Poisson Representable Processes

Malin P. Forsström[*], Nina Gantert[†], Jeffrey E. Steif[‡]

January 24, 2024

In celebration of Geoffrey Grimmett's 70th birthday

Abstract

Motivated by Alain-Sol Sznitman's interlacement process, we consider the set of $\{0,1\}$-valued processes which can be constructed in an analogous way, namely as a union of sets coming from a Poisson process on a collection of sets. Our main focus is to determine which processes are representable in this way. Some of our results are as follows. (1) All positively associated Markov chains and a large class of renewal processes are so representable. (2) Whether an average of two product measures, with close densities, on n variables, is representable is related to the zeroes of the polylogarithm functions. (3) Using (2), we show that a number of tree indexed Markov chains as well as the Ising model on $\mathbb{Z}^d$, $d \geq 2$, for certain parameters are not so representable. (4) The collection of permutation invariant processes which are representable corresponds exactly to the set of infinitely divisible random variables on $[0,\infty]$ via a certain transformation. (5) The supercritical (low temperature) Curie-Weiss model is not representable for large n.

Contents

[*]Email: palo@chalmers.se
Address: Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden

[†]Email: nina.gantert@tum.de
Address: SoCIT, Department of Mathematics, 85748 Garching b. München, Boltzmannstr. 3, Germany

[‡]Email: steif@chalmers.se
Address: Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden

1 Introduction

Let S be a finite or countably infinite set and let ν be a σ-finite measure on $\mathcal{P}(S)\backslash\{\emptyset\}$ where $\mathcal{P}(S)$ is the power set of S. This generates a $\{0,1\}$-valued process $X^\nu = \{X_i^\nu\}_{i\in S}$ defined as follows.

We first consider the Poisson process Y^ν on $\mathcal{P}(S)\backslash\{\emptyset\}$ with intensity measure ν (see [9] for the definition). Note that $\mathcal{P}(S)\backslash\{\emptyset\}$ can be viewed as an open subset of $\{0,1\}^S$ and hence has a nice topology and Borel structure. $Y^\nu(\omega)$ is then a collection $\{B_j\}_{j\in I}$ of nonempty subsets (perhaps with repetitions) of S. Note that $|I| < \infty$ a.s. if $\nu(\mathcal{P}(S)\backslash\{\emptyset\}) < \infty$ and that $|I| = \infty$ a.s. if $\nu(\mathcal{P}(S)\backslash\{\emptyset\}) = \infty$.

Finally, we define $\{X_i^\nu\}_{i\in S}$ by

$$X_i^\nu = \begin{cases} 1 & \text{if } i \in \cup_{j\in I} B_j \\ 0 & \text{otherwise.} \end{cases}$$

To see that X_i^ν is a random variable, one observes that $X_i^\nu = 1$ if and only if

$$Y^\nu \cap \mathcal{S}_i \neq \emptyset,$$

where

$$\mathcal{S}_i := \{T \in \mathcal{P}(S) \colon i \in T\}$$

and the above is an event by definition of a Poisson process since $\mathcal{S}_i$ is a open set in $\mathcal{P}(S)\backslash\{\emptyset\}$. Loosely speaking, X^ν is obtained by taking the union of the sets arising in the Poisson process, and identifying this with the corresponding $\{0,1\}$-sequence.

Definition 1.1. *We let $\mathcal{R}$ denote the set of all processes $(X_i)_{i\in S}$ which are equal (in distribution) to X^ν for some ν.*

Understanding which X are in $\mathcal{R}$ seems to be an interesting question and will be the main focus of this paper. We will for the most part deal with three different situations:

1. S is a finite set,

2. S is $\mathbb{Z}^d$ for some $d \geq 1$ and ν (and hence also X^ν) is translation invariant under the natural $\mathbb{Z}^d$-action on $\mathcal{P}(S)\backslash\{\emptyset\}$ and

3. S is infinite and ν (and hence also X^ν) is invariant under all finite permutations of $\mathcal{P}(S)\backslash\{\emptyset\}$.

We give an alternative but equivalent description of this model in the case when $|S| < \infty$ which has a more combinatorial flavor. Let S be a finite set and for each nonempty subset T of S, let $p(T) \in [0,1]$. Now, for each $\emptyset \neq T \subseteq S$, we independently "choose" T with probability $p(T) \in [0,1]$ and we let X be the union of the "chosen" T's which we identify with a $\{0,1\}$-valued process $\{X_i\}_{i \in S}$. The correspondence between this formulation and the earlier one is that $p(T)$ is simply the probability that a Poisson random variable with parameter $\nu(\{T\})$ is nonzero.

We introduce the following natural notation. For $A \subseteq S$, we let

$$\mathcal{S}_A^\cup := \bigcup_{i \in A} \mathcal{S}_i \qquad (= \{S : A \cap S \neq \emptyset\})$$

and

$$\mathcal{S}_A^\cap := \bigcap_{i \in A} \mathcal{S}_i \qquad (= \{S : A \subseteq S\}).$$

We observe that for any $A \subseteq S$, we have that

$$P\big(X^\nu(A) \equiv 0\big) = e^{-\nu(\mathcal{S}_A^\cup)}.$$

We now discuss a number of examples. In these examples and throughout the rest of the paper, for $p \in [0,1]$, we will let Π_p denote a product measure with 1's having density p.

Example 1.2. Let $a > 0$, and let

$$\nu = \sum_{i \in S} a\delta_{\{i\}}.$$

Then $X^\nu \sim \Pi_{1-e^{-a}}$.

Example 1.3. Let $a > 0$, and let $\nu = a\delta_S$. Then X^ν has distribution

$$(1 - e^{-a})\delta_{\mathbf{1}} + e^{-a}\delta_{\mathbf{0}},$$

where $\mathbf{1}$ ($\mathbf{0}$) is the configuration consisting of all 1's (0's). If $S = \mathbb{Z}^d$, then this yields a (trivial) non-ergodic process.

Example 1.4. Let $S = \mathbb{Z}$, and let

$$\nu = \sum_{i \in \mathbb{Z}} \delta_{\{i,i+1\}}.$$

Then X^ν is the image of an i.i.d. sequence under a block map. More precisely, if $(Y_i)_{i \in \mathbb{Z}} \sim \Pi_{1-e^{-1}}$, then $X^\nu \stackrel{d}{=} \big(\max(Y_i, Y_{i+1})\big)_{i \in \mathbb{Z}}$.

Example 1.5. Let $S = \mathbb{Z}$, and let

$$\nu = \sum_{i \in \mathbb{Z},\, n \geq 0} a_n \delta_{\{i, i+n\}}$$

for some $(a_n)_{n \geq 0}$. It is easy to see (similar to Example 1.3) that X^ν is a Bernoulli shift, i.e. a factor of an i.i.d. sequence. However, we don't know if it is a finitary factor of an i.i.d. sequence. The Borel-Cantelli Lemma immediately gives that $X^\nu \equiv 1$ a.s. if and only if $\sum_{n=0}^{\infty} a_n = \infty$.

Example 1.6. The well studied random interlacement process in $\mathbb{Z}^d$ for $d \geq 3$, introduced by Alain-Sol Sznitman, falls into this context; see [6] for the definition and some of what is known. For those familiar with this, we actually need to massage it slightly so that it falls into our context. In the random interlacement process, we have a Poisson process over random walk realizations modulo time shifts which are transient in both forward and backward time. If we now map such a trajectory (modulo time shifts) to its range and then take their union, we then obtain a process in $\mathcal{R}$ since the push-forward of a Poisson process is a Poisson process. The ν in [6, Theorem 5.2] would provide us with our ν (after pushing forward). The random interlacement process has an intensity parameter which just corresponds to scaling the measure ν. It was in fact this model which provided the motivation for our paper.

Example 1.7. The union of the discrete loops that arise in a random walk loop soup corresponds to a process in $\mathcal{R}$. The random walk loop soup is a well studied object in relation to the Brownian loop soup and the discrete Gaussian free field. It was introduced in [10] and is defined in the following way: the rooted loop measure μ^{RW} assigns to each (nearest neighbor) random walk loop in $\mathbb{Z}^2$ of length $2n$ the measure $(1/2n)4^{-2n}$ and the measure ν is given by $\lambda\mu^{\mathrm{RW}}$ where $\lambda \in (0, \infty)$ is an intensity parameter.

While we are limiting ourselves to countable sets, in continuous space, similar constructions (e.g., Boolean models, Poisson cylinder models) play a crucial role in stochastic geometry. We also want to mention that the idea of enriching a graph by attaching a Poissonian number of independent finite subgraphs comes up naturally in the analysis of random graphs, see for instance [2] and [3].

The paper is organized as follows. Our main focus is to determine, for a given $\{0,1\}$-valued process, if it belongs to $\mathcal{R}$. On the way to answering this question, we give some properties of processes in $\mathcal{R}$. It is easy to show that processes in $\mathcal{R}$ are positively associated. It turns out that they also have the so-called downward FKG property (but not necessarily the FKG property), see Theorem 2.4. Also, if X is a collection of i.i.d. $\{0,1\}$-valued random variables, then $X \in \mathcal{R}$. Taking $S = \mathbb{Z}$, it is natural to ask if Markov chains or renewal processes are in $\mathcal{R}$. We show in Section 3 that indeed all positively associated Markov chains are in $\mathcal{R}$, and describe the corresponding measure ν, see Theorem 3.1. On the way to this result, we give a necessary and sufficient condition for a renewal process to be in $\mathcal{R}$, see Theorem 3.5. In Section 4, we consider processes X on $\{0,1\}^{\mathbb{N}}$ which are invariant under finite permutations, and we ask if they are in $\mathcal{R}$. In this section, we first prove a version of de Finetti's Theorem for possibly infinite measures (which turned out to be known) that is of independent interest, see Theorem 4.4. We then show that the

$X \in \mathcal{R}$ are in one-to-one correspondence with infinitely divisible distributions on $[0, \infty)$, see Theorem 4.5. Interesting such examples include the $\{0,1\}$-sequences coming from classical urn models, see Examples 4.9 and 4.10. In Section 5.1, we investigate the finite permutation invariant case. Taking a sequence $X_n \in \{0,1\}^n$, $n \geq 1$, such that each X_n is permutation invariant, we show that in order to have $X_n \in \mathcal{R}$ for all $n \geq 1$, it is necessary that the arithmetic means of the X_n's concentrate, see Theorem 5.1. As an immediate consequence, we obtain that for temperatures less than the critical one, the Curie-Weiss model is not in $\mathcal{R}$ for large n, see Theorem 5.2. We then study finite averages of m product measures, and we give sufficent conditions for $X \in \mathcal{R}$ for $m \geq 2$ and for $X \notin \mathcal{R}$ for $m = 2$, see Theorem 5.6 for the latter case. The proof is quite technical, given that the statement is only about an average of two product measures, but Section 6 relies crucially on this result. In Section 6, we consider tree-indexed Markov chains on infinite trees and we give conditions on the parameters such that the process is not in $\mathcal{R}$. With a similar argument, we show that the Ising model on $\mathbb{Z}^d$ for $d \geq 2$ is not in $\mathcal{R}$ for a certain range of parameters. In Section 7.3, we consider stationary processes X^ν on $\{0,1\}^{\mathbb{N}}$. We give a necessary and sufficient condition on ν such that X^ν is ergodic, see Theorem 7.3, and a sufficent condition on ν such that X^ν is a Bernoulli shift, see Theorem 7.5. We end in Section 8 with some open questions.

2 Background definitions, and some first properties and examples

We begin by recalling some basic definitions.

Definition 2.1. *A probability measure μ on $\{0,1\}^S$ is said to be positively associated if for all increasing sets A and B,*

$$\mu(A \cap B) \geq \mu(A)\mu(B).$$

Definition 2.2. *A probability measure μ on $\{0,1\}^S$ is said to satisfy the FKG property if for all $I \subseteq S$ and all $\{a_i\}_{i \in I}$ with each $a_i \in \{0,1\}$, the conditional measure*

$$\mu(\cdot \mid X_i = a_i \text{ for } i \in I)$$

on $\{0,1\}^{S \setminus I}$ is positively associated.

Definition 2.3. *A probability measure μ on $\{0,1\}^S$ is said to satisfy the downwards FKG property if in the definition of the FKG property, we only require the positive association of the conditional measure when each $a_i = 0$.*

Clearly the FKG property implies the downward FKG property which in turn implies positive association. Note that while positive association and the FKG property are unaffected by reversing 0's and 1's, this is not the case with the downward FKG property.

The following result demonstrates the very different roles played by the 0's and 1's for our X^ν.

Theorem 2.4. *For every S and ν, X^ν has the downward FKG property. However, it does not necessarily satisfy the FKG property.*

Proof. We begin the proof by showing that X^ν has positive association for all ν's. In the case of finite S, it is immediate that X^ν is given by increasing functions of i.i.d. random variables and hence by Harris' Theorem has positive association. An easy approximation argument gives the result for general sets S.

We now prove the downward FKG property. When one conditions on the event that X^ν is zero on some subset $A \subseteq S$, one is conditioning on the event that no element in $\mathcal{S}_A^\cup$ occurred in Y^ν. The conditional distribution of Y^ν then becomes $Y^{\nu|_{(\mathcal{S}_A^\cup)^c}}$ and hence the conditional distribution of X^ν is still of our form and hence is positively associated by the first part of this proof.

Finally, we give an example of an X^ν which does not have the FKG property. Let $S = \{1, 2, 3\}$ and ν give weight $\log 2$ to each of $\{1, 2\}$ and $\{2, 3\}$. Then Y^ν is one of the following collection of sets each having probability $1/4$.

(a) $\emptyset$,

(b) $\{\{1, 2\}\}$,

(c) $\{\{2, 3\}\}$, and

(d) $\{\{1, 2\}, \{2, 3\}\}$.

If we condition on $x_2 = 1$, then we know Y^ν is one of the last three each then with conditional probability $1/3$. Now, it is immediate that conditioned on $x_2 = 1$, we have $x_1 = 1$ with probability $2/3$, $x_3 = 1$ with probability $2/3$ and $x_1 = x_3 = 1$ with probability $1/3$, which is less than $4/9$. Hence Y^ν is not FKG. $\square$

Remark 2.5. *The example in the proof of Theorem 2.4 also gives an example of an $X \in \mathcal{R}$ such that $X \mid X_2 = 1$ is not in $\mathcal{R}$.*

Remark 2.6. *The upper invariant measure for the contact process exhibits similar behavior to the above example. It is not FKG (see [11]) but it is downwards FKG (see [1]).*

Proposition 2.7. *Assume ν is a translation invariant measure on $\mathcal{P}(\mathbb{Z}) \setminus \{\emptyset\}$ that gives positive weight to an infinite subset S which is not periodic. Then $X^\nu \equiv 1$ a.s.*

Proof. Since S is not periodic, all of its translates are distinct and have the same ν-weight. We show $X^\nu(0) = 1$ almost surely. Since S is infinite, there are an infinite number of translates of S containing 0 and so at least one of these will occur almost surely. $\square$

The next proposition says that the overlap property of the sets that ν charges describes pairwise correlations in a simple way.

