A mini course on percolation theory

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Abstract. These are lecture notes based on a mini course on percolation which was given at the Jyväskylä summer school in mathematics in Jyväskylä, Finland, August 2009. The point of the course was to try to touch on a number of different topics in percolation in order to give people some feel for the field. These notes follow fairly closely the lectures given in the summer school. However, some topics covered in these notes were not covered in the lectures (such as continuity of the percolation function above the critical value) while other topics covered in detail in the lectures are not proved in these notes (such as conformal invariance).

Contents

1. Introduction

Percolation is one of the simplest models in probability theory which exhibits what is known as *critical phenomena*. This usually means that there is a natural parameter in the model at which the behavior of the system drastically changes. Percolation theory is an especially attractive subject being an area in which the major problems are easily stated but whose solutions, when they exist, often require ingenious methods. The standard reference for the field is [12]. For the study of percolation on general graphs, see [23]. For a study of critical percolation on the hexagonal lattice for which there have been extremely important developments, see [36].

In the standard model of percolation theory, one considers the the d-dimensional integer lattice which is the graph consisting of the set $\mathbb{Z}^d$ as vertex set together with an edge between any two points having Euclidean distance 1. Then one fixes a parameter p and declares each edge of this graph to be *open* with probability p and investigates the structural properties of the obtained random subgraph consisting of $\mathbb{Z}^d$ together with the set of open edges. The type of questions that one is interested in are of the following sort.

Are there infinite components? Does this depend on p? Is there a critical value for p at which infinite components appear? Can one compute this critical value? How many infinite components are there? Is the probability that the origin belongs to an infinite component a continuous function of p?

The study of percolation started in 1957 motivated by some physical considerations and very much progress has occurred through the years in our understanding. In the last decade in particular, there has been tremendous progress in our understanding of the 2-dimensional case (more accurately, for the hexagonal lattice) due to Smirnov's proof of conformal invariance and Schramm's SLE processes which describe critical systems.

2. The model, nontriviality of the critical value and some other basic facts

2.1. Percolation on $\mathbb{Z}^2$: The model

We now define the model. We start with the graph $\mathbb{Z}^2$ which, as a special case of that described in the introduction, has vertices being the set $\mathbb{Z}^2$ and edges between pairs of points at Euclidean distance 1. We will construct a random subgraph of $\mathbb{Z}^2$

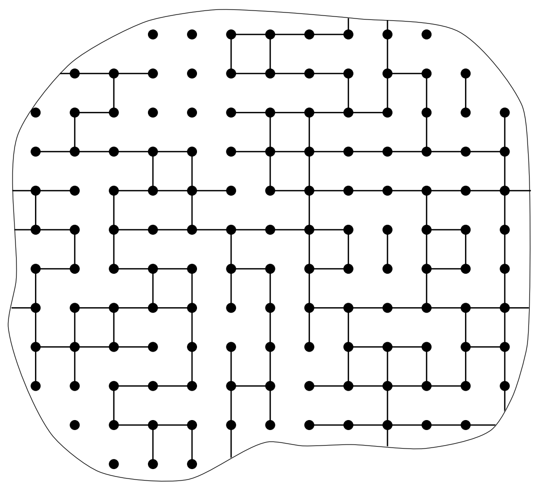

FIGURE 1. A percolation realization (from [12])

as follows. Fix $p \in [0,1]$ which will be the crucial parameter in the model. Letting each edge be independenly *open* with probability p and *closed* with probability $1-p$, our random subgraph will be defined by having the same vertex set as $\mathbb{Z}^2$ but will only have the edges which were declared open. We think of the open edges as retained or present. We will think of an edge which is open as being in state 1 and an edge which is closed as being in state 0. See Figure 1 for a realization.

Our first basic question is the following: What is the probability that the origin $(0,0)$ (denoted by 0 from now on) can reach infinitely many vertices in our random subgraph? By Exercise 2.1 below, this is the same as asking for an infinite self-avoiding path from 0 using only open edges. Often this function (of p), denoted here by $\theta(p)$, is called the *percolation function.* See Figure 2.

One should definitely not get hung up on any measure-theoretic details in the model (essentially since there are no measure theoretic issues to speak of) but nonetheless I say the above more rigorously. Let E denote the edge set of $\mathbb{Z}^2$. Let $\Omega = \prod_{e \in E}\{0,1\}$, F be the σ-algebra generated by the cylinder sets (the events

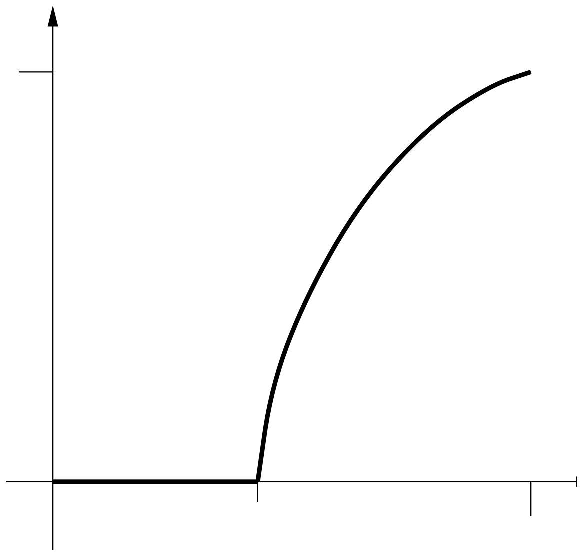

Figure 2. The percolation function (from [12])

which only depend on a finite number of edges) and let $\mathrm{P}_p = \prod_{e \in E} \mu_p$ where $\mu_p(1) = p, \mu_p(0) = 1 - p$. The latter is of course a product measure. Ω is the set of possible outcomes of our random graph and P_p describes its distribution. This paragraph can be basically ignored and is just for sticklers but we do use the notation P_p to describe probabilities when the parameter used is p.

Let $C(x)$ denote the component containing x in our random graph; this is just the set of vertices connected to x via a path of open edges. Of course $C(x)$ depends on the realization which we denote by ω but we do not write this explicitly. We abbreviate $C(0)$ by C. Note that $\mathrm{P}_p(|C| = \infty) = \theta(p)$.

Exercise 2.1: For any subgraph of $\mathbb{Z}^2$ (i.e., for every ω), show that $|C| = \infty$ if and only if there is a self-avoiding path from 0 to ∞ consisting of open edges (i.e., containing infinitely many edges).

Exercise 2.2: Show that $\theta(p)$ is nondecreasing in p. Do NOT try to compute anything!

Exercise 2.3: Show that $\theta(p)$ cannot be 1 for any $p < 1$.

2.2. The existence of a nontrivial critical value

The main result in this section is that for p small (but positive) $\theta(p) = 0$ and for p large (but less than 1) $\theta(p) > 0$. In view of this (and exercise 2.2), there is a *critical value* $p_c \in (0, 1)$ at which the function $\theta(p)$ changes from being 0 to being positive. This illustrates a so-called *phase transition* which is a change in the global behavior of a system as we move past some critical value. We will see later (see Exercises 2.7 and 2.8) the elementary fact that when $\theta(p) = 0$, a.s. there is no infinite component anywhere while when $\theta(p) > 0$, there is an infinite component somewhere a.s.

Let us finally get to proving our first result. We mention that the method of proof of the first result is called the *first moment method*, which just means you bound the probability that some nonnegative integer-valued random variable is positive by its expected value (which is usually much easier to calculate). In the proof below, we will implicitly apply this first moment method to the number of self-avoiding paths of length n starting at 0 and for which all the edges of the path are open.

Theorem 2.1. *If $p < 1/3$, $\theta(p) = 0$.*

Proof. : Let F_n be the event that there is a self-avoiding path of length n starting at 0 using only open edges. For any given self-avoiding path of length n starting at 0 in $\mathbb{Z}^2$ (not worrying if the edges are open or not), the probability that all the edges of this given path are open is p^n. The number of such paths is at most $4(3^{n-1})$ since there are 4 choices for the first step but at most 3 choices for any later step. This implies that $\mathrm{P}_p(F_n) \leq 4(3^{n-1})p^n$ which $\to 0$ as $n \to \infty$ since $p < 1/3$. As $\{|C| = \infty\} \subseteq F_n \quad \forall n$, we have that $\mathrm{P}_p\{|C| = \infty\} = 0$; i.e., $\theta(p) = 0$. □

Theorem 2.2. *For p sufficiently close to 1, we have that $\theta(p) > 0$.*

Proof. : The method of proof to be used is often called a *contour or Peierls argument*, the latter named after the person who proved a phase transition for another model in statistical mechanics called the Ising model.

The first key thing to do is to introduce the so-called *dual graph* $(\mathbb{Z}^2)^*$ which is simply $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$. This is nothing but our ordinary lattice translated by the vector $(\frac{1}{2}, \frac{1}{2})$. See Figure 3 for a picture of $\mathbb{Z}^2$ and its dual $(\mathbb{Z}^2)^*$. One then sees that there is an obvious 1-1 correspondence between the edges of $\mathbb{Z}^2$ and those of $(\mathbb{Z}^2)^*$. (Two corresponding edges cross each other at their centers.)

