y} \Lambda(y')^{1/\alpha} S_{y'}^{(\alpha)}\}.$$

Since the $S_y^{(\alpha)}$'s are nonnegative, the events $\{\mathcal{E}_{y,\alpha}\}_{y \in J}$ are disjoint for any $\alpha > 0$. If we define

$$Y_{y,\alpha} \sim S_\alpha \left(1, \frac{2\Lambda(y)}{\Lambda(y) + \sum_{y' \in J, y' \neq y} \Lambda(y')} - 1, 0 \right)$$

then it follows from Property 1.2.1 on p. 10 of [10] that

$$P\left(\mathcal{E}_{y,\alpha}\right) = P(Y_{y,\alpha} > 0).$$

Next, Equation 2.2.30 on p. 79 in [13] (see also p. 40 in [10]) implies that if $S^{(\alpha,\beta)} \sim S_\alpha(1,\beta,0)$, then

$$\lim_{\alpha \to 0} P\left(S^{(\alpha,\beta)} > 0\right) = \frac{1+\beta}{2}.$$

(Here one needs to note that the representation for β in Equation 2.2.30 is not the same as ours but that the ratio of the two approaches 1 as $\alpha \to 0$.) It follows that

$$\lim_{\alpha \to 0} P(Y_{y,\alpha} > 0) = \Lambda(y)$$

and hence in particular that

$$\lim_{\alpha \to 0} \sum_{y \in J} P(\mathcal{E}_{y,\alpha}) = \lim_{\alpha \to 0} \sum_{y \in J} P(Y_{y,\alpha} > 0) = 1.$$

It is clear that for each $y \in J$

$$\mathcal{E}_{y,\alpha} \subseteq \{X_\alpha^0 = y\}$$

and the result follows. The last two statements are clear. $\qquad\square$

Remark 10.1. The result here can be extended quite alot. For example, if Λ is a spectral probability measure on $\mathbb{S}_{n-1}$, $X_\alpha \sim S_\alpha(\Lambda)$ and $E \subseteq \mathbb{S}^{n-1}$ is such that $\Lambda(\partial E) = 0$, then

$$\lim_{\alpha \to 0} P(X_\alpha / \|X_\alpha\| \in E) = \Lambda(E).$$

We finally give the proof of Theorem 1.16.

73

Proof of Theorem 1.16. (i). By (4), it follows immediately that there is a critical value $\alpha_c \in [0,2]$ with the desired property and so we need to show that $\alpha_c > 0$. We first recall the example from [12] (Proposition 2.13) which is uniform distribution on the $2n$ configurations where the number of 1's is either one or $n-1$; the point of this example is that it is not a color process but it has nonnegative pairwise correlations which in fact are strictly positive if and only if $n \geq 5$. Since we want strictly positive pairwise correlations, when $n = 4$, we modify this slightly by giving weight 0.01 to each of the two constant configurations and taking away the same total amount of mass from the original eight configurations; this will then have strictly positive pairwise correlations. Next, using Theorem 1.15, we can find a fully transitive symmetric stable vector whose zero-threshold is sufficiently close to the latter vector so that it has positive pairwise correlations and the probability of having all 1's is at most 0.02. Such a vector cannot be a color process since the probability of having all 1's is always at least $1/8$ for a color process.

(ii). We begin by attempting to show that $\alpha_c < 2$. If this were not the case, there would exist a sequence α_n approaching two and a sequence of vectors $\{X_n\}$, where X_n is a fully transitive symmetric α_n-stable vector with scale 1 of length n whose zero-threshold has pairwise nonnegative correlations but does not have a color representation. By compactness, choose a a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ converging to a random vector X_∞. X_∞ is necessarily a fully transitive Gaussian vector with mean zero and standard deviation $1/\sqrt{2}$ and some nonnegative pairwise correlation $a \in [0,1]$.

If $a \in (0,1)$, in view of the proof of Theorem 3.2, X_∞^0 has a color representation $q = \{q_\sigma\}$ which gives *strictly* positive probability to each partition. It follows that any random vector which is sufficiently close to X_∞^0 and which has a formal solution necessarily has a color representation. To see this latter implication, one can fix a set of free variables to be as they are in q in which case the linear operator becomes a bijection onto its image and then use continuity. It follows that $\{X_{n_i}^0\}$, which converges to X_∞^0, has a color representation for large i, a contradiction.

If $a = 1$, then, for large i, $\{X_{n_i}^0\}$ gives the constant configurations probability close to 1 which implies, by Theorem 2.2, that $\{X_{n_i}^0\}$ is a color process, again a contradiction.