Proposition 2.8. *Assume that $X = (X_s)_{s \in S} = X^\nu$. Then, for any $k, \ell \in S$, we have*

$$P(X_k = 0, X_\ell = 0) = P(X_k = 0)P(X_\ell = 0)e^{\nu(\mathcal{S}_k \cap \mathcal{S}_\ell)}.$$

More generally, if A and B are disjoint subsets of S, we have

$$P(\bigcap_{k\in A}\{X_k=0\}\cap\bigcap_{\ell\in B}\{X_\ell=0\})=P(\bigcap_{k\in A}\{X_k=0\})P(\bigcap_{\ell\in B}\{X_\ell=0\})e^{\nu(\bigcup_{k\in A}\mathcal{S}_k\cap\bigcup_{\ell\in B}\mathcal{S}_\ell)}.$$

Proof. We prove only the first statement. The second statement is proved in the same way. By definition,

$$\begin{aligned}P(X_k=0,X_\ell=0)&=e^{-\nu(\mathcal{S}_k\cup\mathcal{S}_\ell)}=e^{-[\nu(\mathcal{S}_k)+\nu(\mathcal{S}_\ell)-\nu(\mathcal{S}_k\cap\mathcal{S}_\ell)]}\\&=P(X_k=0)P(X_\ell=0)e^{\nu(\mathcal{S}_k\cap\mathcal{S}_\ell)}.\end{aligned}$$

□

Corollary 2.9. *For any S and ν, if X^ν is pairwise independent, then it is an independent process.*

Proof. By inclusion-exclusion, it suffices to show that for any finite $A\subseteq S$, one has

$$P(X^\nu_A\equiv 0)=\prod_{a\in A}P(X^\nu_a=0).$$

The assumption of pairwise independence together with Proposition 2.8 implies that for $a,b\in A$, $a\neq b$, $\nu(\mathcal{S}_a\cap\mathcal{S}_b)=0$. This yields $\nu(\mathcal{S}^{\cup}_A)=\sum_{a\in A}\nu(\mathcal{S}_a)$ and hence

$$P(X^\nu_A\equiv 0)=e^{-\nu(\mathcal{S}^{\cup}_A)}=e^{-\sum_{a\in A}\nu(\mathcal{S}_a)}=\prod_{a\in A}P(X^\nu_a=0).$$

□

The proof of the following useful lemma will be left to the reader.

Lemma 2.10. *For any S, ν and ν', one has that*

$$X^{\nu+\nu'}\stackrel{d}{=}\max\{X^\nu,X^{\nu'}\},$$

where the latter two processes are assumed to be independent.

Lemma 2.11. *Given $\{0,1\}$-valued random variables $X=(X_k)_{k\in[n]}$, if there is a nonnegative measure ν on $\mathcal{P}([n])\backslash\{\emptyset\}$ such that*

$$P\big(X(K)\equiv 0\big)=e^{-\nu(\mathcal{S}^{\cup}_K)},\quad K\subseteq[n],$$

then $X^\nu\stackrel{d}{=}X$.

Proof. If ν is nonnegative, X^ν exists. Since X and X^ν agree on events of the form $X_K\equiv 0$, they must agree on all events, and hence the desired conclusion follows. □

Lemma 2.12. *Let $X = (X_1, X_2, \dots, X_n)$ be $\{0,1\}$-valued random variables such that $P\big(X(I) \equiv 0\big) > 0$ for all $I \subseteq [n]$. Then there is a unique signed measure ν on $\mathcal{P}([n]) \setminus \{\emptyset\}$ that satisfies*

$$\nu\big(\mathcal{S}_I^{\cup}\big) = -\log P\big(X(I) \equiv 0\big), \quad I \subseteq [n] \tag{2.1}$$

and is given by (2.3). *(Note that by Lemma 2.11, if such a nonnegative measure ν exists, then $X = X^\nu$.)*

Proof. If a signed measure ν exists then, since n is finite, we have

$$\nu\big(\mathcal{S}_I^{\cup}\big) = \sum_{J \subseteq [n] \colon J \cap I \neq \emptyset} \nu(J)$$

and hence (2.1) is equivalent to the system of linear equations given by

$$\sum_{J \subseteq [n] \colon J \cap I \neq \emptyset} \nu(J) = -\log P\big(X(I) \equiv 0\big), \quad I \subseteq [n]. \tag{2.2}$$

The desired conclusion will thus follow if we can show that this system of linear equations always has a unique (possibly signed) solution $(\nu(J))_{J \subseteq [n], J \neq \emptyset}$. To this end, we first rewrite (2.2) as

$$\begin{aligned}\sum_{\substack{J \subseteq [n] \setminus I \colon \\ J \neq \emptyset}} \nu(J) &= \sum_{\substack{J \subseteq [n] \colon \\ J \neq \emptyset}} \nu(J) - \sum_{J \subseteq [n] \colon J \cap I \neq \emptyset} \nu(J) \\ &= -\log P\big(X([n]) \equiv 0\big) - \Big(-\log P\big(X(I) \equiv 0\big)\Big), \quad I \subseteq [n].\end{aligned}$$

Equivalently, this becomes

$$\sum_{\substack{J \subseteq I \colon \\ J \neq \emptyset}} \nu(J) = -\log P\big(X([n]) \equiv 0\big) + \Big(\log P\big(X([n] \setminus I) \equiv 0\big)\Big), \quad I \subseteq [n].$$

By the Möbius inversion theorem, we see that this equation is equivalent to

$$\begin{aligned}\nu(K) &= \sum_{I \subseteq K} (-1)^{|K|-|I|} \Big(-\log P\big(X([n]) \equiv 0\big) + \log P\big(X([n] \setminus I) \equiv 0\big)\Big) \\ &= \sum_{I \subseteq K} (-1)^{|K|-|I|} \log P\big(X([n] \setminus I) \equiv 0\big), \qquad \emptyset \neq K \subseteq [n].\end{aligned} \tag{2.3}$$

This concludes the proof. $\square$

Lemma 2.13. *Assume that $X = X^\nu$ and that $(X_i)_{i \in [n]} \overset{d}{=} (X_{\sigma(i)})_{i \in [n]}$ for some $\sigma \in S_n$. Then $\nu = \nu \circ \sigma$.*

Proof. From Lemma 2.12, we know that if $X = X^\nu$ exists, then ν is unique. If $X = X^\nu$ then $X = X_\sigma = X^{\nu \circ \sigma}$, and hence we must have $\nu = \nu \circ \sigma$. $\square$

The proof of the following lemma is left to the reader, the third part of whose proof uses a simple compactness argument.

Lemma 2.14. *Consider a process $X = \{X_s\}_{s\in S}$.*

(a) If $X = X^\nu$ and $B \subseteq S$, then there is ν_B such that $X|_B = X^{\nu_B}$. Moreover, for any non-empty measurable subset $\mathcal{A} \subseteq \mathcal{P}(B)$, we have $\nu_B(\mathcal{A}) = \nu\big(\{A' \in \mathcal{P}(S)\colon A' \cap B \in \mathcal{A}\}\big)$.

(b) If $X = X^\nu$ and $B \subseteq S$, then there is a measure $\nu_{B,0}$ on $\mathcal{P}(B) \smallsetminus \{\emptyset\}$ such that $X|\big\{X(B^c) \equiv 0\big\} = X^{\nu_{B,0}}$. Moreover, $\nu_{B,0} = \nu|_{\mathcal{P}(B)}$.

(c) If there exist $S_1 \subseteq S_2 \subseteq, \ldots$ such that $S = \bigcup_i S_i$ and $X_{S_n} = X^{\nu_n}$ for some ν_n, then $X = X^\nu$ for some ν. (The projection of ν on to each S_n will simply be ν_n.)

We first point out that when $n = 2$, provided we have positive association, X is always of this form. In particular, one can check that $X = X^\nu$, where for $\emptyset \neq J \subseteq \{1, 2\}$ we set $\nu(J) = -\log\big(1 - p(J)\big)$ where

$$\begin{cases} p(\{1,2\}) = 1 - P(X_1 = 0)P(X_2 = 0)/P(X_1 = X_2 = 0) \\ p(\{1\}) = 1 - (P(X_1 = X_2 = 0))/P(X_2 = 0) \\ p(\{2\}) = 1 - (P(X_1 = X_2 = 0))/P(X_1 = 0). \end{cases}$$

Sticking with $n = 2$ in the nonpositively associated case, it is interesting to see which sets the representing signed measure ν gives negative weight to.

Example 2.15. Let X be $(1, 0)$ or $(0, 1)$ each with probability $(1 - \varepsilon)/2$ and equal to $(0, 0)$ or $(1, 1)$ each with probability $\varepsilon/2$, where $\varepsilon < 1/2$. Then

$$P\big(X(1) = 0\big) = P\big(X(2) = 0\big) = 1/2 \quad \text{and} \quad P\big(X(\{1, 2\}) \equiv 0\big) = \varepsilon/2.$$

Consequently,

$$\nu(\mathcal{S}_1 \cup \mathcal{S}_2) = -\log \varepsilon/2 = \log 2 - \log \varepsilon = 2\nu\big(\{1\}\big) + \nu\big(\{1, 2\}\big)$$

and

$$\nu(\mathcal{S}_1) = -\log 1/2 = \log 2 = \nu\big(\{1\}\big) + \nu\big(\{1, 2\}\big)$$

and hence

$$\nu\big(\{1\}\big) = -\log \varepsilon > 0 \quad \text{and} \quad \nu\big(\{1, 2\}\big) = \log 2 + \log \varepsilon < 0.$$

When we now move to $n = 3$, it is already the case that positive association does not imply that X is of our form as the following example shows.

Example 2.16. Choose $\sigma \in S_3$ uniformly at random, and define $X = (X_1, X_2, X_3)$ by $X_j = \mathbf{1}(\sigma(j) = j)$. Then

$$\begin{cases} P(X \equiv 1) = 1/6 \\ P\big(X = (1, 1, 0)\big) = P\big(X = (1, 0, 1)\big) = P\big(X = (0, 1, 1)\big) = 0 \\ P\big(X = (1, 0, 0)\big) = P\big(X = (0, 1, 0)\big) = P\big(X = (0, 0, 1)\big) = 1/6 \\ P(X \equiv 0) = 1/3. \end{cases}$$

The random vector X defined above is known to be positively associated. On the other hand, one easily verifies that it is not of our form.

Remarks 2.17.

(i) *Jeff Kahn ([8]) proved the much stronger and much more difficult fact that the above random vector cannot be expressed as increasing functions of i.i.d. random variables.*

(ii) *We will see another example later on of a positively associated process which is not in $\mathcal{R}$ for $n = 3$; it will in fact be an average of two product measures.*

We provide a further interesting example for $n = 3$ which is positively associated but not in $\mathcal{R}$.

Example 2.18. Let X, Y be i.i.d. 0 or 1 each with probability 1/2. Consider (X, Y, XY). This is positively associated since the vector is given by increasing functions of i.i.d. random variables. Next, if $XY = 0$, then either X or Y is equal to zero. Consequently,

$$P(X = Y = 1 \mid XY = 0) = 0$$

and hence (X, Y, XY) is not downward FKG. By Theorem 2.4, we conclude that this is not in $\mathcal{R}$.

The unique signed measure ν which satisfies $X = X^\nu$ is given by

$$\begin{cases} \nu\big(\{1\}\big) = \nu\big(\{2\}\big) = \log 2 \\ \nu\big(\{3\}\big) = \nu\big(\{1,3\}\big) = \nu\big(\{2,3\}\big) = 0 \\ \nu\big(\{1,2\}\big) = \log 3/4 < 0 \\ \nu\big(\{1,2,3\}\big) = \log 4/3. \end{cases}$$

The next result tells us that with some additional symmetries, positive association implies that we are Poisson generated when $n = 3$.

Theorem 2.19. *Consider a probability measure μ on $\{0,1\}^3$ which is invariant under permutations and interchanging 0 and 1. Then the following are equivalent.*

(a) *μ has positive association.*

(b) *μ is an increasing function of i.i.d. random variables.*

(c) *μ satisfies the FKG property.*

(d) *μ is in $\mathcal{R}$.*

Proof. The set of measures μ as above is just a one parameter family since $p_1 := P(X \equiv 1) \leq 1/2$ determines the measure given all of the symmetries. Since (c) implies (b) implies (a) and (d) implies (b) implies (a) in general, we need only show that (a) implies (c) and (a) implies (d).

It is elementary to check that μ is positively associated if and only if $p_1 \geq 1/8$. Under this assumption, it is easy to verify that the FKG property holds (one way to see this is that the model is then the ferromagnetic Curie-Weiss model).

To see that (a) implies (d), one simply checks that the following ν measure works. Letting $p_2 := P\big(X(1) = X(2) = 1,\, X(3) = 0\big)$, one verifies that $p_2 = (1 - 2p_1)/6$ which is then at most $1/8$. ν gives each of the three singletons weight $\log(\frac{p_1+p_2}{p_1})$, each of the three doubletons weight $\log(\frac{p_1}{2(p_1+p_2)^2})$ and the unique three element set weight $\log(\frac{8(p_1+p_2)^3}{p_1})$. One can check that the first and third terms are always non-negative while the second term is non-negative if and only if $p_1 \geq 1/8$. $\square$

This model has a similar flavor to the so-called divide and color model (see [16]) but they are certainly different. In the latter model, one takes a random partition of S (with any distribution) and then assigns all the elements in each partition element either 1 or 0 with probability p and $1 - p$. This is done independently for different clusters. We now give some examples illustrating the difference between these concepts.

Example 2.20 (Example 2.17 in [16]). Consider the divide and color process X corresponding to the two partitions, $(12, 3, 4)$ and $(1, 2, 34)$ being chosen with equal probability. Letting $A = \{X_1 = X_2 = 1\}$ and $B = \{X_3 = X_4 = 1\}$, one checks that these are increasing but negatively correlated events and hence this does not have positive association. Consequently, $X \notin \mathcal{R}$ by Theorem 2.4.