Given a realization of open and closed edges of $\mathbb{Z}^2$, we obtain a similar realization for the edges of $(\mathbb{Z}^2)^*$ by simply calling an edge in the dual graph open if

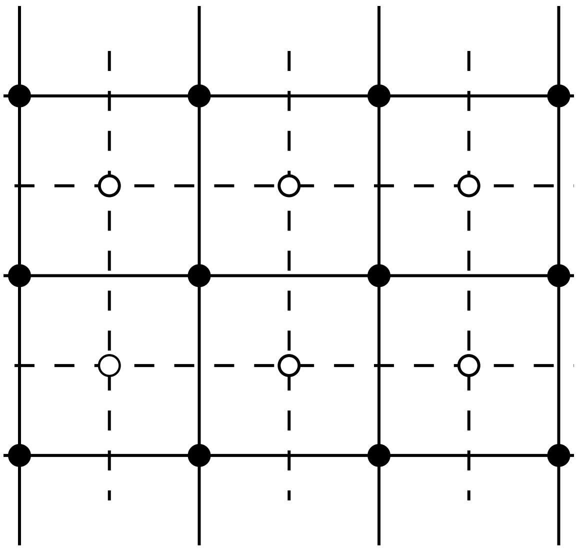

FIGURE 3. $\mathbb{Z}^2$ and its dual $(\mathbb{Z}^2)^*$ (from [12])

and only if the corresponding edge in $\mathbb{Z}^2$ is open. Observe that if the collection of open edges of $\mathbb{Z}^2$ is chosen according to P_p (as it is), then the distribution of the set of open edges for $(\mathbb{Z}^2)^*$ will trivially also be given by P_p.

A key step is a result due to Whitney which is pure graph theory. No proof will be given here but by drawing some pictures, you will convince yourself it is very believable. Looking at Figures 4 and 5 is also helpful.

Lemma 2.3. *$|C| < \infty$ if and only if $\exists$ a simple cycle in $(\mathbb{Z}^2)^*$ surrounding 0 consisting of all closed edges.*

Let G_n be the event that there is a simple cycle in $(\mathbb{Z}^2)^*$ surrounding 0 having length n, all of whose edges are closed. Now, by Lemma 2.3, we have

$$\mathrm{P}_p(|C| < \infty) = \mathrm{P}_p(\cup_{n=4}^{\infty} G_n) \leq \sum_{n=4}^{\infty} \mathrm{P}_p(G_n) \leq \sum_{n=4}^{\infty} n4(3^{n-1})(1-p)^n$$

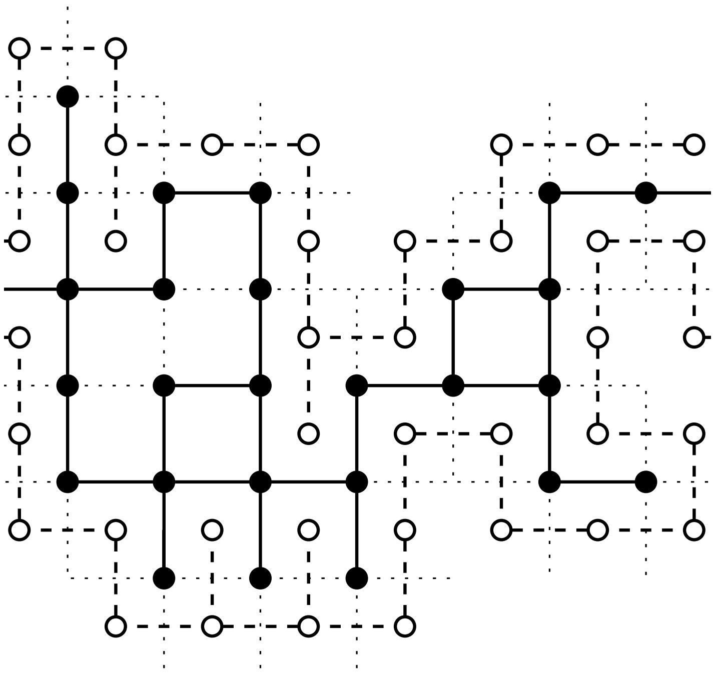

FIGURE 4. Whitney (from [12])

since one can show that the number of cycles around the origin of length n (not worrying about the status of the edges) is at most $n4(3^{n-1})$ (why?, see Exercise 2.4) and the probability that a given cycle has all its edges closed is $(1-p)^n$. If $p > \frac{2}{3}$, then the sum is $< \infty$ and hence it can be made arbitrarily small if p is chosen close to 1. In particular, the sum can be made less than 1 which would imply that $P_p(|C| = \infty) > 0$ for p close to 1. $\square$

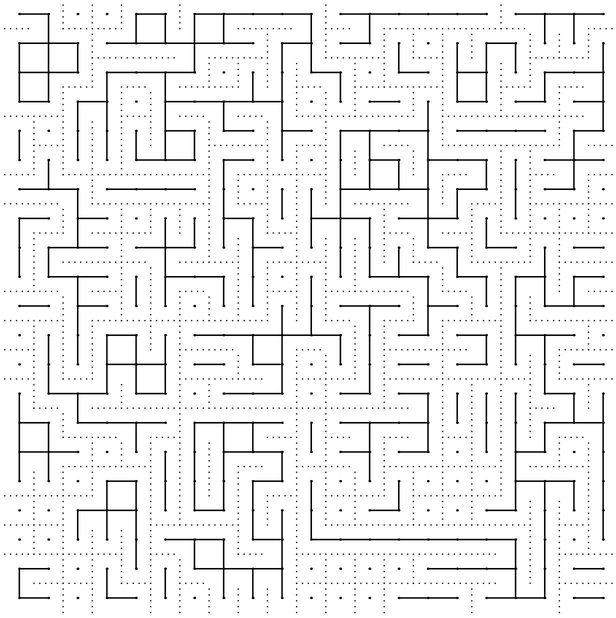

FIGURE 5. Whitney (picture by Vincent Beffara)

Remark:
We actually showed that $\theta(p) \to 1$ as $p \to 1$.

Exercise 2.4: Show that the number of cycles around the origin of length n is at most $n4(3^{n-1})$.

Exercise 2.5: Show that $\theta(p) > 0$ for $p > \frac{2}{3}$.
Hint: Choose N so that $\sum_{n \geq N}^{\infty} n4(3^{n-1})(1-p)^n < 1$. Let E_1 be the event that all edges are open in $[-N, N] \times [-N, N]$ and E_2 be the event that there are no simple cycles in the dual surrounding $[-N, N]^2$ consisting of all closed edges. Look now at $E_1 \cap E_2$.

It is now natural to define the *critical value* p_c by

$$p_c := \sup\{p : \theta(p) = 0\} = \inf\{p : \theta(p) > 0\}.$$

With the help of Exercise 2.5, we have now proved that $p_c \in [1/3, 2/3]$. (The model would not have been interesting if p_c were 0 or 1.) In 1960, Harris [16] proved that $\theta(1/2) = 0$ and the conjecture made at that point was that $p_c = 1/2$ and there was indications that this should be true. However, it took 20 more years before there was a proof and this was done by Kesten [18].

Theorem 2.4. [18] *The critical value for* $\mathbb{Z}^2$ *is* $1/2$.

In Chapter 7, we will give the proof of this very fundamental result.

2.3. Percolation on $\mathbb{Z}^d$

The model trivially generalizes to $\mathbb{Z}^d$ which is the graph whose vertices are the integer points in $\mathbb{R}^d$ with edges between vertices at distance 1. As before, we let each edge be open with probability p and closed with probability $1-p$. Everything else is defined identically. We let $\theta_d(p)$ be the probability that there is a self-avoiding open path from the origin to ∞ for this graph when the parameter p is used. The subscript d will often be dropped. The definition of the critical value in d dimensions is clear.

$$p_c(d) := \sup\{p : \theta_d(p) = 0\} = \inf\{p : \theta_d(p) > 0\}.$$

In $d = 1$, it is trivial to check that the critical value is 1 and therefore things are not interesting in this case. We saw previously that $p_c(2) \in (0, 1)$ and it turns out that for $d > 2$, $p_c(d)$ is also strictly inside the interval $(0, 1)$.

Exercise 2.6: Show that $\theta_{d+1}(p) \geq \theta_d(p)$ for all p and d and conclude that $p_c(d+1) \leq p_c(d)$. Also find some lower bound on $p_c(d)$. How does your lower bound behave as $d \to \infty$?

Exercise 2.7. Using Kolmogorov's 0-1 law (which says that all tail events have probability 0 or 1), show that P_p (some $C(x)$ is infinite) is either 0 or 1. (If you are unfamiliar with Kolmogorov's 0-1 law, one should say that there are many (relatively easy) theorems in probability theory which guarantee, under certain circumstances, that a given type of event must have a probability which is either 0 or 1 (but they don't tell which of 0 and 1 it is which is always the hard thing).)