If $a = 0$ (i.e. the limiting Gaussian is an i.i.d. process), we do not know how to obtain a contradition since then the color representation for the limit is concentrated on the all singletons configuration. However, now it is clear that (ii) holds since when we restrict to stables a fixed distance from the i.i.d. Gaussians, any subsequential limit cannot have $a = 0$. $\qquad\square$

11   Open questions

1. Is every discrete Gaussian free field a color process for all values of h? In particular, is this true in the fully symmetric case or in the Markov chain case?

2. Let X be the three-dimensional discrete Gaussian free field indexed by $\mathbb{Z}^3$. (This is obtained by taking the limit, as $n \to \infty$ of the discrete Gaussian free field on $[-n,n]^3$ with zero boundary conditions.)

(a) By an easy limiting argument, the results here imply that X^0 is a color process. Does some or do all color representations contain an infinite class? (If they exist, each such class should have zero density.)

(b) Is X^h a color process for sufficiently large h?

3. Consider a fully supported Gaussian vector $X := (X_1, \ldots, X_n)$ with mean zero and variance one or a symmetric stable vector $X := (X_1, \ldots, X_n)$ all of whose marginals are equal. If $0 \leq h_1 < h_2$, is it the case that X^{h_1} being a color process implies that X^{h_2} is a color process?

4. Are there natural conditions which guarantee the existence of a color representation for small h for Gaussian vectors?

5. Is there a Gaussian vector X such that X has a color representation for large h but not for $h = 0$?

6. In Lemma 7.7, can the assumption of being inverse Stieltjes be removed?

7. Can Theorem 1.8 be extended to general n?

8. In Theorem 1.16, show that for all n, $\alpha_c(n) < 2$. Determine $\lim_{n \to \infty} \alpha_c(n)$. Is the limit 2?

9. How do the sets of zero threshold stables vary as we vary α? In particular, is the containment as α decreases strict? What happens when $h > 0$?

10. Find sufficient and necessary conditions on Λ so that X_1^h and X_2^h are nonnegatively correlated for all h.

Acknowledgements

References

[1] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, (1970), National Bureau of Standards Applied Mathematics Series, 55.

[2] Ming Dai and Arunava Mukherjea, *Identification of the parameters of a multivariate normal vector by the distribution of the maximum*, Journal of Theoretical Probability, Vol. 14, No. 1 pp. 267–298 (2001).

[3] Enkelejd Hashorva, *Asymptotics and bounds for multivariate Gaussian tails*, Journal of Theoretical Probability, Vol. 18, No. 1 pp. 79–97 (2005).

[4] Charles R. Johnson and Ronald L. Smith, *Path product matrices*, Linear and Multilinear Algebra, Vol. 46, Issue 3, pp. 117-191 (1999).

[5] Titus Lupu, *From loop clusters and random interlacement to the free field*, Annals of Probability, Volume 44, Number 3, pp. 2117–2146 (2016).

[6] Titus Lupu and Wendelin Werner, *A note on Ising random currents, Ising-FK, loop-soups and the Gaussian free field*, Electronic Communications in Probability, Vol. 21, paper no. 13, (2016).

[7] Thomas L. Markham, *Nonnegative matrices whose inverses are M-matrices*, Proceedings of the American Mathematical Society, Vol. 36, No. 2, pp. 326-330 (1972).

[8] Ilya Molchanov, *Convex and star-shaped sets associated with multivariate stable distributions, I: Moments and densities*, Journal of Multivariate Analysis, Volume 100, Issue 10, pp. 2195-2213 (2009).

[9] Robert J. Plemmons, *M-matrix characterizations. 1 – nonsingular M-matrices*, Linear Algebra and Its Applications, Vol. 18, Issue 2, pp. 175-188 (1977).

[10] Gennady Samorodnitsky and Murad S. Taqqu, *Stable non-Gaussian random processes, Stochastic models with infinite variance*, (1994), Chapman & Hall.

[11] William Sheppard, *On the application of the theory of error to cases of normal distribution and normal correlation*, Philosophical Transactions of the Royal Society of London, Series A, Vol. 192, pp. 101–167, 531, (1899).

[12] Jeffrey E. Steif and Johan Tykesson, *Generalized divide and color models*, arXiv:1702.04296v1 [math.PR] , (2017).

[13] Vladimir M. Zolotarev, *One-dimensional Stable Distributions*, Volume 65 of "Translations of mathematical monographs", American Mathematical Society. Translation from the 1983 Russian edition.