Example 2.21. X_1, X_2, X_3, X_4 be i.i.d. with $P(X_1 = 0) = 1/2$. For $n \in \{1, 2, 3\}$, let $Y_n := \max(X_n, X_{n+1})$. We first leave it to the reader to check that (Y_1, Y_2, Y_3) is Poisson generated by using

$$\nu\big(\{1, 2\}\big) = \nu\big(\{2, 3\}\big) = \nu\big(\{1\}\big) = \nu\big(\{3\}\big) = \log 2.$$

However, we now argue that (Y_1, Y_2, Y_3) is not a divide and color process. To see this, note that $P(Y_1 = 1) = 3/4$ and hence any divide and color model would need to be made with $p = 3/4$. Assume (Y_1, Y_2, Y_3) is a divide and color model. Since Y_1 and Y_3 are independent, they cannot be in the same partition element. Hence the only possible partitions are $(1, 2, 3), (12, 3), (1, 23)$. By symmetry, these are given masses p, $(1 - p)/2$, and $(1 - p)/2$ for some p. This implies that

$$P(Y_1 = Y_2 = Y_3 = 0) = p(1/4)^3 + (1 - p)(1/4)^2.$$

Since, by definition, we have $P(Y_1 = Y_2 = Y_3 = 0) = 1/16$, it follows that $p = 0$. Next, note that

$$1/8 = P(Y_1 = Y_2 = 0) = 1/2(1/4) + 1/2(1/4)^2 = 1/8 + 1/32,$$

a contradiction.

The following lemma states that the set $\mathcal{R}$ is closed in the set of all random vectors.

Lemma 2.22. *Let S be countable and let $X_n \in \{0, 1\}^S$ be a sequence of random vectors that converges in distribution to a random vector $X \in \{0, 1\}^S$. If $X_n \in \mathcal{R}$ for every n, then $X \in \mathcal{R}$.*

Proof. Assume first that $|S| < \infty$. For $n \geq 1$, since $X_n \in \mathcal{R}$, there is a measure ν_n on $\mathcal{P}(S) \smallsetminus \{\emptyset\}$ such that $X_n = X^{\nu_n}$. Allowing now our measures to take the value ∞, we can extract a convergent subsequence $(\nu_{n'})$ of (ν_n) converging to some ν which is allowed to take the value ∞. Now $X_{n'} = X^{\nu_{n'}}$ converges to both X and to X^ν and hence $X \in \mathcal{R}$. Applying Lemma 2.14, we obtain the desired conclusion for any S. $\square$

It turns out that domination from below by product measures for translation invariant processes on $\mathbb{Z}^d$ which belong to $\mathcal{R}$ has a simple characterization.

Proposition 2.23. *Let ν be a translation invariant measure on $\mathcal{P}(\mathbb{Z}^d) \setminus \{\emptyset\}$. Then $X^\nu \geq \Pi_p$ if and only if*

$$\nu(\mathcal{S}_A^\cup) \geq -|A| \log(1-p)$$

for all boxes A of the form $\{-n, \ldots, n\}^d$.

Proof. This follows immediately from Theorem 2.4 and [12, Theorem 4.1] using the fact that for any box A, $P\big(X^\nu(A) \equiv 0\big) = e^{-\nu(\mathcal{S}_A^\cup)}$. $\square$

3 Markov and renewal processes

In this section, we begin by proving the following result, which shows that all positively associated Markov chains on $\{0,1\}^\mathbb{Z}$ are in $\mathcal{R}$.

To simplify notation in what follows, given a stationary process X, we define

$$c_k \coloneqq P\big(X_0 = 0, X_k = 0\big), \quad k \geq 0,$$

Note that if $c_0 = P\big(X_0 = 0\big) > 0$, then by positive association, we have $c_k \geq c_0^2 > 0$ for all $k > 0$.

Theorem 3.1. *Let X be a non-trivial stationary positively associated Markov chain on $\{0,1\}^\mathbb{Z}$. Then $X \in \mathcal{R}$ and ν is given by*

$$\nu(K) = \begin{cases} \log \frac{c_{|K|-1} c_{|K|+1}}{c_{|K|}^2} & \text{if } K \text{ is a finite interval, and} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Remark 3.2. *We note that if X is a non-constant positively associated $\{0,1\}$-valued Markov chain, then its transition matrix can be written as*

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 1 - p(1-r) & p(1-r) \\ pr & 1 - pr \end{pmatrix}, \tag{3.2}$$

for some $p, r \in (0,1)$. Here p is the probability that the Markov chain rerandomizes (otherwise it stays fixed) and r is the probability it moves to zero when it rerandomizes. Hence, as easily checked,

$$c_k = r(1-p)^k + r^2\big(1 - (1-p)^k\big), \quad k \geq 0. \tag{3.3}$$

The proof of Theorem 3.1 will use the following lemma involving renewal processes, which we now define.

Definition 3.3. *Let X be non-trivial a $\{0,1\}$-valued process on $\mathbb{Z}$. We say that X is a renewal process (with respect to 0) if there is a sequence $(b_n)_{n\geq 1}$ of non-negative real numbers such that $\sum_{n=1}^\infty b_n \leq 1$ and any $(a_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$*

$$P\big(\min\{j \geq 1\colon X_{k+j} = 0\} = n \mid X_k = 0, (X_{k-i})_{i=1}^\infty = (a_{k-i})_{i=1}^\infty\big) = b_n, \qquad \textit{for all } k, n \in \mathbb{Z}.$$

We will in addition always assume that $\sum_{n=1}^\infty b_n = 1$ and that the process is stationary.

Lemma 3.4. *Assume that $X = X^\nu \in \mathcal{R}$ for some translation invariant measure ν. Then X is a renewal process if and only if ν is supported on finite intervals of $\mathbb{Z}$.*

Proof. We show "only if" direction; the "if" direction is left to the reader.

Assume that $X = X^\nu$ is a renewal process. Then, by Theorem 2.4, X is positively associated, and hence

$$P(X_{n+1} = 0 \mid X_n = 0) > 0 \quad \forall n \in \mathbb{Z}. \tag{3.4}$$

Let $n \in \mathbb{Z}$. Then, by definition, we have

$$e^{-\nu(\mathcal{S}_{n+1} \smallsetminus \mathcal{S}_n)} = P(X_{n+1} = 0 \mid X_n = 0) > 0.$$

Since X is a renewal process, we also have

$$P(X_{n+1} = 0 \mid X_n = 0) = P(X_{n+1} = 0 \mid X_n = 0,\, X_{n-1} = 0,\, \ldots) = e^{-\nu(\mathcal{S}_{n+1} \smallsetminus \bigcup_{j\leq n} \mathcal{S}_j)}.$$

Combining the two above equations, we obtain

$$\nu(\mathcal{S}_{n+1} \smallsetminus \mathcal{S}_n) = \nu(\mathcal{S}_{n+1} \smallsetminus \bigcup_{j\leq n} \mathcal{S}_j) < \infty.$$

In particular, this implies that for any $j \leq n$, we have

$$\nu(\mathcal{S}_{n+1} \cap \mathcal{S}_n^c \cap \mathcal{S}_j) = 0.$$

This implies that ν is supported on intervals. Assume now for contradiction that ν has support on infinite intervals. Then, without loss of generality, we can assume that

$$\nu\Big(\big\{[k,\infty) \cap \mathbb{Z}\colon k \in \mathbb{Z}\big\}\Big) > 0.$$

Since $X = X^\nu$, with strictly positive probability there is $k \in \mathbb{Z}$ such that $X_j = 1$ for all $j \geq k$, and hence X cannot be recurrent. This concludes the proof. $\square$

Proof of Theorem 3.1. Assume first that that $X = X^\nu \in \mathcal{R}$. Since X is stationary, ν is translation invariant, and by Lemma 3.4, ν is supported on finite intervals. Letting $k \geq 1$, it follows that

$$\begin{aligned} c_k = P(X_0 = X_k = 0) &= e^{-\nu(\mathcal{S}^\cup_{\{0,k\}})} = e^{-\nu(\mathcal{S}_0)} e^{-\nu(\mathcal{S}_k)} e^{\nu(\mathcal{S}^\cap_{\{0,k\}})} = e^{-\nu(\mathcal{S}_0)} e^{-\nu(\mathcal{S}_0)} e^{\nu(\mathcal{S}^\cap_{\{0,k\}})} \\ &= c_0^2 e^{\nu(\mathcal{S}^\cap_{\{0,k\}})} = c_0^2 \prod_{j>k} e^{(j-k)\nu([j])}. \end{aligned}$$

From this it follows that

$$\frac{c_{k-1}c_{k+1}}{c_k^2} = \frac{c_0^2 \prod_{j>k-1} e^{(j-(k-1))\nu([j])} c_0^2 \prod_{j>k+1} e^{(j-(k+1))\nu([j])}}{c_0^2 \prod_{j>k} e^{(j-k)\nu([j])} c_0^2 \prod_{j>k} e^{(j-k)\nu([j])}} = e^{\nu([k])}.$$

Consequently, ν is given by (3.1).

Using (3.3), it is easy to check that X being positively associated implies that the following inequality holds.

$$c_{k-1}c_{k+1} \geq c_k^2 \quad \forall k \geq 1. \tag{3.5}$$

Let ν be given by (3.1). Since ν is translation invariant and supported only on finite intervals, it follows from Lemma 3.4 that X^ν is a renewal process. Next, note that for any $k \geq 0$, we have, for all $m \geq k+1$,

$$\begin{aligned}\sum_{j=k+1}^{m} (j-k) \log \frac{c_{j+1}c_{j-1}}{c_j^2} &= \log c_k - (m+1-k)\log c_m + (m-k)\log c_{m+1} \\ &= \log c_k - \log c_m + (m-k)\log \frac{c_{m+1}}{c_m}.\end{aligned} \tag{3.6}$$

Using (3.3), one can verify that

$$\lim_{m\to\infty} c_m = c_0^2 \quad \text{and} \quad \lim_{m\to\infty} m \log \frac{c_{m+1}}{c_m} = 0. \tag{3.7}$$

Letting $m \to \infty$ in (3.6), we obtain

$$P(X_0^\nu = X_k^\nu = 0) = c_0^2 e^{\sum_{j>k}(j-k)\nu([j])} = c_0^2 e^{\log c_k - \log c_0^2} = c_k,$$

where the first equality follows as in the first display of the proof. Since X^ν and X are both renewal processes, this shows that $X^\nu = X$. □

We now state and prove a more general version of Theorem 3.1 which is valid for all renewal processes.

Theorem 3.5. *Let X be a renewal process with $P(X_0 = 0) > 0$. Then $X \in \mathcal{R}$ if and only if*

$$c_{k-1}c_{k+1} \geq c_k^2 \quad \forall k \geq 1. \tag{3.8}$$

Moreover, in this case, $X = X^\nu$, where ν is the translation invariant measure given by (3.1).

The proof of Theorem 3.1 already proves Theorem 3.5 if we can show that (3.7) also holds in this more general setting. This is the purpose of the following lemma.

Lemma 3.6. *Assume X is a renewal process with $c_0 > 0$ such that* (3.8) *holds. Then*

(a) $(c_k)_{k\geq 0}$ *is decreasing,*

(b) $\lim_{k\to\infty} c_k = c_0^2$, *and*

(c) $\lim_{k\to\infty} k\log(c_{k+1}/c_k) = 0$.

Proof.

(a) Since (3.8) holds and $c_0 > 0$, we have $c_k \in (0,1)$ for all $k \geq 0$.

For any $k \geq 1$, we have

$$\frac{c_{k+1}c_{k-1}}{c_k^2} \geq 1 \Leftrightarrow \frac{c_{k-1}}{c_k} \geq \frac{c_k}{c_{k+1}}.$$

Consequently, the sequence $(\frac{c_{k-1}}{c_k})_{k\geq 1}$ is decreasing and converges to a limit $a \in [0, c_0/c_1]$ as $k \to \infty$. We will now show that $a \geq 1$. To this end, assume for contradiction that $a < 1$. Then there is $j \geq 1$ such that $c_k/c_{k+1} < (1+a)/2 < 1$ for all $k \geq j$, and hence $c_k > c_j\big(2/(1+a)\big)^{k-j}$ for all $k \geq j$. Since $c_j > 0$, this implies that $\lim_{k\to\infty} c_k = \infty$, contradicting that $c_k < 1$ for all $k \geq 0$. Hence we must have $a \geq 1$. Since $(c_{k-1}/c_k)_{k\geq 1}$ is decreasing, it follows that $c_{k-1}/c_k \geq a = 1$ for all $k \geq 1$, and hence $(c_k)_{k\geq 0}$ is decreasing. This completes the proof of (a).

(b) Since, by (a), $(c_k)_{k\geq 0}$ is decreasing, let c_∞ denote its limit. Let

$$a_k := P\big(\min\{j \geq 0\colon X_j = 0\} = k\big), \qquad k \geq 0.$$

Then, for any $k \geq 1$, we have (due to stationarity)

$$c_0 = P(X_k = 0) = \sum_{j=0}^{k} a_j (c_{k-j}/c_0).$$

Since $\sum_{j=0}^{\infty} a_j = 1$, $c_0 > 0$, and $c_k \searrow c_\infty$, we obtain (b).

(c) To this end, note that for any $m \geq 1$, we have

$$\begin{aligned} S_m &:= \sum_{k=1}^{m} k \log \frac{c_{k-1}c_{k+1}}{c_k^2} = \log c_0 + m\log c_{m+1} - (m+1)\log c_m \\ &= \log c_0 - \log c_m + \log(c_{m+1}/c_m)^m \overset{\text{(b)}}{\leq} \log c_0 - \log c_m + 0 \\ &\nearrow \log c_0 - \log c_0^2 = -\log c_0. \end{aligned}$$

Since each term in the sum S_m is non-negative, S_m is increasing in m. Since $(S_m)_{m\geq 1}$ is increasing and bounded from above by $-\log c_0$, its limit $\lim_{m\to\infty} S_m$ exists and is bounded from above by $-\log c_0$. Since, by (b), the limit of $(c_m)_{m\geq 0}$ also exists, and

$$S_m - (\log c_0 - \log c_m) = \log(c_{m+1}/c_m)^m,$$

it follows that $\lim_{m\to\infty}(c_{m+1}/c_m)^m$ exists. Next, note that since $c_0 \geq c_1 \geq \cdots \geq c_0^2$ and $\sum_{m=1}^{\infty} 1/m = \infty$, we must have $\liminf_{m\to\infty} m(c_m - c_{m+1}) = 0$. Since

$$\begin{aligned} &\liminf_{m\to\infty} m(c_m - c_{m+1}) = 0 \Rightarrow \liminf_{m\to\infty} m(1 - c_{m+1}/c_m) = 0 \\ &\quad \Rightarrow \liminf_{m\to\infty} (1 - (1 - c_{m+1}/c_m))^m = 1 \Leftrightarrow \liminf_{m\to\infty} (c_{m+1}/c_m)^m = 1 \end{aligned}$$

and $\lim_{m\to\infty}(c_{m+1}/c_m)^m$ exists, it follows that

$$\lim_{m\to\infty}(c_{m+1}/c_m)^m = \liminf_{m\to\infty}(c_{m+1}/c_m)^m = 1.$$

This establishes (c), and thus completes the proof.