Exercise 2.8. Show that $\theta(p) > 0$ if and only if P_p (some $C(x)$ is infinite)=1.

Nobody expects to ever know what $p_c(d)$ is for $d \geq 3$ but a much more interesting question to ask is what happens at the critical value itself; i.e. is $\theta_d(p_c(d))$ equal to 0 or is it positive. The results mentioned in the previous section imply that it is 0 for $d = 2$. Interestingly, this is also known to be the case for $d \geq 19$ (a highly nontrivial result by Hara and Slade ([15]) but for other d, it is viewed as one of the major open questions in the field.

Open question: For $\mathbb{Z}^d$, for intermediate dimensions, such as $d = 3$, is there percolation at the critical value; i.e., is $\theta_d(p_c(d)) > 0$?

Everyone expects that the answer is no. We will see in the next subsection why one expects this to be 0.

2.4. Elementary properties of the percolation function

Theorem 2.5. *$\theta_d(p)$ is a right continuous function of p on $[0, 1]$. (This might be a good exercise to attempt before looking at the proof.)*

Proof. : Let $g_n(p) := \mathrm{P}_p$ (there is a self-avoiding path of open edges of length n starting from the origin). $g_n(p)$ is a polynomial in p and $g_n(p) \downarrow \theta(p)$ as $n \to \infty$. Now a decreasing limit of continuous functions is always upper semi-continuous and a nondecreasing upper semi-continuous function is right continuous. □

Exercise 2.9. Why does $g_n(p) \downarrow \theta(p)$ as $n \to \infty$ as claimed in the above proof? Does this convergence hold uniformly in p?

Exercise 2.10. In the previous proof, if you don't know what words like upper semi-continuous mean (and even if you do), redo the second part of the above proof with your hands, not using anything.

A much more difficult and deeper result is the following due to van den Berg and Keane ([4]).

Theorem 2.6. *$\theta_d(p)$ is continuous on $(p_c(d), 1]$.*

The proof of this result will be outlined in Section 4. Observe that, given the above results, we can conclude that there is a jump discontinuity at $p_c(d)$ if and only if $\theta_d(p_c(d)) > 0$. Since nice functions should be continuous, we should believe that $\theta_d(p_c(d)) = 0$.

3. Uniqueness of the infinite cluster

In terms of understanding the global picture of percolation, one of the most natural questions to ask, assuming that there is an infinite cluster, is how many infinite clusters are there?

My understanding is that before this problem was solved, it was not completely clear to people what the answer should be. Note that, for any $k \in 0, 1, 2, \ldots, \infty$, it is trivial to find a realization ω for which there are k infinite clusters. (Why?). The following theorem was proved by Aizenman, Kesten and Newman ([1]). A much simpler proof of this theorem was found by Burton and Keane ([8]) later on and this later proof is the proof we will follow.

Theorem 3.1. *If $\theta(p) > 0$, then $\mathrm{P}_p(\exists$ a unique infinite cluster $) = 1$.*

Before starting the proof, we tell (or remind) the reader of another 0-1 Law which is different from Kolmogorov's theorem and whose proof we will not give. I will not state it in its full generality but only in the context of percolation. (For people who are familiar with ergodic theory, this is nothing but the statement that a product measure is ergodic.)

Lemma 3.2. *If an event is* translation invariant*, then its probability is either 0 or 1. (This result very importantly assumes that we are using a product measure, i.e, doing things independently.)*

Translation invariance means that you can tell whether the event occurs or not by looking at the percolation realization but not being told where the origin is. For example, the events 'there exists an infinite cluster' and 'there exists at least 3 infinite clusters' are translation invariant while the event '$|C(0)| = \infty$' is not.

Proof of Theorem 3.1. Fix p with $\theta(p) > 0$. We first show that the number of infinite clusters is nonrandom, i.e. it is constant a.s. (where the constant may depend on p). To see this, for any $k \in \{0, 1, 2, \ldots, \infty\}$, let E_k be the event that the number of infinite cluster is exactly k. Lemma 3.2 implies that $\mathrm{P}_p(E_k) = 0$ or 1 for each k. Since the E_k's are disjoint and their union is our whole probability space, there is some k with $\mathrm{P}_p(E_k) = 1$, showing that the number of infinite clusters is a.s. k.

The statement of the theorem is that the k for which $\mathrm{P}_p(E_k) = 1$ is 1 assuming $\theta(p) > 0$; of course if $\theta(p) = 0$, then k is 0. It turns out to be much easier to rule out all finite k larger than 1 than it is to rule out $k = \infty$. The easier part, due to Newman and Schulman ([25]), is stated in the following lemma. Before reading the proof, the reader is urged to imagine for herself why it would be absurd that, for example, there could be 2 infinite clusters a.s.

Lemma 3.3. *For any* $k \in \{2, 3, \ldots\}$*, it cannot be the case that* $\mathrm{P}_p(E_k) = 1$*.*

Proof. : The proof is the same for all k and so we assume that $\mathrm{P}_p(E_5) = 1$. Let $F_M = \{$there are 5 infinite clusters and each intersects $[-M, M]^d\}$. Observe that $F_1 \subseteq F_2 \subseteq \ldots \subseteq \ldots$ and $\cup_i F_i = E_5$. Therefore $\mathrm{P}_p(F_i) \stackrel{i \to \infty}{\longrightarrow} 1$. Choose N so that $\mathrm{P}_p(F_N) > 0$.

Now, let $\tilde{F}_N$ be the event that all infinite clusters touch the boundary of $[-N, N]^d$. Observe that (1) this event is measurable with respect to the edges outside of $[-N, N]^d$ and that (2) $F_N \subseteq \tilde{F}_N$. Note however that these two events are not the same. We therefore have that $\mathrm{P}_p(\tilde{F}_N) > 0$. If we let G be the event that all the edges in $[-N, N]^d$ are open, then G and $\tilde{F}_N$ are independent and hence $\mathrm{P}_p(G \cap \tilde{F}_N) > 0$. However, it is easy to see that $G \cap \tilde{F}_N \subseteq E_1$ which implies that $\mathrm{P}_p(E_1) > 0$ contradicting $\mathrm{P}_p(E_5) = 1$. □

It is much harder to rule out infinitely many infinite clusters, which we do now. This proof is due to Burton and Keane. Let Q be the # of infinite clusters. Assume $\mathrm{P}_p(Q = \infty) = 1$ and we will get a contradiction. We call z an "encounter point" (e.p.) if

1. z belongs to an infinite cluster C and
2. $C \backslash \{z\}$ has no finite components and exactly 3 infinite components.

See Figure 6 for how an encounter points looks.

Lemma 3.4. *If* $\mathrm{P}_p(Q = \infty) = 1$*, then* $\mathrm{P}_p(0$ *is an e.p.* $) > 0$*.*

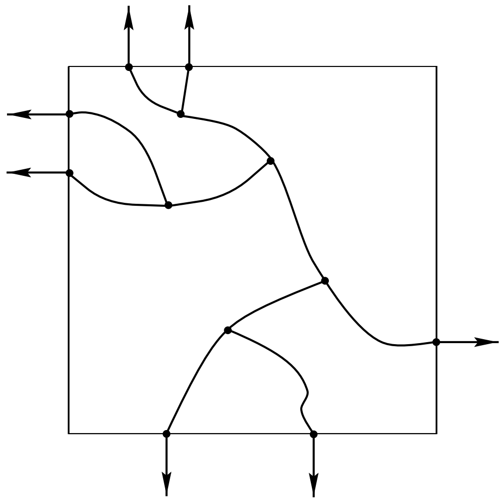

FIGURE 6. An encounter point (from [12])

Proof. : Let $F_M = \{\text{at least 3 infinite clusters intersect } [-M, M]^d\}$.

Since $F_1 \subseteq F_2 \subseteq \ldots$ and $\cup_i F_i = \{\exists \geq 3 \text{ infinite clusters}\}$, we have, under the assumption that $P_p(Q = \infty) = 1$, that $P_p(F_i) \overset{i\to\infty}{\longrightarrow} 1$ and so we can choose N so that $P_p(F_N) > 0$.

Now, let $\tilde{F}_N$ be the event that outside of $[-N, N]^d$, there are at least 3 infinite clusters all of which touch the boundary of $[-N, N]^d$. Observe that (1) this event is measurable with respect to the edges outside of $[-N, N]^d$ and that (2) $F_N \subseteq \tilde{F}_N$ and so $P_p(\tilde{F}_N) > 0$. Now, if we have a configuration with at least 3 infinite clusters all of which touch the boundary of $[-N, N]^d$, it is easy to see that one can find a configuration within $[-N, N]^d$ which, together with the outside configuration, makes 0 an e.p. By independence, this occurs with positive probability and we have $P_p(0 \text{ is an e.p. }) > 0$. [Of course the configuration we need inside depends on the outside; convince yourself that this argument can be made completely precise.] $\square$

Let $\delta = P_p$ (0 is an encounter point) which we have seen is positive under the assumption that $P_p(Q = \infty) = 1$. The next key lemma is the following.

Lemma 3.5. *For any configuration and for any N, the number of encounter points in $[-N,N]^d$ is at most the number of outer boundary points of $[-N,N]^d$.*

Remark: This is a completely deterministic statement which has nothing to do with probability.

Before proving it, let's see how we finish the proof of Theorem 3.1. Choose N so large that $\delta(2N+1)^d$ is strictly larger than the number of outer boundary points of $[-N,N]^d$. Now consider the number of e.p.'s in $[-N,N]^d$. On the one hand, by Lemma 3.5 and the way we chose N, this random variable is always strictly less than $\delta(2N+1)^d$. On the other hand, this random variable has an expected value of $\delta(2N+1)^d$, giving a contradiction. □

We are left with the following.

Proof of Lemma 3.5.

Observation 1: For any finite set S of encounter points contained in the same infinite cluster, there is at least one point s in S which is *outer* in the sense that all the other points in S are contained in the same infinite cluster after s is removed. To see this, just draw a picture and convince yourself; it is easy.

Observation 2: For any finite set S of encounter points contained in the same infinite cluster, if we remove all the elements of S, we break our infinite cluster into at least $|S|+2$ infinite clusters. This is easily done by induction on $|S|$. It is clear if $|S|=1$. If $|S|=k+1$, choose, by observation 1, an outer point s. By the induction hypothesis, if we remove the points in $S\backslash s$, we have broken the cluster into at least $k+2$ clusters. By drawing a picture, one sees, since s is outer, removal of s will create at least one more infinite cluster yielding $k+3$, as desired.

Now fix N and order the infinite clusters touching $[-N,N]^d$, $C_1, C_2, \ldots, C_k$, assuming there are k of them. Let j_1 be the number of encounter points inside $[-N,N]^d$ which are in C_1. Define $j_2, \ldots, j_k$ in the same way. Clearly the number of encounter points is $\sum_{i=1}^k j_i$. Looking at the first cluster, removal of the j_1 encounter points which are in $[-N,N]^d \cap C_1$ leaves (by observation 2 above) at least $j_1+2 \geq j_1$ infinite clusters. Each of these infinite clusters clearly intersects the outer boundary of $[-N,N]^d$, denoted by $\partial[-N,N]^d$. Hence $|C_1 \cap \partial[-N,N]^d| \geq j_1$. Similarly, $|C_i \cap \partial[-N,N]^d| \geq j_i$. This yields

$$|\partial[-N,N]^d| \geq \sum_{i=1}^k |C_i \cap \partial[-N,N]^d| \geq \sum_{i=1}^k j_i.$$

□

4. Continuity of the percolation function

The result concerning uniqueness of the infinite cluster that we proved will be crucial in this section (although used in only one point).

Proof of Theorem 2.6. Let $\tilde{p} > p_c$. We have already established right continuity and so we need to show that $\lim_{\pi \nearrow \tilde{p}} \theta(\pi) = \theta(\tilde{p})$. The idea is to couple all percolation realizations, as p varies, on the same probability space. This is not so hard. Let $\{X(e) : e \in E^d\}$ be a collection of independent random variables indexed by the edges of $\mathbb{Z}^d$ and having uniform distribution on $[0,1]$. We say $e \in E^d$ is p-open if $X(e) < p$. Let P denote the probability measure on which all of these independent uniform random variables are defined.

Remarks:
(i) $\mathrm{P}(e \text{ is p}-\text{open }) = p$ and these events are independent for different e's. Hence, for any p, the set of e's which are p-open is just a percolation realization with parameter p. In other words, studying percolation at parameter p is the same as studying the structure of the p-open edges.
(ii) However, as p varies, everything is defined on the same probability space which will be crucial for our proof. For example, if $p_1 < p_2$, then

$$\{e : e \text{ is } p_1 \text{ open}\} \subseteq \{e : e \text{ is } p_2 \text{ open}\}.$$

Now, let C_p be the p-open cluster of the origin (this just means the cluster of the origin when we consider edges which are p-open). In view of Remark (ii) above, obviously $C_{p_1} \subseteq C_{p_2}$ if $p_1 < p_2$ and by Remark (i), for each p, $\theta(p) = \mathrm{P}(|C_p| = \infty)$. Next, note that

$$\lim_{\pi \nearrow \tilde{p}} \theta(\pi) = \lim_{\pi \nearrow \tilde{p}} \mathrm{P}(|C_\pi| = \infty) = \mathrm{P}(|C_\pi| = \infty \text{ for some } \pi < \tilde{p}).$$

The last equality follows from using countable additivity in our big probability space (one can take π going to $\tilde{p}$ along a sequence). (Note that we have expressed the limit that we are interested in as the probability of a certain event in our big probability space.) Since $\{|C_\pi| = \infty \text{ for some } \pi < \tilde{p}\} \subseteq \{|C_{\tilde{p}}| = \infty\}$, we need to show that

$$\mathrm{P}(\{|C_{\tilde{p}}| = \infty\} \backslash \{|C_\pi| = \infty \text{ for some } \pi < \tilde{p}\}) = 0.$$

If it is easier to think about, this is the same as saying

$$\mathrm{P}(\{|C_{\tilde{p}}| = \infty\} \cap \{|C_\pi| < \infty \text{ for all } \pi < \tilde{p}\}) = 0.$$

Let α be such that $p_c < \alpha < \tilde{p}$. Then a.s. there is an infinite α-open cluster I_α (not necessarily containing the origin).

Now, if $|C_{\tilde{p}}| = \infty$, then, by Theorem 3.1 applied to the $\tilde{p}$-open edges, we have that $I_\alpha \subseteq C_{\tilde{p}}$ a.s. If $0 \in I_\alpha$, we are of course done with $\pi = \alpha$. Otherwise, there is a $\tilde{p}$-open path ℓ from the origin to I_α. Let $\mu = \max\{X(e) : e \in \ell\}$ which is $< \tilde{p}$. Now, choosing π such that $\mu, \alpha < \pi < \tilde{p}$, we have that there is a π open path from 0 to I_α and therefore $|C_\pi| = \infty$, as we wanted to show. □

5. The critical value for trees: the second moment method

Trees, graphs with no cycles, are much easier to analyze than Euclidean lattices and other graphs. Lyons, in the early 90's, determined the critical value for any tree and also determined whether one percolates or not at the critical value (both scenarios are possible). See [21] and [22]. Although this was done for general trees, we will stick here to a certain subset of all trees, namely the *spherically symmetric* trees, which is still a large enough class to be very interesting.

A spherically symmetric tree is a tree which has a root ρ which has a_0 children, each of which has a_1 children, etc. So, all vertices in generation k have a_k children.

Theorem 5.1. *Let A_n be the number of vertices in the nth generation (which is of course just $\prod_{i=0}^{n-1} a_i$). Then*

$$p_c(T) = 1/[\liminf_n A_n^{1/n}].$$

Exercise 5.1 (this exercise explains the very important and often used second moment method). Recall that the first moment method amounts to using the (trivial) fact that for a nonnegative integer valued random variable X, $\mathrm{P}(X > 0) \le E[X]$.
(a). Show that the "converse" of the first moment method is false by showing that for nonnegative integer valued random variables X, $E[X]$ can be arbitrarily large with $\mathrm{P}(X > 0)$ arbitrarily small. (This shows that you will never be able to show that $\mathrm{P}(X > 0)$ is of reasonable size based on knowledge of only the first moment.)
(b). Show that for any nonnegative random variable X

$$\mathrm{P}(X > 0) \ge \frac{E[X]^2}{E[X^2]}.$$

(This says that if the mean is large, then you can conclude that $\mathrm{P}(X > 0)$ might be "reasonably" large **provided** you have a reasonably good upper bound on the second moment $E[X^2]$.) Using the above inequality is called the *second moment method* and it is a very powerful tool in probability.

We will see that the first moment method will be used to obtain a lower bound on p_c and the second moment method will be used to obtain an upper bound on p_c.

Proof of Theorem 5.1.
Assume $p < 1/[\liminf_n A_n^{1/n}]$. This easily yields (an exercise left to the reader) that $A_n p^n$ approaches 0 along some subsequence $\{n_k\}$. Now, the probability that there is an open path to level n is at most the expected number of vertices on the nth level connected to the root which is $A_n p^n$. Hence the probability of having an open path to level n_k goes to 0 and hence the root percolates with probability 0. Therefore $p_c(T) \ge 1/[\liminf_n A_n^{1/n}]$.