□

Proof of Theorem 3.5. Replacing (3.7) with Lemma 3.6, the proof of Theorem 3.1 gives the desired conclusion. □

Lemma 3.4 tells us that for positively associated Markov chains, the corresponding ν is supported on finite intervals. Interestingly, a similar result holds for Markov random fields.

Proposition 3.7. *Let $X = X^\nu$ be a $\{0,1\}$-valued process on a connected graph that satisfies the Markov property and is such that for all finite sets A,*

$$P\big(X(A) \equiv 0 \mid X(\partial A) \equiv 0\big) > 0.$$

Let $\mathcal{D}$ be the set of all disconnected subsets of S, at least one of whose components is finite. Then $\nu(\mathcal{D}) = 0$.

Proof. By assumption, we have for any finite set A

$$P\big(X(A) \equiv 0 \mid X(\partial A) \equiv 0\big) = e^{-\nu(\mathcal{S}_A^\cup \smallsetminus \mathcal{S}_{\partial A}^\cup)} > 0.$$

Since X satisfies the Markov property, we also have

$$P\big(X(A) \equiv 0 \mid X(\partial A) \equiv 0\big) = P\big(X(A) \equiv 0 \mid X(A^c) \equiv 0\big) = e^{-\nu(\mathcal{S}_A^\cup \smallsetminus \mathcal{S}_{A^c}^\cup)} > 0.$$

Combining the two above equations, we obtain

$$\nu(\mathcal{S}_A^\cup \smallsetminus \mathcal{S}_{\partial A}^\cup) = \nu(\mathcal{S}_A^\cup \smallsetminus \mathcal{S}_{A^c}^\cup) < \infty$$

which easily yields

$$\nu\big((\mathcal{S}_A^\cup \cap \mathcal{S}_{A^c}^\cup) \smallsetminus \mathcal{S}_{\partial A}^\cup\big) = 0.$$

This concludes the proof. □

Remark 3.8. *With Theorem 3.1 in mind, it is natural to ask whether*

1. *all positively associated tree-indexed Markov chains are in $\mathcal{R}$, and if*
2. *the Ising model on $\mathbb{Z}^d$, $d \geq 2$, is in $\mathcal{R}$.*

We answer both these questions negatively in Theorem 6.1 and Theorem 6.3.

4 Infinite permutation invariant processes

In this section, we consider (possibly infinite) measures ν on $\{0,1\}^{\mathbb{N}}$ which are invariant under finite permutations and the associated processes X^ν which are permutation invariant probability measures on $\{0,1\}^{\mathbb{N}}$. Also, in this section, we let Π_p denote the product measure on $\{0,1\}^S$ with density p and we recall de Finetti's Theorem which states that each $\{0,1\}$-valued process X indexed by $\mathbb{N}$ which is invariant under finite permutations is an average of product measures; i.e. there is a (unique) probability measure μ on $[0,1]$ such that the distribution of X is

$$\int_0^1 \Pi_p \, d\mu(p).$$

We will identify X with μ.

Example 4.1. If $X \sim \Pi_p$, i.e. if $\mu = \delta_p$, then $X = X^\nu$ for the (infinite) measure ν defined by

$$\nu(A) = \begin{cases} -\log(1-p) & \text{if } A = \{j\} \text{ for some } j \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

Example 4.2. For all $p \in [0,1]$, if $X \sim \alpha\Pi_p + (1-\alpha)\Pi_1$, i.e. if $\mu = \alpha\delta_p + (1-\alpha)\delta_1$, then $X = X^\nu$ for the (infinite) measure ν defined by

$$\nu(A) = \begin{cases} -\log(1-p) & \text{if } A = \{j\} \text{ for some } j \in \mathbb{N} \\ -\log\alpha & \text{if } A = \mathbb{N}, \\ 0 & \text{else.} \end{cases}$$

Example 4.3. Let $\nu = \Pi_p$. Then

$$X^\nu \stackrel{d}{=} \sum_{j=0}^{\infty} \frac{e^{-1}}{j!} \Pi_{1-(1-p)^j},$$

which is a convex combination of product measures whose densities have an accumulation point at one.

4.1 A de Finetti theorem for infinite measures

To begin here, we need to first understand the permutation invariant (possibly infinite) measures ν on $\{0,1\}^{\mathbb{N}}$. If we stick to probability measures, then this is the classical de Finetti's Theorem above. While this immediately extends to any finite measure, we need to encompass infinite measures as well here. We found the following infinite version of de Finetti's Theorem which is relevant for our specific context. After having done this, we learned that this was already done (in a slightly different and even more general setup) by Harry Crane; see [5, Theorem 2.7]. Despite the result therefore not being new, we include our proof below since it is quite short and written using the same language as the rest of the paper.

Theorem 4.4 (A version of de Finetti's theorem for possibly infinite measures)**.** *Let ν be a permutation invariant measure on $\mathcal{P}(\mathbb{N}) \smallsetminus \{\emptyset\}$. Assume that $\nu(\mathcal{S}_1) < \infty$. Then there is a unique measure σ on $(0,1]$ and $c \geq 0$ such that*

$$\nu = c \sum_{i\in\mathbb{N}} \delta_i + \int_0^1 \Pi_x \, d\sigma(x) \tag{4.1}$$

and

$$\int_0^1 x \, d\sigma(x) < \infty.$$

Note that the assumption that $\nu(\mathcal{S}_1) < \infty$ is needed to not have $X^\nu(A) \equiv 1$ almost surely.

Proof. Let $\mathcal{U} \coloneqq \big\{\{1\},\{2\},\dots\big\}$, and for $j \geq 1$, let $\nu_j \coloneqq \nu|_{\mathcal{S}^\cup_{[j]} \smallsetminus (\mathcal{U} \cup \mathcal{S}^\cup_{[j-1]})}$. Note that by assumption, for each $j \geq 1$, ν_j is a finite measure. Then we can write

$$\nu = c \sum_{i\in\mathbb{N}} \delta_i + \sum_{j=1}^\infty \nu_j.$$

In particular, we have written ν as a sum of measures with disjoint supports.

Define

$$\tau A \coloneqq \{1\} \cup (A+1), \quad A \subseteq \mathbb{N}.$$

and

$$\hat\nu_1(\cdot) \coloneqq \nu_1\big(\tau \circ \cdot\big) = \nu\big(\tau \circ \cdot\big).$$

Then $\hat\nu_1$ is permutation invariant and finite, and hence, by de Finetti's theorem, there is a unique finite measure m on $[0,1]$ such that

$$\hat\nu_1 \overset{d}{=} \int \Pi_x \, dm(x).$$

Note that

$$\|m\| = \|\hat\nu_1\| = \nu\big(\{A\colon \min A = 1, \, |A| > 1\}\big) < \infty.$$

For $j \geq 1$, let

$$\mathcal{A}_j \coloneqq \operatorname{supp}(\nu_j) = \{A \subseteq \mathbb{N}\colon |A| > 1, \, \min A = j\}.$$

For $\mathcal{S} \subseteq \mathcal{A}_j$, we have

$$\nu_j(\mathcal{S}) = \nu(\mathcal{S}) = \nu(\sigma_{1j}\mathcal{S}) = \nu_1(\sigma_{1j}\mathcal{S}) = \hat\nu_1\big(\tau^{-1}(\sigma_{1j}\mathcal{S})\big)$$

$$= \int \Pi_x\big(\tau^{-1}(\sigma_{1j}\mathcal{S})\big) \, dm(x) = \int \Pi_x(\mathcal{S}) \, x^{-1} \, dm(x).$$

Letting $d\sigma(x) = x^{-1} \, dm(x)$, the desired conclusion immediately follows. □

4.2 Characterization in terms of infinite divisibility

We recall that a random variable Z on $[0,\infty)$ is infinitely divisible if its Laplace transform satisfies

$$\mathbb{E}[e^{-tZ}] = e^{-\gamma t} e^{\int_0^\infty (e^{-st}-1)\, d\hat{\sigma}(s)}, \quad t \geq 0 \tag{4.2}$$

where $\gamma \geq 0$, $\hat{\sigma}\big((\varepsilon,\infty]\big)$ is finite for all $\varepsilon > 0$, and $\int_0^1 s\, d\hat{\sigma}(s) < \infty$. Here $\hat{\sigma}$ is called the Levy measure of Z. The notion of an infinitely divisible distribution on $[0,\infty)$ easily extends to the case where the probability that $Z = \infty$ is strictly positive. This corresponds to the law $p\mathcal{L}(Z') + (1-p)\delta_\infty$, where Z' is infinitely divisible.

Our main theorem completely identifies permutation invariant random vectors in $\mathcal{R}$ with infinitely divisible distributions on $[0,\infty]$.

Theorem 4.5.

(a) *Assume that $X \in \mathcal{R}$ satisfies $X = X^\nu$ where ν is as in (4.1). Further, let μ be such that $X \sim \int_0^1 \Pi_p\, d\mu(p)$. Let $\varphi\colon x \mapsto -\log(1-x)$ map $[0,1]$ to $[0,\infty]$, and let $Z = \varphi(Q)$ where $Q \sim \mu$. Then*

   (i) *Z is infinitely divisible,*

   (ii) *$\hat{\sigma} = \sigma \circ \varphi^{-1}$ and*

   (iii) *$\gamma = \nu(\{1\})$,*

   where $\hat{\sigma}$ and γ are as in (4.2).

(b) *Let Z be an infinitely divisible distribution on $[0,\infty]$. Let $\varphi^{-1} = \phi\colon x \mapsto 1-e^{-x}$ map $[0,\infty]$ to $[0,1]$, and let $\mu = \mathcal{L}(\phi(Z))$. Let $X \sim \int_0^1 \Pi_p\, d\mu(p)$. Then*

   (i) *$X \in \mathcal{R}$,*

   (ii) *$\sigma = \hat{\sigma} \circ \phi^{-1}$, and*

   (iii) *$\nu(\{1\}) = \gamma$,*

   where $\hat{\sigma}$ and γ are as in (4.2) and $X = X^\nu$, where ν is as in (4.1).

Lemma 4.6. *Let $X \sim \int_0^1 \Pi_p\, d\mu(p)$. Assume that $X \in \mathcal{R}$ and let $\nu = c\sum_{i\in\mathbb{N}} \delta_{\{i\}} + \int_0^1 \Pi_x\, d\sigma(x)$ be such that $X = X^\nu$. Further, let $Q \sim \mu$, $y_0 := 1 - e^{-\nu(\{1\})}$, and $(Y_j)_{j\geq 1}$ be a Poisson point process with intensity σ. Then*

$$Q \overset{d}{=} 1 - (1-y_0)\prod_{j\geq 1}(1-Y_j).$$

Proof. Assume without loss of generality that $Y_1 \geq Y_2 \geq \dots$ Then, one observes that

$$X^\nu \mid (Y_j)_{j\geq 1} \sim \Pi_{1-(1-y_0)\prod_{j\geq 1}(1-Y_j)}.$$

Since we also have $X^\nu \sim \int_0^1 \Pi_p\, d\mu(p)$, it follows, since the set of permutation invariant measures is a simplex, that

$$1-(1-y_0)\prod_{j\geq 1}(1-Y_j) \sim \mu.$$

This concludes the proof. $\square$

Proof of Theorem 4.5. Let y_0 and $(Y_j)_{j\geq 1}$ be as in Lemma 4.6. Then, by this lemma, we have

$$Q \overset{d}{=} 1-(1-y_0)\prod_{j\geq 1}(1-Y_j),$$

and hence

$$Z=-\log(1-Q)\sim -\log(1-y_0)-\sum_{j\geq 1}\log(1-Y_j).$$

This implies that for $t\geq 0$ we have

$$\begin{aligned}
\mathbb{E}[e^{-tZ}] &= \mathbb{E}[e^{-t(-\log(1-y_0)-\sum_{j\geq 1}\log(1-Y_j))}] = \mathbb{E}[e^{t\log(1-y_0)+t\sum_{j\geq 1}\log(1-Y_j)}] \\
&= e^{t\log(1-y_0)}\mathbb{E}[e^{t\sum_{j\geq 1}\log(1-Y_j)}] = e^{t\log(1-y_0)}e^{\int_0^1 (e^{t\log(1-y)}-1)\,d\sigma(y)} \\
&= e^{t\log(1-y_0)}e^{\int_0^1 ((1-y)^t-1)\,d\sigma(y)} = e^{t\log(1-y_0)}e^{\int_0^\infty (e^{-st}-1)\,d(\sigma\circ\varphi^{-1})(s)},
\end{aligned} \tag{4.3}$$

where we use Campbell's formula in the third to last equality. This concludes the proof of (a).

To see that (b) holds, let ν be defined by

$$\nu := \sum_{j\in\mathbb{N}}\gamma\delta_{\{j\}} + \int_0^1 \Pi_x\,d(\hat\sigma\circ\phi^{-1})(x).$$

Let $X' := X^\nu$. Then, by construction, $X'\in\mathcal{R}$. We will now show that $X\overset{d}{=}X'$. To see this, let μ' be the de Finetti measure of X', $Q'\sim\mu'$, and $Z' := -\log(1-Q')$. Let $Q\sim\mu$ and $Z := -\log(1-Q)$. Since the set of permutation invariant measures is a simplex, it suffices to show that $\mu=\mu'$, or equivalently, that $Z\overset{d}{=}Z'$. Using (4.3) with Z replaced by Z', we obtain $\mathbb{E}[e^{-tZ}]=\mathbb{E}[e^{-tZ'}]$ for all $t\geq 0$. Hence $Z\overset{d}{=}Z'$. This concludes the proof of (b). $\square$

The following result is a direct consequence of Theorem 4.5.