To show the reverse inequality, we need to show that if $p > 1/[\liminf_n A_n^{1/n}]$, then the root percolates with positive probability. Let X_n denote the number of vertices at the nth level connected to the root. By linearity, we know that $E(X_n) = A_n p^n$. If we can show that for some constant C, we have that for all n,

$$E(X_n^2) \leq CE(X_n)^2, \tag{5.1}$$

we would then have by the second moment method exercise that $\mathrm{P}(X_n > 0) \geq 1/C$ for all n. The events $\{X_n > 0\}$ are decreasing and so countable additivity yields $\mathrm{P}(X_n > 0\, \forall n) \geq 1/C$. But the latter event is the same as the event that the root is percolating and one is done.

We now bound the second moment in order to establish (5.1). Letting $U_{v,w}$ be the event that both v and w are connected to the root, we have that

$$E(X_n^2) = \sum_{v,w \in T_n} \mathrm{P}(U_{v,w})$$

where T_n is the nth level of the tree. Now $\mathrm{P}(U_{v,w}) = p^{2n} p^{-k_{v,w}}$ where $k_{v,w}$ is the level at which v and w split. For a given v and k, the number of w with $k_{v,w}$ being k is at most A_n/A_k. Hence (after a little computation) one gets that the second moment above is

$$\leq A_n \sum_{k=0}^{n} p^{2n} p^{-k} A_n/A_k = E(X_n)^2 \sum_{k=0}^{n} 1/(p^k A_k).$$

If $\sum_{k=0}^{\infty} 1/(p^k A_k) < \infty$, then we would have (5.1). We have not yet used that $p > 1/[\liminf_n A_n^{1/n}]$ which we now use. If this holds, then $\liminf_n (p^n A_n)^{1/n} \geq 1 + \delta$ for some $\delta > 0$. This gives that $1/(p^k A_k)$ decays exponentially to 0 and we have the desired convergence of the series. □

Remark: The above theorem determines the critical value but doesn't say what happens at the critical value. However, the proof gives more than this and sometimes tells us what happens even at the critical value. The proof gives that $\sum_{k=0}^{\infty} 1/(p^k A_k) < \infty$ implies percolation at p and in certain cases we might have convergence of this series at p_c giving an interesting case where one percolates *at* the critical value. For example if $A_n \asymp 2^n n^\alpha$ (which means that the ratio of the left and right sides are bounded away from 0 and ∞ uniformly in n and which is possible to achieve) then Theorem 5.1 yields $p_c = 1/2$ but furthermore, if $\alpha > 1$, then $\sum_{k=0}^{n} 1/(p_c^k A_k) \asymp \sum 1/n^\alpha < \infty$ and so we percolate at the critical value. For $\alpha \leq 1$, the above sum diverges and so we don't know what happens at p_c. However, in fact, Lyons showed that $\sum_{k=0}^{n} 1/(p^k A_k) < \infty$ is also a necessary condition to percolate at p. In particular, in the case above with $\alpha \leq 1$, we do not percolate at the critical value. This yields a phase transition in α at a finer scale.

6. Some various tools

In this section, I will state two basic tools in percolation. Although they come up all the time, we have managed to avoid their use until now. However, now we will need them both for the next section, Section 7, where we give the proof that the critical value for $\mathbb{Z}^2$ is 1/2.

6.1. Harris' inequality

Harris' inequality (see [16]) tells us that certain random variables are positively correlated. To state this, we need to first introduce the important property of a function being *increasing*. Let $\Omega := \{0,1\}^J$. There is a partial order on Ω given by $\omega \preceq \omega'$ if $\omega_i \leq \omega_i'$ for all $i \in J$.

Definition 6.1. A function $f : \{0,1\}^J \to \mathbb{R}$ is increasing if $\omega \preceq \omega'$ implies that $f(\omega) \leq f(\omega')$. An event is increasing if its indicator function is increasing.

If J is the set of edges in a graph and x and y are vertices, then the event that there is an open path from x to y is an increasing event.

Theorem 6.2. *Let $X := \{X_i\}_{i\in J}$ be independent random variables taking values 0 and 1. Let f and g be increasing functions as above. Then*

$$E(f(X)g(X)) \geq E(f(X))E(g(X)).$$

To understand this, note that an immediate application to percolation is the following. Let x, y, z and w be 4 vertices in $\mathbb{Z}^d$, A be the event that there is an open path from x to y and B be the event that there is an open path from z to w. Then $\mathrm{P}(A \cap B) \geq \mathrm{P}(A)\mathrm{P}(B)$.

Proof. We do the proof only in the case that J is finite; to go to infinite J is done by approximation. We prove this by induction on $|J|$. Assume $|J| = 1$. Let ω_1 and ω_2 take values 0 or 1. Then since f and g are increasing, we have

$$(f(\omega_1) - f(\omega_2))(g(\omega_1) - g(\omega_2)) \geq 0.$$

Now letting ω_1 and ω_2 be independent with distribution X_1, one can take expectation in the above inequality yielding

$$E[f(\omega_1)g(\omega_1)] + E[f(\omega_2)g(\omega_2)] - E[f(\omega_2)g(\omega_1)] - E[f(\omega_1)g(\omega_2)] \geq 0.$$

This says $2E(f(X_1)g(X_1)) \geq 2E(f(X_1))E(g(X_1))$.

Now assuming it is true for $|J| = k-1$ and f and g are functions of k variables, we have, by the law of total expectation

$$E(f(X_1,\dots,X_k)g(X_1,\dots,X_k)) = E[E(f(X_1,\dots,X_k)g(X_1,\dots,X_k)) \mid X_1,\dots,X_{k-1}]. \tag{6.1}$$

The $k = 1$ case implies that for each $X_1,\dots,X_{k-1}$,

$$E(f(X_1,\dots,X_k)g(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1}) \geq$$
$$E(f(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})E(g(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1}).$$

Hence (6.1) is

$$\geq E[E(f(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})E(g(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})]. \tag{6.2}$$

Now $E(f(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})$ and $E(g(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})$ are each increasing functions of $X_1,\dots,X_{k-1}$ as is easily checked and hence the induction assumption gives that (6.2) is

$$\geq E[E(f(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})]E[E(g(X_1,\dots,X_k) \mid X_1,\dots,X_{k-1})] = E(f(X_1,\dots,X_k))E(g(X_1,\dots,X_k))$$

and we are done. □

Remark: The above theorem is not true for all sets of random variables. Where precisely in the above proof did we use the fact that the random variables are independent?

6.2. Margulis-Russo Formula

This formula has been discovered by a number of people and it describes the derivative with respect to p of the probability under P_p of an increasing event A in terms of the very important concept of influence or pivotality. Let P_p be product measure on $\{0,1\}^J$ and let A be a subset of $\{0,1\}^J$ which is increasing.

Exercise 6.1. Show that if A is an increasing event, then $\mathrm{P}_p(A)$ is nondecreasing in p.

Definition 6.3. Given $i \in J$, and $\omega \in \{0,1\}^J$, let $\omega^{(i)}$ denote the sequence which is equal to ω off of i and is 1 on i and $\omega_{(i)}$ denote the sequence which is equal to ω off of i and is 0 on i. Given a (not necessarily increasing) event A of $\{0,1\}^J$, let $\mathrm{Piv}_i(A)$ be the event, called that 'i is pivotal for A', that exactly one of $\omega^{(i)}$ and $\omega_{(i)}$ is in A. Let $I_i^p(A) = \mathrm{P}_p(\mathrm{Piv}_i(A))$, which we call the influence at level p of i on A.

Remarks: The last definition is perhaps a mouth full but contains fundamentally important concepts in probability theory. In words, $\mathrm{Piv}_i(A)$ is the event that changing the sequence at location i changes whether A occurs. It is an event which is measurable with respect to ω off of i. $I_i^p(A)$ is then the probability under P_p that $\mathrm{Piv}_i(A)$ occurs. These concepts are fundamental in a number of different areas, including theoretical computer science.

Exercise 6.2. Let J be a finite set and let A be the event in $\{0,1\}^J$ that there are an even number of 1's. Determine $\mathrm{Piv}_i(A)$ and $I_i^p(A)$ for each i and p.

Exercise 6.3. Assume that $|J|$ is odd and let A be the event in $\{0,1\}^J$ that there are more 1's than 0's. Describe, for each i, $\mathrm{Piv}_i(A)$ and $I_i^{1/2}(A)$.

The following is the fundamental Margulis-Russo Formula. It was proved independently in [24] and [27].

Theorem 6.4. *Let A be an increasing event in $\{0,1\}^J$ with J a finite set. Then*

$$d(\mathrm{P}_p(A))/dp = \sum_{i\in J} I_i^p(A).$$

Exercise 6.4. Let A be the event in $\{0,1\}^J$ corresponding to at least one of the first two bits being 1. Verify the Margulis-Russo Formula in this case.