Corollary 4.7. *Let $m\geq 2$, $0\leq p_1<\cdots<p_m\leq 1$ and $\alpha_1,\ldots,\alpha_m\in(0,1)$ be such that $\sum_{i=1}^m\alpha_i=1$. Let $X=(X_j)_{j\in\mathbb{N}}\sim\sum_{i=1}^m\alpha_i\Pi_{p_i}$. Then $X\in\mathcal{R}$ if and only if $m=2$ and $p_m=1$.*

Proof. We have $X\sim\int_0^1\Pi_p\,d\mu(p)$, where μ has finite support. Since the only finitely supported infinitely divisible random variables with no support at ∞ are constant, the desired conclusion follows from Theorem 4.5. $\square$

4.3 Examples

Example 4.8. Let σ be the Lebesgue measure on $[0,1]$. Further, let $\nu = \int_0^1 \Pi_x \, d\sigma(x)$, and let μ be such that $X^\nu \sim \int \Pi_p \, d\mu(p)$. Then, by Theorem 4.5(a), for $Q \sim \mu$ we have that $-\log(1-Q)$ is infinitely divisible with $\gamma = 0$ and

$$d\hat{\sigma}(x) = e^{-x} \, d\sigma(x) = e^{-x} \, dx.$$

On the other hand, assume that μ is of the form

$$\mu \sim e^{-1}\delta_0 + (1-e^{-1}) \int_0^1 \Pi_p f(p) \, dp,$$

for some "nice" probability density function f. Let $(Y_i)_{i\geq 1}$ be a Poisson point process with intensity σ, let $U_1, U_2, \dots \sim \mathrm{unif}(0,1)$ be i.i.d., and let $Q \sim \mu$. Then, using Lemma 4.6, for $t > 0$, we have

$$\begin{aligned} P(Q \leq t) &= P\big(1 - \prod_{i\geq 1}(1-Y_i) \leq t\big) = P\big(1-t \leq \prod_{i\geq 1}(1-Y_i)\big) \\ &= \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} P\big(1-t \leq \prod_{i=1}^{k}(1-U_i)\big) = \sum_{k=0}^{\infty} \frac{e^{-1}}{k!} P\big(1-t \leq \prod_{i=1}^{k} U_i\big) \\ &= 1 - \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} P\big(\prod_{i=1}^{k} U_i < 1-t\big) = 1 - \sum_{k=1}^{\infty} \frac{e^{-1}}{k!} \int_0^{1-t} \frac{(-\log u)^{k-1}}{(k-1)!} \, du. \end{aligned}$$

Taking derivatives of both sides, we obtain

$$f(t) = \frac{d}{dt} P(Q \leq t) = e^{-1} \sum_{k=1}^{\infty} \frac{(-\log(1-t))^{k-1}}{k!(k-1)!} = e^{-1} \frac{J_1\left(2\sqrt{-\log(1-t)}\right)}{\sqrt{-\log(1-t)}},$$

where J_1 is the Bessel function of the first kind with order 1.

Example 4.9. Let μ be the uniform measure on $[0,1]$ and let $X \sim \int_0^1 \Pi_p \, d\mu(p)$. Let $Q \sim \mu$ and let $Z := -\log(1-Q)$. Then

$$P(Z \leq t) = P(-\log(1-Q) \leq t) = P(Q \leq 1 - e^{-t}) = 1 - e^{-t},$$

and hence Z has density $f(t) = e^{-t}$. Since the exponential distribution is infinitely divisible, it follows from Theorem 4.5(b) that $X \in \mathcal{R}$. Note that it is well known and easily checked that the exponential distribution with parameter 1 corresponds to $\gamma = 0$ and $d\hat{\sigma}(t) = e^{-t}/t \, dt$ in (4.2). Again using Theorem 4.5(b), we further obtain in (4.1),

$$d\sigma(x) = e^{-(-\log(1-x))}/(-\log(1-x)) \cdot \frac{1}{1-x} \, dx = \frac{1}{-\log(1-x)} \, dx \tag{4.4}$$

and

$$\nu(\{1\}) = 0.$$

Example 4.10. Let μ be a Beta-distribution with parameters $\alpha, \beta \in (0,\infty)$ (i.e. let μ have density $x^{\alpha-1}(1-x)^{\beta-1}B(\alpha,\beta)^{-1}$) and let $Q \sim \mu$. Then $-\log(1-Q)$ is infinitely divisible (see e.g. [17, Example VI.12.21]) with $\gamma = 0$ and Levy measure

$$d\hat{\sigma}(t) = \frac{e^{-\beta t}(1-e^{-\alpha t})}{t(1-e^{-t})}\, dt.$$

Hence, by Theorem 4.5(b), $X \sim \int_0^1 \Pi_p \, d\mu(p)$ is in $\mathcal{R}$. Moreover, we get

$$d\sigma(x) = d\hat{\sigma}(-\log(1-x)) \cdot \frac{1}{1-x} = \frac{(1-x)^{\beta-1}(1-(1-x)^{\alpha})}{-x\log(1-x)}\, dx$$

and $\nu\big(\{1\}\big) = 0$. We recover Example 4.9 by letting $\alpha = \beta = 1$.

5 Finite permutation invariant processes

5.1 The Curie Weiss model and a finite permutation invariant result

In this section, we state and prove a theorem in the finite permutation invariant setting, which we then use to show that the supercritical Curie Weiss model is not in $\mathcal{R}$ for large n.

Theorem 5.1. *For each $n \geq 1$, let $X_n \in \{0,1\}^n$ be permutation invariant. Let $\bar{X}_n :=$ $\big(X_n(1) + \dots X_n(n)\big)/n$. Assume that there is $c \in (0,1)$, such that*

$$\lim_{n\to\infty} P(\bar{X}_n \geq c) = 0 \tag{5.1}$$

and that $X_n \in \mathcal{R}$ for each $n \geq 1$. Then

$$\lim_{n\to\infty} \operatorname{Var}(\bar{X}_n) = 0. \tag{5.2}$$

Proof. Let $\delta \in (0,1)$, and let $\mathcal{S}^\delta$ be the set of all subsets of $[n]$ with at least δn elements.

Let $\nu_n \geq 0$ be such that $X = X^{\nu_n}$ (such a ν_n exists since $X_n \in \mathcal{R}$).

We will first show that $\lim_{n\to\infty} \nu_n(\mathcal{S}^\delta) = 0$. To this end, let $c \in (0,1)$ be such that (5.1) holds and choose k such that $(1-\delta)^k < \frac{1-c}{2}$. We now condition on the event $\mathcal{E}_{k,n}$ that the Poisson random variable corresponding to the number of sets chosen from S^δ for the nth system is at least k. Note that

$$P(\mathcal{E}_{k,n}) = \sum_{\ell \geq k}^{\infty} \frac{e^{-\nu_n(\mathcal{S}^\delta)}\nu_n(\mathcal{S}^\delta)^\ell}{\ell!}.$$

One easily sees that for each $i \in [n]$,

$$P\big(X_n(i) = 1 \mid \mathcal{E}_{k,n}\big) \geq 1 - (1-\delta)^k \geq \frac{1+c}{2}.$$

Applying Markov's inequality conditioned on $\mathcal{E}_{k,n}$, we get

$$P\big(\bar{X}_n \geq c \mid \mathcal{E}_{k,n}\big) = 1 - P\Big(\sum_{i=1}^{n}\big(1 - X_n(i)\big) > (1-c)n \mid \mathcal{E}_{k,n}\Big)$$
$$\geq 1 - \frac{\mathbb{E}\big[\sum_{i=1}^{n}\big(1 - X_n(i)\big) \mid \mathcal{E}_{k,n}\big]}{(1-c)n} \geq 1 - \frac{(1-c)n/2}{(1-c)n} = 1/2.$$

From this, it follows that

$$P(\bar{X}_n \geq c) \geq \frac{P(\mathcal{E}_{k,n})}{2}.$$

Using (5.1), we obtain

$$\lim_{n\to\infty} P(\mathcal{E}_{k,n}) = 0.$$

which implies that $\lim_{n\to\infty} \nu_n(\mathcal{S}^\delta) = 0$. In particular, $\lim_{n\to\infty} \nu_n(\mathcal{S}_{[2]}^\cap) = 0$.

Now note that

$$\mathbb{E}[\bar{X}_n] = \mathbb{E}\big[X_n(1)\big] = 1 - e^{-\nu_n(\mathcal{S}_1)} \leq 1$$

and that

$$\mathbb{E}[\bar{X}_n^2] = n^{-2}\mathbb{E}\Big[\big(\sum_{i=1}^{n} X_n(i)\big)^2\Big] = n^{-2}\Big(n\mathbb{E}[X_n(1)^2] + n(n-1)\mathbb{E}\big[X_n(1)X_n(2)\big]\Big)$$
$$= n^{-2}\big(n\mathbb{E}\big[X_n(1)\big] + n(n-1)\mathbb{E}\big[X_n(1)X_n(2)\big]\big)$$
$$= n^{-2}\Big(n\mathbb{E}[X_n(1)] + n(n-1)\Big(\big(1 - e^{-\nu_n(\mathcal{S}_{[2]}^\cap)}\big) + e^{-\nu_n(\mathcal{S}_{[2]}^\cap)}\big(1 - e^{-(\nu_n(\mathcal{S}_1) - \nu_n(\mathcal{S}_{[2]}^\cap))}\big)^2\Big)\Big)$$
$$\xrightarrow{n\to\infty} 0 + \Big(\big(1 - e^{-0}\big) + e^{-0}\big(1 - e^{-(\nu_n(\mathcal{S}_1) - 0)}\big)^2\Big) = \lim_{n\to\infty} \mathbb{E}[\bar{X}_n]^2.$$

From this the desired conclusion immediately follows. □

Theorem 5.2. *Let $\mu_{J,n}$ be the Curie-Weiss model on $[n]$ with parameter $J = \beta/n$, so that*

$$\mu_{J,n}(\sigma) = Z_{J,n}^{-1} e^{J\sum_{i<j}\sigma_i\sigma_j}, \quad \sigma \in \{-1,1\}^n.$$

Let $X_n \in \{0,1\}^n$ be the corresponding $\{0,1\}$-valued random vector where we have identified -1 with 0. If $\beta > \beta_c$ and n is sufficiently large, then the Curie-Weiss model is not in $\mathcal{R}$.

Proof. It is well known that $\bar{X}_n$ concentrates at $1/2 \pm c_\beta$ as $n \to \infty$, where

1. $c_\beta = 0$ if $\beta \leq \beta_c$.
2. $0 < c_\beta < \frac{1}{2}$ if $\beta > \beta_c$.

Applying Theorem 5.1, the desired conclusion immediately follows. □

5.2 Finite averages of product measures

In Corollary 4.7, we considered finite averages of product measures, and showed that these were not in $\mathcal{R}$ except in very few special cases. In this setting, given $X \notin \mathcal{R}$, there is an $N \geq 3$ such that $X([n]) \in \mathcal{R}$ for $n < N$ and $X([n]) \notin \mathcal{R}$ for $n \geq N$. The reason for the "3" is that any average of product measures X is positively associated and hence by the discussion after Lemma 2.14, $X([2])$ is always in $\mathcal{R}$.

Our averages of product measures have precisely $2m-1$ parameters. The next theorem says that we can reduce it to $2m-2$ parameters. This might not seem like a giant improvement, but when $m = 2$, we then have only two parameters. This is the case which we will analyze in most detail and we can then visualize the phase diagram reasonably well since it is just two-dimensional. In this theorem, we use the following notation. Given $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ such that $1 = x_1 > x_2 > \cdots > x_m \geq 0$, $q \in (0,1]$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1)^m$ such that $\sum_{i=1}^m \alpha_i = 1$, we let

$$X^{q,\mathbf{x},\boldsymbol{\alpha}} \sim \sum_{i=1}^m \alpha_i \Pi_{1-qx_i}.$$

Using this particular form for the product measures is advantageous since, as stated in the following theorem, being representable turns out to be independent of q.

Theorem 5.3. *Let $n \geq 2$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be such that $1 = x_1 > x_2 > \cdots > x_m \geq 0$, $q \in (0,1]$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1)^m$ be such that $\sum_{i=1}^m \alpha_i = 1$. Then $X^{q,\mathbf{x},\boldsymbol{\alpha}}([n]) \in \mathcal{R}$ if and only if $X^{q',\mathbf{x},\boldsymbol{\alpha}}([n]) \in \mathcal{R}$ for all $q' \in (0,1]$.*

The proof of Theorem 5.3 will use the following lemma.

Lemma 5.4. *Let $n \geq 2$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be such that $1 = x_1 > x_2 > \cdots > x_m \geq 0$, $q \in (0,1]$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1)^m$ be such that $\sum_{i=1}^m \alpha_i = 1$. Let ν_n be the unique signed measure corresponding to $X^{q,\mathbf{x},\boldsymbol{\alpha}}([n])$. Then, for $\ell \in [n]$,*

$$\nu_n\big([\ell]\big) = \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \log \sum_{i=1}^m \alpha_i (qx_i)^{n-j} \tag{5.3}$$

$$= \begin{cases} -\log q + \log \frac{\sum_{i=1}^m \alpha_i x_i^{n-1}}{\sum_{i=1}^m \alpha_i x_i^n} & \text{if } \ell = 1 \\ \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \log \sum_{i=1}^m \alpha_i x_i^{n-j} & \text{if } \ell \in \{2, 3, \ldots, n\}. \end{cases} \tag{5.4}$$

Moreover, $\nu_n(\{1\}) \geq -\log q$ and $\nu_n([2]) \geq 0$.

Proof. For $j \in [n]$, we have

$$P\big(X([n] \smallsetminus [j]) \equiv 0\big) = \sum_{i=1}^m \alpha_i (qx_i)^{n-j}.$$

Using (2.3), we obtain (5.3). From this, noting that $\sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} (n-j) \log q = 0$ if $\ell \geq 2$, we obtain (5.4). From this, using the Cauchy-Schwarz inequality for the second inequality, we obtain

$$\nu_n\big(\{1\}\big) = -\log q + \log \frac{\sum_{i=1}^m \alpha_i x_i^{n-1}}{\sum_{i=1}^m \alpha_i x_i^n} \geq -\log q,$$

$$\nu_n(\{1,2\}) = \log \frac{\sum_{i=1}^m \alpha_i x_i^{n-2} \sum_{i=1}^m \alpha_i x_i^n}{(\sum_{i=1}^m \alpha_i x_i^{n-1})^2} \geq 0.$$

□

Proof of Theorem 5.3. Assume that $X^{q,\mathbf{x},\boldsymbol{\alpha}}([n]) \in \mathcal{R}$. Then there is a measure ν on $\mathcal{P}([n]) \smallsetminus \{\emptyset\}$ such that $X^{q,\mathbf{x},\boldsymbol{\alpha}}([n]) = X^\nu$. Let ν' be the unique signed measure corresponding to $X^{q',\mathbf{x},\boldsymbol{\alpha}}$. Let $A \subseteq [n]$ satisfy $|A| > 1$. By Lemma 5.4, we then have that $\nu(A) = \nu'(A)$, and hence, since ν is a non-negative measure it follows that $\nu'(A) \geq 0$. Finally, we note that, again by Lemma 5.4, we have $\nu'(\{1\}) \geq -\log q' \geq 0$. Hence ν' is a non-negative measure, implying that $X^{q',\mathbf{x},\boldsymbol{\alpha}}([n]) \in \mathcal{R}$. This concludes the proof. □

The following proposition begins to give information about the phase diagram for when $X([n]) \in \mathcal{R}$ by explaining what happens when α_1 is sufficiently close to zero or one and when x_2 is sufficiently close to zero (see Figure 1). (Recall that by Theorem 5.3, for any n, $X([n])$ being in $\mathcal{R}$ is independent of q.)

Proposition 5.5. *Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be such that $1 = x_1 > x_2 > \cdots > x_m \geq 0$, $q \in (0,1]$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (0,1)^m$ be such that $\sum_{i=1}^m \alpha_i = 1$. Let $X \sim \sum_{i=1}^m \alpha_i \Pi_{1-qx_i}$ and $n \geq 3$.*

(a) If $m \geq 2$, then $X([n]) \in \mathcal{R}$ if x_2 is sufficiently close to 0.

(b) If $m \geq 2$, then $X([n]) \in \mathcal{R}$ if α_1 is sufficiently close to 1.

(c) If $m = 2$, then $X([n]) \notin \mathcal{R}$ if $x_2 > 0$ and α_1 is sufficiently close to 0. (This is not true in general for $m \geq 3$.)

Proof. Assume first that $x_2 = 0$. In this case, one verifies that

$$\nu_n\big([k]\big) = \begin{cases} -\log q & \text{if } k = 1 \\ -\log \alpha_1 & \text{if } k = n \\ 0 & \text{otherwise,} \end{cases}$$

and hence $X([n]) \in \mathcal{R}$. In the rest of the proof, we therefore assume that $x_2 > 0$.

We first show that (a) holds. To this end, assume that x_2 is small enough so that $\alpha_1^{-1} \sum_{i=2}^m \alpha_i x_i < 1$. Then, using (5.4), a Taylor expansion, and the assumption that

$\sum_{i=1}^m \alpha_i = 1$, we obtain that, for any $k \in \{2, 3, \dots, n\}$,

$$\begin{aligned}
\nu_n\big([k]\big) &= \sum_{j=0}^{\min(k,n-1)} (-1)^{k-j} \binom{k}{j} \log\big(\alpha_1 + \sum_{i=2}^{m} \alpha_i x_i^{n-j}\big) \\
&= \sum_{j=0}^{\min(k,n-1)} (-1)^{k-j} \binom{k}{j} \Big(\log \alpha_1 + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \big(\alpha_1^{-1} \sum_{i=2}^{m} \alpha_i x_i^{n-j}\big)^{\ell}\Big) \\
&= \begin{cases} \alpha_1^{-1} \sum_{i=2}^{m} \alpha_i x_i^{n-k} + O(x_2^{n-k+1}) & \text{if } k \in \{2, 3, \dots, n-1\} \\ -\log \alpha_1 + O(x_2) & \text{if } k = n. \end{cases}
\end{aligned}$$

Finally, we note that by Lemma 5.4, $\nu_n(\{1\}) \geq -\log q$. From this (a) immediately follows.

We now show that (b) holds. To this end, assume that α_1 is close enough to 1 to ensure that $\alpha_1^{-1} \sum_{i=2}^{m} \alpha_i x_i < 1$. Then, for $k \in \{2, 3, \dots, n\}$, as above, it follows that for $\alpha_1 \to 1$,

$$\begin{aligned}
\nu_n\big([k]\big) &= \sum_{j=0}^{\min(k,n-1)} (-1)^{k-j} \binom{k}{j} \Big(\log \alpha_1 + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \big(\alpha_1^{-1} \sum_{i=2}^{m} \alpha_i x_i^{n-j}\big)^{\ell}\Big) \\
&= \alpha_1^{-1} \sum_{i=2}^{m} \alpha_i x_i^{n-k} (1 - x_i)^k + O((1-\alpha_1)^2).
\end{aligned}$$

The last equality, when $k = n$, uses the observation that $\log \alpha_1 + \alpha_1^{-1}(1-\alpha_1) = O((1-\alpha_1)^2)$. Since $\nu_n(\{1\}) \geq -\log q$ by Lemma 5.4, it follows that $X([n]) \in \mathcal{R}$ when α_1 is close to 1. This concludes the proof of (b).

We now show that (c) holds. To this end, assume that $m = 2$, and that α_1 is close enough to 0 to ensure that $\alpha_1 < (1-\alpha_1)x_2^n$. Then, for $k \in \{2, 3, \dots, n\}$, by (5.4) and a Taylor expansion, it follows that for $\alpha_1 \to 0$,

$$\begin{aligned}
\nu_n\big([k]\big) &= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \log\big(\alpha_1 + \alpha_2 x_2^{n-j}\big) \\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \Big(\log(\alpha_2 x_2^{n-j}) + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \big(\alpha_1/(\alpha_2 x_2^{n-j})\big)^{\ell}\Big) \\
&= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \big(\log(\alpha_2 x_2^{n-j}) + \alpha_1/(\alpha_2 x_2^{n-j})\big) + O(\alpha_1^2).
\end{aligned}$$

Noting that

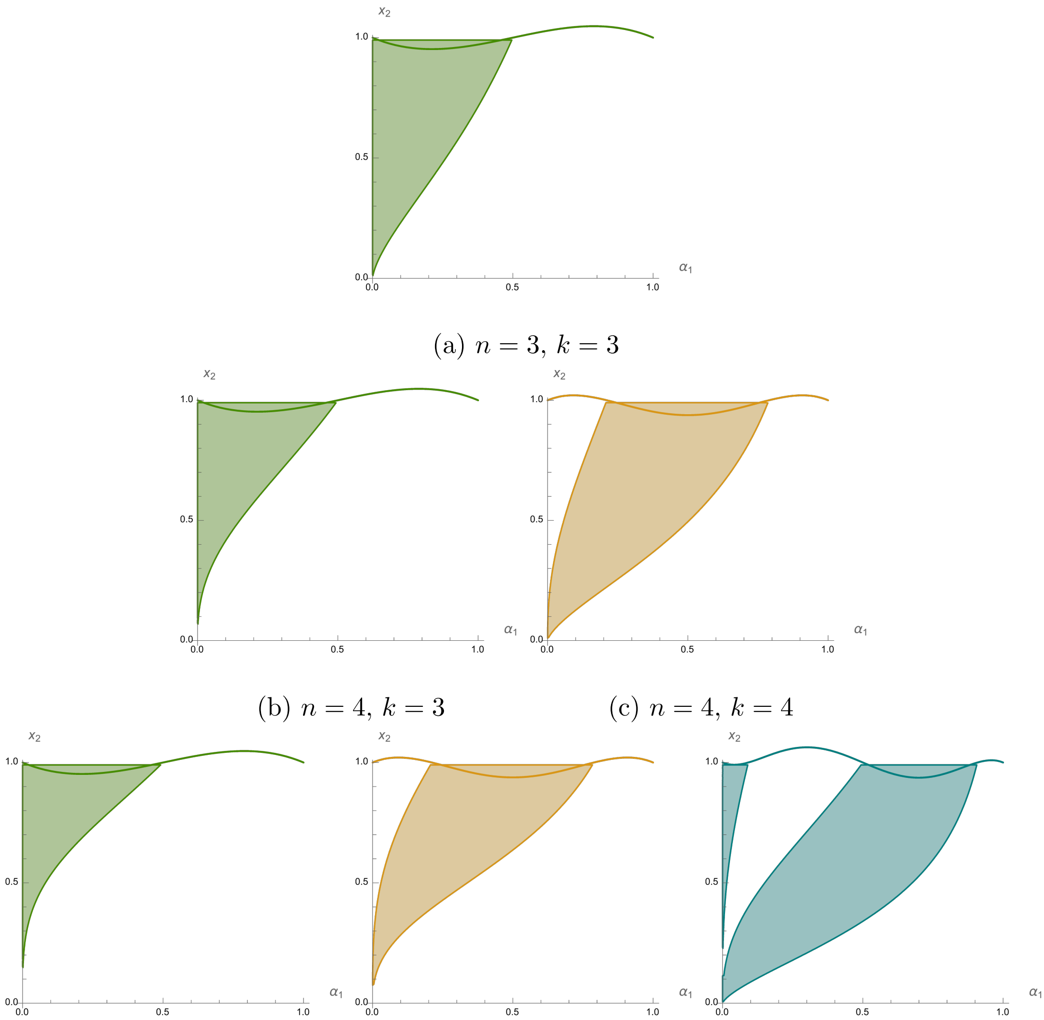
$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \log \alpha_2 = 0 = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n-j) \log x_2,$$

it follows that

$$\nu_n\big([k]\big) = (-1)^k \frac{\alpha_1}{1-\alpha_1} x_2^{-n} (1-x_2)^k + O(\alpha_1^2).$$

This concludes the proof of (c). $\square$

The case where $m = 2$ and x_2 is taken to be close to 1 (corresponding to two product measures having similar densities) is the most intriguing and undergoes a phase transition in α_1 where the critical value corresponds to the largest negative zero of the so-called polylogarithm function as we will see in the following theorem. In addition, this result will later be used to show that certain positively associated tree-indexed Markov chains, as well as the Ising model with certain parameters, are not in $\mathcal{R}$.



(a) $n = 3$, $k = 3$

(b) $n = 4$, $k = 3$ (c) $n = 4$, $k = 4$

(d) $n = 5$, $k = 3$ (e) $n = 5$, $k = 4$ (f) $n = 5$, $k = 5$

Figure 1: In the figures above, we draw the regions where $\nu_n([k]) < 0$ as a function of (our only parameters) x_2 (on the y-axis) and α_1 (on the x-axis) for $m = 2$, $n = 3, 4, 5$, and $k \in \{3, \ldots, n\}$. The curves along the x-axis are the polylogarithm functions which appear in Theorem 5.6(a).

In this result, we will use Li_s to denote the polylogarithm function with index $s \in \mathbb{R}$, defined for $z \in \mathbb{C}$ with $|z| < 1$ by

$$\mathrm{Li}_s(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

and extended to all $z \in \mathbb{C}$ (except for poles) by analytic continuation. When s is a negative integer the function $\mathrm{Li}_s(z)$ is a rational function.

The following properties of the polylogarithm functions are standard and easy to show.

1. For all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$, we have $\mathrm{Li}_j'(x) = \mathrm{Li}_{j-1}(x)/x$.
2. For all $n \geq 2$, Li_{1-n} has exactly n distinct non-positive roots $r_1^{(n)} = 0 > r_2^{(n)} > \cdots > r_n^{(n)} = -\infty$ which satisfy the interlacement property
$$0 = r_1^{(n+1)} = r_1^{(n)} > r_2^{(n+1)} > r_2^{(n)} > \cdots > r_n^{(n+1)} > r_n^{(n)} = r_{n+1}^{(n+1)} = -\infty.$$
3. For all $n \geq 2$, $\mathrm{Li}_{1-n}(x) < 0$ for all $x \in (r_2^{(n)}, 0)$.
4. For all $n \geq 3$, $\mathrm{Li}_{1-n}(x) > 0$ for all $x \in (r_3^{(n)}, r_2^{(n)})$. Hence, for any $n \geq 4$, since $r_2^{(n-1)} \in (r_3^{(n)}, r_2^{(n)})$, we have $\mathrm{Li}_{1-n}(r_2^{(n-1)}) > 0$.
5. $\lim_{n\to\infty} r_2^{(n)} = 0$.

One verifies that $r_2^{(3)} = -1$, $r_2^{(4)} = \sqrt{3} - 2$, and $r_2^{(5)} = 2\sqrt{6} - 5$.

In the next theorem, we describe what happens when $m = 2$ and x_2 is close to 1.

Theorem 5.6. *Let $\mathbf{x} = (x_1, x_2)$ be such that $1 = x_1 > x_2 \geq 0$, $q \in (0, 1]$, and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (0, 1)^2$ be such that $\alpha_1 + \alpha_2 = 1$. Let $X \sim \alpha_1 \Pi_{1-qx_1} + \alpha_2 \Pi_{1-qx_2}$ and $n \geq 3$. (Recall that by Theorem 5.3, for any n, $X([n])$ being in $\mathcal{R}$ is independent of q.)*

(a) Let $k \in \{3, \ldots, n\}$.

- *If $\mathrm{Li}_{1-k}\big(-\alpha_1^{-1}(1 - \alpha_1)\big) < 0$, then $\nu_n\big([k]\big) > 0$ for x_2 sufficiently close to 1.*
- *If $\mathrm{Li}_{1-k}\big(-\alpha_1^{-1}(1 - \alpha_1)\big) > 0$, then $\nu_n\big([k]\big) < 0$ for x_2 sufficiently close to 1.*

(If $\mathrm{Li}_{1-k}(-\alpha_1(1 - \alpha_1)) = 0$, then the sign of $\nu_n([k])$ depends on which zero of Li_{1-k} one is considering.)

(b) $\alpha_1 \geq 1/(1-r_2^{(n)})$ if and only if $X([n]) \in \mathcal{R}$ for x_2 sufficiently close to 1 (see Figure 1).

Proof. We first show that (a) holds. In this case, by (5.3)

$$\begin{aligned}\nu_n([k]) &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\log\bigl(\alpha_1+(1-\alpha_1)x_2^{n-j}\bigr)\\ &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\log\bigl(1+\alpha_1^{-1}(1-\alpha_1)x_2^{n-j})\bigr)\\ &= (-1)^{k+1}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\operatorname{Li}_1\bigl(-\alpha^{-1}(1-\alpha_1)x_2^{n-j}\bigr).\end{aligned}$$

One verifies that for any $\ell \in \{0,1,\dots,k+1\}$, we have

$$\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}(n-j)^{\ell} = \begin{cases} 0 & \text{if } \ell < k\\ k! & \text{if } \ell = k\\ (n-k/2)(k+1)! & \text{if } \ell = k+1.\end{cases}$$

Combining these observations and Property 1 of the polylogarithm functions, we obtain

$$\begin{aligned}&\frac{d^{\ell}}{dx_2^{\ell}}\nu_n([k])|_{x_2=1}\\ &= \begin{cases} 0 & \text{if } \ell < k\\ k!(-1)^{k+1}\operatorname{Li}_{1-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr) & \text{if } \ell = k,\\ \frac{(k+1)!(-1)^{k+1}}{2}\Bigl[(2n-k)\operatorname{Li}_{-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr)-k\operatorname{Li}_{1-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr)\Bigr] & \text{if } \ell = k+1.\end{cases}\end{aligned}$$

(We will not need the case $\ell = k+1$ for (a), but we will need it for (b) and hence include it here.) Using a Taylor expansion of $\nu_n([k])$ around $x_2 = 1$, it follows that $\nu_n([k]) < 0$ for x_2 near 1 if

$$0 > \frac{d^k}{dx_2^k}\nu_n([k])\big|_{x_2=1}(-1)^k = -k!\operatorname{Li}_{1-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr).$$

Hence, if

$$\operatorname{Li}_{1-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr) > 0,$$

then $\nu_n([k]) < 0$ for x_2 near 1. Analogously, if

$$\operatorname{Li}_{1-k}\bigl(-\alpha_1^{-1}(1-\alpha_1)\bigr) < 0,$$

then $\nu_n([k]) > 0$ for x_2 near 1. This completes the proof of (a).