Outline of Proof. Let $\{Y_i\}_{i\in J}$ be i.i.d. uniform variables on $[0,1]$ and let ω^p be defined by $\omega_i^p = 1$ if and only if $Y_i \le p$. Then $\mathrm{P}_p(A) = \mathrm{P}(\omega^p \in A)$ and moreover $\{\omega^{p_1} \in A\} \subseteq \{\omega^{p_2} \in A\}$ if $p_1 \le p_2$. It follows that

$$\mathrm{P}_{p+\delta}(A) - \mathrm{P}_p(A) = \mathrm{P}(\{\omega^{p+\delta} \in A\}\backslash\{\omega^p \in A\}).$$

The difference of these two events contains (with a little bit of thought)

$$\cup_{i\in J}\left(\{\omega^p \in \mathrm{Piv}_i(A)\} \cap \{Y_i \in (p, p+\delta)\}\right) \tag{6.3}$$

and is contained in the union of the latter union (with the open intervals replaced by closed intervals) together with

$$\{Y_i \in [p, p+\delta] \text{ for two different } i\text{'s}\}.$$

This latter event has probability at most $C\delta^2$ for some constant C (depending on $|J|$). Also, the union in (6.3) is disjoint up to an error of at most $C\delta^2$. Hence

$$\mathrm{P}_{p+\delta}(A) - \mathrm{P}_p(A) = \sum_{i\in J} \mathrm{P}(\{\omega^p \in \mathrm{Piv}_i(A)\} \cap \{Y_i \in (p, p+\delta)\}) + O(\delta^2) = \sum_{i\in J} I_i^p(A)\delta + O(\delta^2).$$

(Note that we are using here the fact that the event that i is pivotal is measurable with respect to the other variables.) Divide by δ and let $\delta \to 0$ and we are done. $\square$

Exercise 6.5. (The square root trick.) Let $A_1, A_2, \ldots, A_n$ be increasing events with equal probabilities such that $\mathrm{P}(A_1 \cup A_2 \cup \ldots A_n) \ge p$. Show that

$$\mathrm{P}(A_i) \ge 1 - (1-p)^{1/n}.$$

Hint: Use Harris' inequality.

7. The critical value for $\mathbb{Z}^2$ equals $1/2$

There are a number of proofs of this result. Here we will stick more or less to the original, following [27]. It however will be a little simpler since we will use only the RSW theory (see below) for $p = 1/2$. The reason for my doing this older proof is that it illustrates a number of important and interesting things.

7.1. Proof of $p_c(2) = 1/2$ assuming RSW

First, L-R will stand for left-right and T-B for top-bottom. Let $J_{n,m}$ be the event that there is a L-R crossing of $[0,n] \times [0,m]$. Let $J'_{n,m}$ be the event that there is a L-R crossing of $[0,n] \times (0,m)$ (i.e., of $[0,n] \times [1, m-1]$).

Our first lemma explains a very special property for $p = 1/2$ which already hints that this might be the critical value.

Lemma 7.1. $\mathrm{P}_{1/2}(J_{n+1,n}) = 1/2$.

Proof. This is a symmetry argument using the dual lattice. Let B be the event that there is a B-T crossing of $[1/2, n+1/2] \times [-1/2, n+1/2]$ using closed edges in the dual lattice. By a version of Whitney's theorem, Lemma 2.3, we have that $J_{n+1,n}$ occurs if and only if the event B fails. Hence for any p, $\mathrm{P}_p(J_{n+1,n}) + \mathrm{P}_p(B) = 1$. If $p = 1/2$, we have by symmetry that these two events have the same probability and hence each must have probability 1/2. □

The next theorem is crucial for this proof and is also crucial in Section 9. It will be essentially proved in Subsection 7.2. It is called the Russo Seymour Welsh or RSW Theorem and was proved independently in [31] and [26].

Theorem 7.2. *For all k, there exists c_k so that for all n, we have that*

$$\mathrm{P}_{1/2}(J_{kn,n}) \geq c_k. \tag{7.1}$$

Remarks: There is in fact a stronger version of this which holds for all p which says that if $\mathrm{P}_p(J_{n,n}) \geq \tau$, then $\mathrm{P}_p(J_{kn,n}) \geq f(\tau, k)$ where f does not depend on n or on p. This stronger version together with (essentially) Lemma 7.1 immediately yields Theorem 7.2 above. The proof of this stronger version can be found in [12] but we will not need this stronger version here. The special case dealing only with the case $p = 1/2$ above has a simpler proof due to Smirnov, especially in the context of the so-called triangular lattice. (In [6], an alternative simpler proof of the stronger version of RSW can be found.)

We will now apply Theorem 7.2 to prove that $\theta(1/2) = 0$. This is the first important step towards showing $p_c = 1/2$ and was done in 1960 by Harris (although by a different method since the RSW Theorem was not available in 1960). As already stated, it took 20 more years before Kesten proved $p_c = 1/2$. Before proving $\theta(1/2) = 0$, we first need a lemma which is important in itself.

Lemma 7.3. *Let $O(\ell)$ be the event that there exists an open circuit containing 0 in*

$$\mathrm{Ann}(\ell) := [-3\ell, 3\ell]^2 \backslash [-\ell, \ell]^2.$$

Then there exists $c > 0$ so that for all ℓ,

$$\mathrm{P}_{1/2}(O(\ell)) \geq c.$$

Proof. $\mathrm{Ann}(\ell)$ is the (non–disjoint) union of four $6\ell \times 2\ell$ rectangles. By Theorem 7.2, we know that there is a constant c' so that for each ℓ, the probability of crossing (in the longer direction) a $6\ell \times 2\ell$ rectangle is at least c'. By Harris' inequality, the probability of crossing each of the 4 rectangles whose union is $\mathrm{Ann}(\ell)$ is at least $c := c'^4$. However, if we have a crossing of each of these 4 rectangles, we clearly have the circuit we are after. □

Lemma 7.4. $\theta(1/2) = 0$.

Proof. Let C_k be the event that there is a circuit in $Ann(4^k) + 1/2$ in the dual lattice around the origin consisting of closed edges. A picture shows that the C_k's are independent and Lemma 7.3 implies that $\mathrm{P}_{1/2}(C_k) \geq c$ for all k for some $c > 0$. It follows that $\mathrm{P}_{1/2}(C_k \text{ i.o. }) = 1$ where as usual, C_k i.o. means that there are infinitely many k so that C_k occurs. However, the latter event implies by Lemma 2.3 that the origin does not percolate. Hence $\theta(1/2) = 0$. $\square$

The next lemma is very interesting in itself in that it gives a *finite criterion* which implies percolation.

Proposition 7.5. *(Finite size criterion) For any p, if there exists an n such that*

$$\mathrm{P}_p(J'_{2n,n}) \geq .98,$$

then $\mathrm{P}_p(|C(0)| = \infty) > 0$.

To prove this proposition, one first establishes

Lemma 7.6. *For any $\epsilon \leq .02$, if $\mathrm{P}_p(J'_{2n,n}) \geq 1 - \epsilon$, then $\mathrm{P}_p(J'_{4n,2n}) \geq 1 - \epsilon/2$.*

Proof. Let B_n be the event that there exists an L-R crossing of $[n, 3n] \times (0, n)$. Let C_n be the event that there exists an L-R crossing of $[2n, 4n] \times (0, n)$. Let D_n be the event that there exists a T-B crossing of $[n, 2n] \times [0, n]$. Let E_n be the event that there exists a T-B crossing of $[2n, 3n] \times [0, n]$.

We have $\mathrm{P}_p(D_n) = \mathrm{P}_p(E_n) \geq \mathrm{P}_p(B_n) = \mathrm{P}_p(C_n) = \mathrm{P}_p(J'_{2n,n}) \geq 1 - \epsilon$. Therefore $\mathrm{P}_p(J'_{2n,n} \cap B_n \cap C_n \cap D_n \cap E_n) \geq 1 - 5\epsilon$. By drawing a picture, one sees that the above intersection of 5 events is contained in $J'_{4n,n}$. Letting $\tilde{J}'_{4n,n}$ denote the event that exists a L-R crossing of $[0, 4n] \times (n, 2n)$, we have that $J'_{4n,n}$ and $\tilde{J}'_{4n,n}$ are independent, each having probability at least $1 - 5\epsilon$ and so

$$\mathrm{P}_p(J'_{4n,n} \cup \tilde{J}'_{4n,n}) = 1 - (1 - \mathrm{P}_p(J'_{4n,n}))^2 \geq 1 - 25\epsilon^2.$$

This is at least $1 - \epsilon/2$ if $\epsilon \leq .02$. Since $J'_{4n,n} \cup \tilde{J}'_{4n,n} \subseteq J'_{4n,2n}$, we are done. $\square$

Proof of Proposition 7.5.
Choose n_0 such that $\mathrm{P}_p(J'_{2n_0,n_0}) \geq .98$. Lemma 7.6 and induction implies that for all $k \geq 0$

$$\mathrm{P}_p(J'_{2^{k+1}n_0, 2^k n_0}) \geq 1 - \frac{.02}{2^k}$$

and hence

$$\sum_{k \geq 0} \mathrm{P}_p((J'_{2^{k+1}n_0, 2^k n_0})^c) < \infty.$$

We will define events $\{H_k\}_{k \geq 0}$ where H_k will be a crossing like $J'_{2^{k+1}n_0, 2^k n_0}$ except in a different location and perhaps with a different orientation. Let $H_0 = J'_{2n_0,n_0}$.