We now show that (b) holds. Note that $\alpha_1 < 1/(1-r_2^{(n)}) \Leftrightarrow -\alpha_1^{-1}(1-\alpha_1) < r_2^{(n)} = r_2^{(n)}$. If $\alpha_1 > 1/(1-r_2^{(n)})$, then $\operatorname{Li}_{1-k}(-\alpha_1(1-\alpha_1)) < 0$ for all $k \in \{2,3,\dots,n\}$, and so, by (a), we have $\nu_n([k]) > 0$ for x_2 close to 1 for all $k \in \{1,2,\dots,n\}$. Hence $X([n]) \in \mathcal{R}$ for x_2 close to 1. Conversely, if $\alpha_1 < 1/(1-r_2^{(n)})$, then there is at least one $k \in \{3,4,\dots,n\}$ such that $-\alpha_1^{-1}(1-\alpha_1) \in (r_3^{(k)}, r_2^{(k)})$, and hence $\operatorname{Li}_{1-k}(-\alpha_1(1-\alpha_1)) > 0$. For this k, by (a), we have $\nu_n([k]) < 0$ for x_2 close to 1, and hence $X([n]) \notin \mathcal{R}$ for x_2 close to 1. Finally, if $\alpha_1 = 1/(1 - r_2^{(n)})$, then by the above, we have that $\nu_n([k]) > 0$ for x_2 close to 1 for all $k \in \{1, 2, \ldots, n-1\}$. Noting that for this α_1, we have $\mathrm{Li}_{1-n}(-\alpha_1^{-1}(1-\alpha_1)) = 0$ and $\mathrm{Li}_{-n}\big(-\alpha_1^{-1}(1-\alpha_1)\big) > 0$, and so we obtain $\nu_n([n]) > 0$ for x_2 close to 1 by using a Taylor expansion of $\nu_n([n])$ around $x_2 = 1$ of degree $n+1$ and Property 4 of the polylogarithm functions. Hence $X([n]) \in \mathcal{R}$ for x_2 close to 1 . Finally, if $\alpha_1 = 1/(1 - r_2^{(n)})$, then by the above, we have that $\nu_n([k]) > 0$ for x_2 close to 1 for all $k \in \{1, 2, \ldots, n-1\}$. Noting that for this α_1, we have $\mathrm{Li}_{1-n}(-\alpha_1^{-1}(1-\alpha_1)) = 0$ and $\mathrm{Li}_{-n}\big(-\alpha_1^{-1}(1-\alpha_1)\big) > 0$, and so we obtain $\nu_n([n]) > 0$ for x_2 close to 1 by using a Taylor expansion of $\nu_n([n])$ around $x_2 = 1$ of degree $n+1$ and Property 4 of the polylogarithm functions. Hence $X([n]) \in \mathcal{R}$ for x_2 close to 1. Finally, if $\alpha_1 = 1/(1 - r_2^{(n)})$, then by the above, we have that $\nu_n([k]) > 0$ for x_2 close to 1 for all $k \in \{1, 2, \ldots, n-1\}$. Noting that for this α_1, we have $\mathrm{Li}_{1-n}(-\alpha_1^{-1}(1-\alpha_1)) = 0$ and $\mathrm{Li}_{-n}\big(-\alpha_1^{-1}(1-\alpha_1)\big) > 0$, and so we obtain $\nu_n([n]) > 0$ for x_2 close to 1 by using a Taylor expansion of $\nu_n([n])$ around $x_2 = 1$ of degree $n+1$ and Property 4 of the polylogarithm functions. Hence $X([n]) \in \mathcal{R}$ for x_2 close to 1. This concludes the proof of (b). $\square$

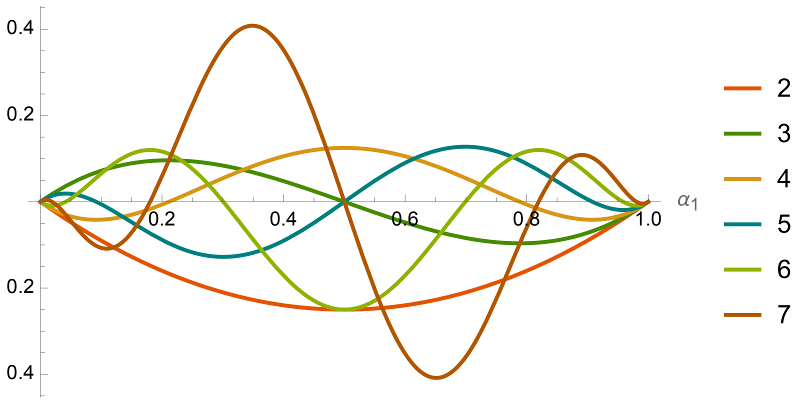


Figure 2: In the above figure, we draw $\mathrm{Li}_{1-k}(-\alpha_1^{-1}(1-\alpha_1))$ for $k = 2, 3, 4, 5, 6, 7$ and $\alpha_1 \in (0, 1)$. Using Theorem 5.6, one verifies that when $n = 3, 4, 5, 6, 7$ we get the following thresholds in α_1 for getting $X([n]) \in \mathcal{R}$ when x_2 is close to one: $\frac{1}{2}$, $\frac{1}{2} + \frac{1}{2\sqrt{3}}$, $\frac{1}{2} + \frac{1}{\sqrt{6}}$, $\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\sqrt{105}+15}{30}}$, $\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\sqrt{15}+10}{15}}$ (approximately equal to 0.5, 0.788675, 0.908248, 0.958684, 0.98085 respectively).

6 Tree-indexed Markov chains and the Ising model

In this section, we apply results from the previous section to obtain results for tree-indexed Markov chains and the Ising model on $\mathbb{Z}^d$, $d \geq 2$.

Theorem 6.1. *Let $d \geq 3$ and let $\mathcal{T}$ be a tree where some vertex o has at least d infinite disjoint paths emanating from it. Let $X^{p,r}$ be the positively associated tree-indexed Markov chain on $\mathcal{T}$ where p and r as in Remark 3.2 belong to $(0,1)$. If $r < 1/(1 - r_2^{(d)})$, where $r_2^{(d)}$ is as in the paragraph before Theorem 5.6, then $X^{p,r} \notin \mathcal{R}$.*

Note that since $r_2^{(d)}$ approaches 0 as $d \to \infty$, we have that for any $p, r \in (0,1)$, there is some d such that $X^{p,r} \notin \mathcal{R}$.

Proof. For $k \geq 1$, let $L_k = \{\ell_1^{(k)}, \ell_2^{(k)}, \ldots, \ell_d^{(k)}\}$ be a set of vertices in $\mathcal{T}$ such that for each $j \in [d]$,

(i) $\mathrm{dist}(\ell_j^{(k)}, o) = k$, and

(ii) the paths between o and $\ell_j^{(k)}$, $j \in [d]$, are disjoint.

Note that conditioned on $X_o^{p,r}$, the random variables $X^{p,r}_{\ell_1^{(k)}}, \ldots, X^{p,r}_{\ell_d^{(k)}}$ are independent and identically distributed. Moreover, if we let

$$\begin{cases} p_0^{(k)} := P(X^{p,r}_{\ell_1^{(k)}} = 1 \mid X_o^{p,r} = 0) \\ p_1^{(k)} := P(X^{p,r}_{\ell_1^{(k)}} = 1 \mid X_o^{p,r} = 1), \end{cases}$$

then

$$X|_{L_k} \sim P(X_o = 0)\Pi_{p_0^{(k)}} + P(X_o = 1)\Pi_{p_1^{(k)}}.$$

Note that $p_0^{(k)} < p_1^{(k)}$ and that

$$\lim_{k\to\infty} \frac{1 - p_1^{(k)}}{1 - p_0^{(k)}} = 1.$$

Using Theorem 5.6(b), it follows that for $r < \frac{1}{1 - r_2^{(d)}}$ and for any sufficiently large k, $X|_{L_k} \notin \mathcal{R}$, and hence, by Lemma 2.14(a), $X \notin \mathcal{R}$. $\square$

Remark 6.2. *Interestingly, we can show that if the tree $\mathcal{T}$ consists of one vertex with three infinite rays emanating from the vertex, then, for r close enough to 1, one has that for all $p \in (0,1)$, the Markov chain indexed by $\mathcal{T}$ belongs to $\mathcal{R}$. In light of Theorem 6.1, this tree therefore exhibits a phase transition in r, which we do not believe holds for the binary tree. This will be elaborated on and extended in future work. In fact, one can check that for the tree $\mathcal{T}$ with four vertices and three leaves, the set of (p, r) for which the tree-indexed Markov chain on T is not in $\mathcal{R}$ is a non-trivial subset of $(0,1)^2$.*

Theorem 6.3. *Let X be the Ising model on $\mathbb{Z}^d$, $d \geq 2$, (identifying -1 with 0), and coupling constant $J > 0$. Then, if*

$$\frac{e^{2dJ}}{e^{2dJ} + e^{-2dJ}} \leq \frac{1}{1 - r_2^{(2d)}}$$

where $r_2^{(2d)}$ is as in the paragraph before Theorem 5.6, we have that $X \notin \mathcal{R}$.

Proof. Let $o := (0, 0, \ldots, 0)$ and for $k \geq 1$, let

$$L_k := \{(\pm k, 0, \ldots, 0), (0, \pm k, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, \pm k)\}.$$

Next, let C_k consist of o and the $2d$ direct paths from o to the points in L_k and finally, let $X^{(k)}$ be the restriction of X conditioned to be zero on the set $\mathbb{Z}^2 \backslash C_k$, to C_k. Then, conditioned on $X_o^{(k)}$ the random variables $\{X_\ell^{(k)}\}_{\ell \in L_k}$ are independent and identically distributed. Moreover, if we, for $\ell \in L_k$, let

$$\begin{cases} p_0^{(k)} := P(X_\ell^{(k)} = 1 \mid X_o^{(k)} = 0) = P\big(X_\ell = 1 \mid X_o = 0,\, X(\mathbb{Z}^2 \backslash C_k) \equiv 0\big) \\ p_1^{(k)} := P(X_\ell^{(k)} = 1 \mid X_o^{(k)} = 1) = P\big(X_\ell = 1 \mid X_o = 1,\, X(\mathbb{Z}^2 \backslash C_k) \equiv 0\big) \end{cases}$$

then

$$X^{(k)}|_{L_k} \sim P(X_o^{(k)} = 0)\Pi_{p_0^{(k)}} + P(X_o^{(k)} = 1)\Pi_{p_1^{(k)}}.$$

Note that since $J > 0$, for every $k \geq 1$, $p_0^{(k)} \neq p_1^{(k)}$, and moreover

$$\lim_{k \to \infty} \frac{1 - p_1^{(k)}}{1 - p_0^{(k)}} = 1.$$

Finally, observe that $P(X_o^{(k)} = 0) \leq \frac{e^{2dJ}}{e^{2dJ} + e^{-2dJ}}$. Using Theorem 5.6(b), it follows that for any sufficiently large k, $X^{(k)}|_{L_k} \notin \mathcal{R}$, and hence, by Lemma 2.14(a) and (b), $X \notin \mathcal{R}$. $\square$

Remark 6.4. *For $d = 2$, if we look at the critical value $J_c = 1/2 \log(1 + \sqrt{2}) = 0.440687$, one has that $\frac{e^{4J_c}}{e^{4J_c} + e^{-4J_c}} > \frac{3+\sqrt{3}}{6} = \frac{1}{1 - r_2^{(4)}}$ and hence the above theorem is only applicable within a subset of the subcritical regime (although we have no suspicions whatsoever that the conclusion fails somewhere). For higher dimensions, it is known that $J_c(d) \asymp \frac{1}{d}$ as $d \to \infty$ and since $r_2^{(2d)}$ goes to 0 as $d \to \infty$, we can conclude that for all sufficiently high dimensions, the above theorem rules out being in $\mathcal{R}$ throughout the subcritical regime and partly into the supercritical regime. The above result can with small modifications be extended to the Ising model with an external field.*

7 The stationary case

In this section, we consider the stationary case.

After having obtained the results in this section, we learned from Nachi Avraham-Re'em and Michael Björklund that most of the results in this section follow from known results in the theory of so-called Poisson suspensions, an area within ergodic theory and primarily within infinite measure ergodic theory. See for example [13, 14]. Since this section is only four pages, we decided to leave it as is, providing fairly direct proofs of the results stated, in the spirit of the rest of the paper, rather than introducing the notion of a Poisson suspension and refer to the relevant theorems in the literature. For example, while Theorem 7.5 would follow from the known and easy result that "Poisson suspensions of dissipative systems are Bernoulli", the proof here gives an easy factor map from an i.i.d. process to our system.

We begin by giving both sufficient and necessary conditions on ν for X^ν to be ergodic. For simplicity, we stick to one-dimensional processes.

We start with the simplest necessary condition as a warm-up.

Proposition 7.1. *Assume that ν is finite and not the zero measure. Then X^ν is not ergodic.*

Proof. Letting $\tilde{\nu}$ be ν renormalized and $Y = (Y_i)$ be the $\{0,1\}$-stationary process with law given by $\tilde{\nu}$ (recall that we identify $\{0,1\}$-sequences with subsets of $\mathbb{Z}$ in the natural way). Then the distribution of X^ν can be expressed as

$$\max\{Y^1, Y^2, \ldots, Y^N\},$$

where $Y^1, Y^2, \ldots$ are independent copies of Y, N is a Poisson random variable with mean $\|\nu\|$ independent of $Y^1, Y^2, \ldots$ and max is taken pointwise. As a consequence, the distribution of X^ν can be expressed as an average of the distributions of $\max\{Y^1, Y^2, \ldots, Y^k\}$ for $k \geq 1$, where each of the latter is a stationary process. Clearly $X^\nu \equiv 0$ with positive probability and since, ν is not the zero measure, $P(Y \equiv 0) < 1$ which easily implies that $P(X^\nu \equiv 0) < 1$. Hence X^ν is not ergodic. $\square$

Remark 7.2. *If we let μ_k be the law of $\max\{Y^1, Y^2, \ldots, Y^k\}$ for $k \geq 1$ and μ_0 the Dirac measure on the sequence consisting only of 0's, the distribution of X^ν is given by*

$$\sum_{k=0}^{\infty} \frac{\|\nu\|^k e^{-\|\nu\|}}{k!} \mu_k.$$

If Y is ergodic, then the above will often, but not always, yield the ergodic decomposition of X^ν. Reasons why it does not in general are

(1) if $Y \equiv 1$ a.s., then (and only then) $\mu_1 = \mu_k$ for all $k \geq 1$, and

(2) it is possible that μ_k is not ergodic and must be further decomposed but this cannot happen if Y is weak-mixing.

The next theorem strengthens Proposition 7.1 and provides two equivalent conditions for ergodicity. Before stating the theorem, we define

$$\mathcal{Z} := \Big\{\eta \in \{0,1\}^{\mathbb{Z}} \colon \lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \eta_k}{n} = 0\Big\}.$$

In other words, $\mathcal{Z}$ consists of the configurations with zero density. It is immediate that $\mathcal{Z}$ is a translation invariant set.