Then let H_1 be a T-B crossing of $(0,2n_0)\times[0,4n_0]$, H_2 be a L-R crossing of $[0,8n_0]\times(0,4n_0)$, H_3 be a T-B crossing of $(0,8n_0)\times[0,16n_0]$, etc. Since the probability of H_k is the same as for $J'_{2^{k+1}n_0,2^kn_0}$, the Borel-Cantelli lemma implies that a.s. all but finitely many of the H_k's occur. However, it is clear geometrically that if all but finitely many of the H_k's occur, then there is percolation. □

Our main theorem in this section is

Theorem 7.7. $p_c = 1/2$.

Before doing the proof, we make a digression and explain how the concept of a *sharp threshold* yields the main result as explained in [28]. In order not to lose track of the argument, this digression will be short and more details concerning this approach will be discussed in subsection 7.3.

The underlying idea is that increasing events A which 'depend on lots of random variables' have "sharp thresholds" meaning that the function $\mathrm{P}_p(A)$, as p increases from 0 to 1, goes very sharply from being very small to being very large.

Exercise 7.1. Let A be the event in $\{0,1\}^J$ corresponding to the first bit being a 1. Note that $\mathrm{P}_p(A) = p$ and hence does not go quickly from 0 to 1 but then again A does not depend on a lot of random variables. Look at what happens if $n = |J|$ is large and A is the event that at least half the random variables are 1.

Definition 7.8. A sequence of increasing events A_n *has a sharp threshold* if for all $\epsilon > 0$, there exists N such that for all $n \ge N$, there is an interval $[a, a+\epsilon] \subseteq [0,1]$ (depending on n) such that

$$\mathrm{P}_p(A_n) < \epsilon \text{ for } p \le a$$

and

$$\mathrm{P}_p(A_n) > 1-\epsilon \text{ for } p \ge a+\epsilon.$$

We claim that if the sequence of events $J'_{2n,n}$ has a sharp threshold, then $p_c \le 1/2$. The reason for this is that if p_c were $1/2+\delta$ with $\delta > 0$, then, since the probability of $J'_{2n,n}$ at $p = 1/2$ is not too small due to Theorem 7.2, a sharp threshold would tell us that, for large n, $\mathrm{P}_p(J'_{2n,n})$ would get very large way before p reaches $1/2+\delta$. However, Proposition 7.5 would then contradict the definition of the critical value. Slightly more formally, we have the following.

Proof of Theorem 7.7 **assuming** $\{J'_{2n,n}\}$ has a sharp threshold.
First, Lemma 7.4 implies that $p_c \ge 1/2$. Assume $p_c = 1/2+\delta_0$ with $\delta_0 > 0$. Then Theorem 7.2 tells us there is $c > 0$ such that

$$\mathrm{P}_{1/2}(J'_{2n,n}) \ge c$$

for all n. Let $\epsilon = \min\{\delta_0/2, .02, c\}$. Choose N as given in the definition of a sharp threshold and let $[a, a+\epsilon]$ be the corresponding interval for $n = N$. Since $\mathrm{P}_{1/2}(J'_{2N,N}) \ge c \ge \epsilon$, a must be $\le 1/2$. Hence $1/2+\delta_0/2 \ge a+\delta_0/2 \ge a+\epsilon$ and hence $P_{1/2+\delta_0/2}(J'_{2N,N}) \geq 1-\epsilon \geq .98$. By Proposition 7.5, we get $P_{1/2+\delta_0/2}(|C(0)| = \infty) > 0$, a contradiction. $\square$

We now follow Kesten's proof as modified by Russo ([27]). We even do a slight modification of that so that the stronger version of RSW is avoided; this was explained to me by A. Bálint.

Proof of Theorem 7.7.
Note that Lemma 7.4 implies that $p_c \geq 1/2$. Assume now that $p_c = 1/2 + \delta_0$ with $\delta_0 > 0$. Let V_n be the number of pivotal edges for the event $J_{4n,n}$. The key step is the following proposition.

Proposition 7.9. *If* $p_c = 1/2 + \delta_0$ *with* $\delta_0 > 0$*, then*

$$\lim_{n\to\infty} \inf_{p\in[1/2,1/2+\delta_0/2]} E_p[V_n] = \infty.$$

Assuming this proposition, the Margulis-Russo formula would then give

$$\lim_{n\to\infty} \inf_{p\in[1/2,1/2+\delta_0/2]} (d/dp)P_p(J_{4n,n}) = \infty.$$

Since $\delta_0 > 0$ by assumption, this of course contradicts the fact that these probabilities are bounded above by 1. $\square$

Proof of Proposition 7.9.
Since $J'_{4n,n/2}$ is an increasing event, Theorem 7.2 implies that

$$\inf_{p\in[1/2,1/2+\delta_0/2],n} P_p(J'_{4n,n/2}) := \epsilon_1 > 0.$$

Next, letting U_n be the event that there is a B-T dual crossing of $[2n + 1/2, 4n - 1/2] \times [-1/2, n + 1/2]$ consisting of closed edges, we claim that

$$\inf_{p\in[1/2,1/2+\delta_0/2],n} P_p(U_n) := \epsilon_2 > 0.$$

The reason for this is that $P_p(U_n)$ is minimized for all n at $p = 1/2 + \delta_0/2$, Proposition 7.5 implies that $P_{1/2+\delta_0/2}(J'_{2n,n}) \leq .98$ for all n since $1/2+\delta_0/2 < p_c$, and the fact that the event $J'_{2n,n}$ translated to the right distance $2n$ and U_n are complementary events.

Now, if U_n occurs, let σ be the right-most such crossing. (You need to think about this and convince yourself it is reasonable that, when such a path exists, there exists a right-most path; I would not worry about a precise proof of this which can be topologically quite messy). Since σ is the right-most crossing, given the path σ, we know nothing about what happens to the left of σ. (Although you do not need to know this, it is worth pointing out that this is some complicated analogue of what is known as a stopping time and the corresponding strong Markov property.) Therefore, conditioned on σ, there is probability at least ϵ_1 that there is a path of open edges from the left of $[0, 4n] \times [0, n/2]$ all the way to 1 step to the left of σ. Note that if this happens, then there is an edge one step away from the end of this path which is pivotal for $J_{4n,n}$! Let γ be the lowest such path if one exists. Conditioned on both σ and γ, we know nothing about the area to the "top-left" of these curves. Let q be the point where σ and γ meet. For each n, consider the annuli $Ann(4^k) + 1/2 + q$ (this is our previous annulus but centered around q) but only for those k's where $4^k \leq n/2$. Lemma 7.3 together with the fact that the events $O(\ell)$ are increasing implies that there is a fixed probability ϵ_3, independent of n and $p \in [1/2, 1/2 + \delta_0/2]$ and k (with $4^k \leq n/2$), such that with probability at least ϵ_3, there is an open path from γ to 1 step to the left of σ running within $Ann(4^k) + 1/2 + q$ in this "top-left" area. (For different k's, these events are independent but this is not needed). Note that each k where this occurs gives us a different pivotal edge for the event $J_{4n,n}$. Since the number of k's satisfying $4^k \leq n/2$ goes to infinity with n, the proposition is established. □

7.2. RSW

In this section, we prove Theorem 7.2. However, it is more convenient to do this for site percolation on the triangular lattice or equivalently the hexagonal lattice instead. (In site percolation, the sites of the graph are independently declared to be white or black with probability p and one asks for the existence of an infinite path of white vertices; edges are not removed.)

(The reader will trust us that the result is equally true on $\mathbb{Z}^2$ and is welcome to carry out the details.) This simpler proof of RSW for $p = 1/2$ on the triangular lattice was discovered by Stas Smirnov and can be found in [13] or in [36].

We first quickly define what the triangular lattice and hexagonal lattice are. The set of vertices consists of those points $x + e^{i\pi/3}y$ where x and y are integers. Vertices having distance 1 have an edge between them. See Figure 7 which depicts the triangular lattice. The triangular lattice is equivalent to the hexagonal lattice where the vertices of the triangular lattice correspond to the hexagons. Figure 7 shows how one moves between these representations. It is somehow easier to visual the hexagonal lattice compared to the triangular lattice. We mention that the duality in Lemma 2.3 is much simpler in this context. For example, there is a sequence of white hexagons from the origin out to infinity if and only if there is no path encircling the origin consisting of black hexagons. So, in working with the hexagonal lattice, one does not need to introduce the dual graph as we did for $\mathbb{Z}^2$.

Outline of proof of Theorem 7.2 for $p = 1/2$ for the hexagonal lattice instead. (We follow exactly the argument in [36].)