Theorem 7.3. *Assume that $P(X^\nu \equiv 1) < 1$, or equivalently, that $\nu(\mathcal{S}_0) < \infty$. Then the following are equivalent.*

(i) ν cannot be expressed as $\nu_1 + \nu_2$ where ν_1 and ν_2 are translation invariant and ν_2 is a nonzero finite measure

(ii) $\nu(\mathcal{Z}^c) = 0$

(iii) X^ν is ergodic.

Proof. (iii) implies (i).
Assume that $\nu = \nu_1 + \nu_2$ where ν_1 and ν_2 are translation invariant and ν_2 is a nonzero finite measure. By Lemma 2.10, if we let X^{ν_1} and X^{ν_2} be independent, then X^ν and $\max\{X^{\nu_1}, X^{\nu_2}\}$ have the same distribution. Since ν_2 is a finite measure, $X^{\nu_2} \equiv 0$ with positive probability, and hence the density of X^ν is the same as the density of X^{ν_1} with positive probability. Since ν_2 is nonzero, X^{ν_2} has a positive density with positive probability and since X^{ν_1} and X^{ν_2} are independent and $P(X^{\nu_1} \equiv 1) < 1$, the density of X^ν is strictly larger than the density of X^{ν_1} with positive probability. Together this implies that X^ν cannot be ergodic.

(i) implies (ii).
It suffices to show that for any $\delta > 0$,

$$\nu\Big(\Big\{\eta \in \{0,1\}^{\mathbb{Z}} \colon \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} \eta_k}{n} \geq \delta\Big\}\Big) = 0.$$

To do this, we will show that $\nu(A_\delta) = 0$ where

$$A_\delta := \big\{\eta \in \{0,1\}^{\mathbb{Z}} \colon \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \eta_k}{n} \geq \delta\big\}, \quad \delta > 0$$

and then apply [15, Theorem 2] to conclude the previous statement. Using Fatou's Lemma, we get

$$\delta\nu(A_\delta) \leq \int \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \eta_k}{n} \, d\nu|_{A_\delta} \leq \liminf_{n\to\infty} \int \frac{\sum_{k=0}^{n-1} \eta_k}{n} \, d\nu|_{A_\delta} = \nu|_{A_\delta}(\mathcal{S}_0) < \infty.$$

Hence $\nu(A_\delta) < \infty$, and so, by assumption, we can conclude that $\nu(A_\delta) = 0$ as follows. Since A_δ is translation invariant, we can write $\nu = \nu|_{A_\delta} + (\nu - \nu|_{A_\delta})$ where the two summands are translation invariant. Using the assumption (i), the desired conclusion follows.

(ii) implies (iii).
We first show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P\big(X_0^\nu = X_k^\nu = 0\big) = P\big(X_0^\nu = 0\big)^2. \tag{7.1}$$

By the first statement in Proposition 2.8, this is equivalent to showing that

$$\lim_{n\to\infty} P\big(X_0^\nu = 0\big)^2 \cdot \frac{1}{n}\sum_{k=0}^{n-1}(e^{\nu(\mathcal{S}_0\cap\mathcal{S}_k)} - 1) = 0.$$

Since $P(X^\nu \equiv 1) < 1$, we have

$$\sup_k \nu(\mathcal{S}_0 \cap \mathcal{S}_k) \leq \nu(\mathcal{S}_0) < \infty,$$

and hence there exists $c > 0$ so that for all $n \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(e^{\nu(\mathcal{S}_0\cap\mathcal{S}_k)} - 1) \leq \frac{c}{n}\sum_{k=0}^{n-1}\nu(\mathcal{S}_0\cap\mathcal{S}_k) = c\int \mathbf{1}_{\mathcal{S}_0}\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_{\mathcal{S}_k}d\nu \tag{7.2}$$

By our key assumption, the random sum $\frac{1}{n}\sum_{k=0}^{n-1}\mathbf{1}_{\mathcal{S}_k}$ approaches 0 ν-a.e. Since $\nu(\mathcal{S}_0) < \infty$ we can now apply the Bounded Convergence Theorem to conclude that the right hand side of (7.2) approaches 0 as $n \to \infty$. This shows that (7.1) holds.

From here, there are two ways to complete the proof. One quick way to do this is that one can observe that Lemma 2.10 implies that X^ν is max-infinitely divisible (see [7] for the definition) and then apply Theorem 1.2 from this latter reference. However, one can proceed directly as follows. Let S_1 and S_2 be two finite subsets of $\mathbb{Z}$. Let A be the event that $X^\nu(S_1) \equiv 0$ and B be the event that $X^\nu(S_2) \equiv 0$. One can show, analogously to the first step, that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}P(A\cap T^{-k}B) = P(A)P(B) \tag{7.3}$$

where T^{-k} is a k - step left shift. One does this by applying the full statement of Proposition 2.8 and then easily modifying the argument above. Using inclusion-exclusion, one can conclude (7.3) for any cylinder sets A and B. This yields the ergodicity. $\square$

Remark 7.4. *It is easy to see that for positively associated processes, ergodicity implies weak-mixing. Alternatively, this is stated in [7, Theorem 1.2] for max-infinitely divisible processes. In addition, by [7, Theorem 1.1], to prove mixing, it suffices to prove decaying pairwise correlations which amounts to showing that* $\lim_{n\to\infty}\nu(\mathcal{S}^{\cap}_{\{0,n\}}) = 0$.

We next give a sufficient condition on ν which yields much stronger ergodic behavior.

Theorem 7.5. *Assume that ν is such that ν-a.s. each $S \in \mathcal{P}(\mathbb{Z})$ has a smallest element. Then X^ν is a Bernoulli Shift; i.e. it is a factor of i.i.d.'s.*

Proof. Let $\mathcal{T}_k$ be the set of subsets whose smallest element is k. Then ν is concentrated on $\bigcup \mathcal{T}_k$ and by translation invariance, the sets $\mathcal{T}_k$ are disjoint and all have the same ν-measure.

Case 1: $\nu(\mathcal{T}_k) = \infty$ for some (and hence all) k.

In this case, it easy to see that $X^\nu \equiv 1$ a.s., and hence we are done.

Case 2: $\nu(\mathcal{T}_k) = 0$ for some (and hence all) k.

In this case ν is the zero measure and $X^\nu \equiv 0$ a.s. and hence we are done.

Case 3: $\nu(\mathcal{T}_k) \in (0, \infty)$ for some (and hence all) k.

We will represent X^ν as a translation invariant function of random variables $\{U_n\}_{n \in Z}$, each uniform on the unit interval $[0, 1]$ as follows. For each k, we can use U_k to generate a Poisson point process $\mathcal{P}_k$ on $\mathcal{T}_k$ with intensity measure $\nu_{\mathcal{T}_k}$. Then, letting $X_n = 1$ if and only if $n \in \cup_{k \leq n} \mathcal{P}_k$, we have that X has the same distribution as X^ν and we are done. $\square$

Remarks 7.6.

1. *If there is a uniform bound on the radius of the subsets where ν is supported, then the above of course holds and the map is a block map.*
2. *If there is a uniform bound only on the sizes of the subsets where ν is supported (a weaker assumption), then the above of course holds but this is not necessarily a block map. In fact, if there is not a uniform bound on the radius of the subsets where ν is supported, then the above map is not finitary.*
3. *Much of the above extends to $\mathbb{Z}^d$.*

Theorem 7.7. *Assume for $\mathbb{Z}^d$ that ν is supported on sets S for which there exists $(k_1, \ldots, k_d)$ so that $i_j \geq k_j$ for each j for all $(i_1, \ldots, i_d) \in S$. Then X^ν is a Bernoulli Shift; i.e. it is a factor of i.i.d.s. In particular, if ν is supported on finite sets, then X^ν is a Bernoulli Shift.*

Remark 7.8. *The assumption loosely says that ν is concentrated on sets which are contained inside of a "north-east" quadrant.*

Proof of Theorem 7.7. We only outline the proof since it requires only minor modifications of the previous proof. We also only do this for $d = 2$. Let $\mathcal{T}_{k_1,k_2}$ be the set of subsets $\mathbb{Z}^d$ where $i_1 \geq k_1$ and $i_2 \geq k_2$ for all $(i_1, i_2) \in S$ and (k_1, k_2) is maximal with respect to the lexographic order with this property. We only deal with the non-trivial case where $\nu(\mathcal{T}_{k_1,k_2}) \in (0, \infty)$. We will represent X as a translation invariant function of random variables $\{U_{k_1,k_2}\}$, uniform on the interval $[0, 1]$. For each (k_1, k_2), we can use $\{U_{k_1,k_2}\}$ to generate a Poisson point process on $\mathcal{T}_{k_1,k_2}$. Then one proceeds as in the 1-dimensional case. $\square$

In view of the results in this section, it becomes clear that, from an ergodic theoretic point of view, the most interesting case is when ν is supported on infinite zero density sets. This is analogous to what happened when one studied divide and color models in [16].

The following provides us with such an example.

Example 7.9. Let X be the random interlacements process on $\mathbb{Z}^3$. Then, by construction, $X = X^\nu$, where ν is the corresponding measure with support on the set of all simple random walks trajectories, transient in both directions, in $\mathbb{Z}^3$, and with the property that the mass assigned to the set of all simple random walks that intersects a given finite set is finite and non-zero. Let ν' be ν restricted to the x-axis. Since a random walk on $\mathbb{Z}^2$ is recurrent, ν' is supported on infinite 0 density sets. Letting X' be the restriction of X to the x-axis, we of course have $X' = X^{\nu'}$. By Theorem 7.3, X' is ergodic. In fact, in [4], the full interlacement process X was shown to be a Bernoulli shift which implies that X' is also a Bernoulli shift.

8 Questions

1. Is the subcritical (high temperature) Curie-Weiss model in $\mathcal{R}$ for large n?
2. Is the Ising model in $\mathbb{Z}^d$ for $J > 0$ always not in $\mathcal{R}$?
3. Is there any nontrivial tree-indexed Markov chain on an infinite regular tree (other than $\mathbb{Z}$) that is in $\mathcal{R}$?
4. Take i.i.d. bond percolation on $\mathbb{Z}^d$, $d > 1$, $p > p_c$. Take $X(v) = 1$ if v belongs to the infinite component and 0 otherwise. Is $X \in \mathcal{R}$? Note, it is known (see [1]) that the law of X is downward FKG. If $X = X^\nu$ for some ν, it is easy to see that ν must then be supported on sets which have no finite components.
5. Is the upper invariant measure for the contact process in $\mathcal{R}$? It is known to be downwards FKG.
6. Given ν, can the behavior of $X^{c\nu}$ depend in an essential way on c? This is clearly a fairly vague question; one example where one sees this kind of phenomenon is in the percolation properties of the interlacement process.
7. We have seen that if $\nu = \sum_{i\in\mathbb{Z},n\geq 1} a_n \delta_{i,i+n}$ with $\sum a_n < \infty$, then X^ν is a factor of i.i.d.'s. Is X^ν in fact a finitary factor of i.i.d.'s?
8. Related to Proposition 2.23, what can one say concerning the class of product measures which dominate X^ν for a translation invariant measure ν on $\mathcal{P}(\mathbb{Z}^d) \setminus \{\emptyset\}$?

Acknowledgements

Our foremost thanks go to Barney, without whom, this project would almost certainly not have been completed or might have taken an additional hundred years. In addition, there are many people we consulted for questions, references, and discussion. For these, we thank David Aldous, Nachi Avraham-Re'em, Christian Berg, Stein Andreas Bethuelsen, Michael Björklund, Harry Crane, Persi Diaconis, Svante Janson, Bob Pego, Konrad Schmüdgen, Jan Stochel, Jerzy Stochel, and Aernout van Enter. We especially acknowledge Christian

Berg and Svante Janson whose responses to a question about moment sequences led us to a better understanding of the permutation invariant case. Lastly, the initial idea of this project stemmed from attending Johan Tykesson's excellent lectures on the interlacement process.

The second author was supported as a jubilee professor at Chalmers University of Technology. The third author acknowledges the support of the Swedish Research Council, grant no. 2020-03763.

References

[1] van den Berg, J., Häggström, O., Kahn, J. Some conditional correlation inequalities for percolation and related processes. Random Struct. Algorithms 29 (2006), no.4, 417–435.

[2] Bloznelis, M., Leskelä, L. Clustering and percolation on superpositions of Bernoulli random graphs. Random Struct. Algorithms (2023) 283–342.

[3] Bollobás, B., Janson, S., Riordan, O. Sparse random graphs with clustering, Random Struct. Algorithms 38(3) (2011), 269–323.

[4] Borbényi, M., Ráth, B., Rokob, S. Random interlacement is a factor of i.i.d., Electron. J. Probab. 28 (2023).

[5] Crane, H. Combinatorial Lévy processes, Ann. Appl. Probab. 28(1) (2018), 285–339.

[6] Drewitz, A., Ráth, B., Sapozhnikov, A. An introduction to random interlacements. Springer Briefs Math. Springer, Cham, (2014). x+120 pp.

[7] Kabluchko, Z., Schlather, M. Ergodic properties of max-infinitely divisible processes, Stoch. Process. Their Appl., 120 (2010), 281–295.

[8] Kahn, J. A note on positive association, preprint, arXiv:2210.08653 (2022).

[9] Last, G., Penrose, M. Lectures on the Poisson Process (Institute of Mathematical Statistics Textbooks). Cambridge University Press. (2017)

[10] Lawler, G. and Trujillo Ferreras, J. Random walk loop soup, Trans. Amer. Math. Soc. 359, no.2, (2007) 767–787.

[11] Liggett, T.M. Survival and coexistence in interacting particle systems, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 420, Kluwer Academic Publishers Group, Dordrecht (1994), 209–226.

[12] Liggett, T.M., Steif, J.E. Stochastic domination: the contact process, Ising models and FKG measures, Annales de l'IHP Probabilités et statistiques 42 (2), 40(2006), 223–243.

[13] Roy, E. Ergodic properties of Poissonian ID processes, Ann. Probab. 35 (2007), no. 2, 551–576.

[14] Roy, E. Poisson suspensions and infinite ergodic theory. Ergod. Th. & Dynam. Sys. 29(2) (2009), 667–683.

[15] Sato, R. On a decomposition of transformations in infinite measure spaces, Pac. J. Math., Vol. 44, No. 2 (1973), 733–738.

[16] Steif, J.E., Tykesson, J. Generalized divide and color models, ALEA, Lat. Am. J. Probab. Math. Stat. 16 (2019), 1–57.

[17] Steutel, F.W., van Harn, K. Infinite Divisibility of Probability Distributions on the Real Line (1st ed.) (2003). CRC Press.