The key step is to prove the following lemma. (The modification for the definition of $J_{n,m}$ for the triangular lattice should be pretty clear.)

Lemma 7.10.

$$\mathrm{P}_{1/2}(J_{2n,m}) \geq (1/4)(\mathrm{P}_{1/2}(J_{n,m}))^2.$$

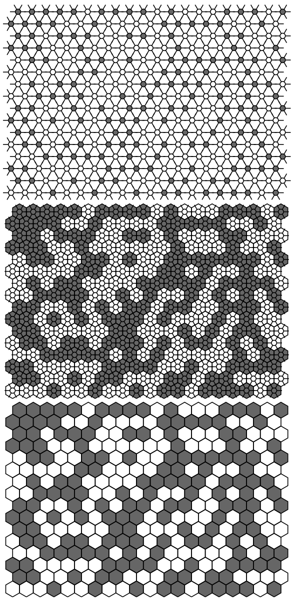

FIGURE 7. From sites to cells (picture by O. Schramm and provided by W. Werner)

Proof. Write P for $P_{1/2}$. If a L-R crossing of $[0,n] \times [0,m]$ exists, let γ be the "highest" one. γ has a type of "Markov property" which says the following. If g is a L-R path (not necessarily open) of $[0,n] \times [0,m]$ touching the left and right sides only once, then the event $\{\gamma = g\}$ is independent of the percolation process "below" g.

If g is such a L-R path (not necessarily open) of $[0,n] \times [0,m]$, let g' be the reflection of g about $x = n$ (which is then a L-R crossing of $[n,2n] \times [0,m]$. Assume g does not touch the x-axis (a similar argument can handle that case as well). Let σ be the part of the boundary of $[0,2n] \times [0,m]$ which is on the left side and below g (this consists of 2 pieces, the x-axis between 0 and n and the positive y-axis below the left point of g). Let σ' be the reflection of σ about $x = n$. Symmetry and duality gives that the probability that there is an open path from right below g to σ' is $1/2$. Call this event $A(g)$.

Observe that if g is a path as above which is open and $A(g)$ occurs, then we have a path from the left side of $[0,2n] \times [0,m]$ to either the right side of this box or to the bottom boundary on the right side; call this latter event F. We obtain

$$\mathrm{P}(F) = \mathrm{P}(\cup_g (F \cap \{\gamma = g\})) \geq \mathrm{P}(\cup_g (A(g) \cap \{\gamma = g\})).$$

$A(g)$ and $\{\gamma = g\}$ are independent and we get that the above is therefore

$$1/2 \sum_g \mathrm{P}(\{\gamma = g\}) = \mathrm{P}(J_{n,m})/2.$$

If F' is defined to be the reflection about $x = n$ of the event F, then by Theorem 6.2, $\mathrm{P}(F \cap F') \geq (1/4)(\mathrm{P}(J_{n,m}))^2$. Finally, one observes that $F \cap F' \subseteq J_{2n,m}$. □

Continuing with the proof, we first note that the analogue of Lemma 7.1 for the triangular lattice is that the probability of a crossing of white (or black) hexagons of the $2n \times 2n$ rhombus in Figure 8 is exactly $1/2$ for all n. With some elementary geometry, one sees that a crossing of such a rhombus yields a crossing of a $n \times \sqrt{3}n$ rectangle and hence $\mathrm{P}(J_{n,\sqrt{3}n}) \geq 1/2$ for all n. From here, Lemma 7.10 gives the result by induction. □

7.3. Other approaches.

We briefly mention in this section other approaches to computing the critical value of $1/2$.

Exercise 7.2. (Outline of alternative proof due to Yu Zhang for proving $\theta(1/2) = 0$ using uniqueness of the infinite cluster; this does not use RSW).

Step 1: Assuming that $\theta(1/2) > 0$, show that the probability that there is an open path from the right side of $[-n,n] \times [-n,n]$ to infinity touching this box only once (call this event E) approaches 1 as $n \to \infty$. (Hint: Use the square-root trick of the previous section, Exercise 6.5.)

Step 2: Let F be the event analogous to E but using the left side instead, G the analogous event using closed edges in the dual graph and the top of the box and H the analogous event to G using the bottom of the box. Using step 1, show that for large n $\mathrm{P}(E \cap F \cap G \cap H) > 0$.

Step 3: Show how this contradicts uniqueness of the infinite cluster.

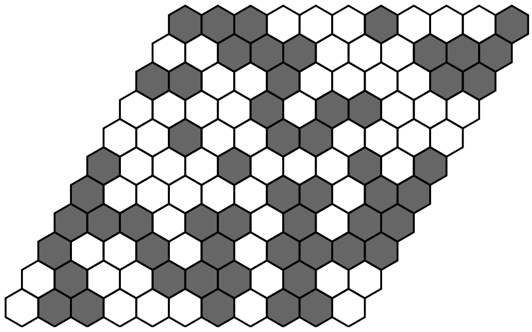

FIGURE 8. White top-to-bottom crossing vs. black horizontal crossing (picture provided by W. Werner)

In the next section, the following very nontrivial result is mentioned but not proved.

Theorem 7.11. *In any dimension, if $p < p_c(d)$, then there exists $c = c(p) > 0$ so that the probability that there is an open path from the origin to distance n away is at most e^{-cn}.*

Exercise 7.3. Use Theorem 7.11 and Lemma 7.1 to show that $p_c(Z^2) \leq 1/2$.

Alternatively, there are at present fairly sophisticated and general results which imply sharp thresholds, which we have seen is the key to proving that the critical value is 1/2. An early version of such a result comes from [28], where the following beautiful result was proved.

Theorem 7.12. *Let A_n be a sequence of increasing events. If (recall Definition 6.3)*

$$\lim_{n\to\infty} \sup_{p,i} I_i^p(A_n) = 0,$$

then the sequence $\{A_n\}$ has a sharp threshold.

Exercise 7.4. Show that $\lim_{n\to\infty} \sup_{p,i} I_i^p(J'_{2n,n}) = 0$.
Hint: Use the fact that $\theta(1/2) = 0$. (This exercise together with Theorem 7.12 allows us, as we saw after Definition 7.8, to conclude that $p_c(Z^2) \leq 1/2$.)

Theorem 7.12 is however nontrivial to prove. In [6], the threshold property for crossings is obtained by a different method. The authors realized that it could be deduced from a sharp threshold result in [10] via a symmetrization argument. A key element in the proof of the result in [10] is based on [17], where it is shown that for $p = 1/2$ if an event on n variables has probability bounded away from 0 and 1, then there is a variable of influence at least $\log(n)/n$ for this event. (The proof of this latter very interesting result uses Fourier analysis and the concept of hypercontractivity). As pointed out in [10], the argument in [17] also yields that if all the influences are small, then the sum of all the influences is very large (provided again that the variance of the event is not too close to 0). A sharpened version of this latter fact was also obtained in [35] after [17]. Using this, one can avoid the symmetrization argument mentioned above. One of the big advantages of the approach in [6] is that it can be applied to other models. In particular, with the help of the sharp threshold result of [10], it is proved in [7] that the critical probability for a model known as Voronoi percolation in the plane is 1/2. It seems that at this time neither Theorem 7.12 nor other approaches can achieve this. This approach has also been instrumental for handling certain dependent models.

8. Subexponential decay of the cluster size distribution

Lots of things *decay exponentially* in percolation when you are away from criticality but not everything. I will first explain three objects related to the percolation cluster which do decay exponentially and finally one which does not, which is the point of this section. If you prefer, you could skip down to Theorem 8.1 if you only want to see the main point of this section.

For our first exponential decay result, it was proved independently by Menshikov and by Aizenman and Barsky that in any dimension, if $p < p_c$, the probability of a path from the origin to distance n away decays exponentially. This was a *highly nontrivial* result which had been one of the major open questions in percolation theory in the 1980's. It easily implies that the expected size of the cluster of the origin is finite for $p < p_c$.

Exercise 8.1. Prove this last statement.

It had been proved by Aizenman and Newman before this that if the expected size of the cluster of the origin is finite at parameter p, then the cluster size has an exponential tail in the sense that

$$\mathrm{P}_p(|C| \geq n) \leq e^{-cn}$$

for all n for some $c > 0$ where c will depend on p. This does not follow from the Menshikov and Aizenman-Barsky result since that result says that the radius of the cluster of the origin decays exponentially while $|C|$ is sort of like the radius raised to the dth power and random variables who have exponential tails don't necessary have the property that their powers also have exponential tails.

The above was all for the subcritical case. In the supercritical case, the radius, conditioned on being finite, also decays exponentially, a result of Chayes, Chayes and Newman. This says that for $p > p_c$,

$$\mathrm{P}_p(A_n) \leq e^{-cn}$$

for all n for some $c = c(p) > 0$ where A_n is the event that there is an open path from the origin to distance n away and $|C| < \infty$. Interestingly, it turns out on the other hand that the cluster size, conditioned on being finite, does not have an exponential tail in the supercritical regime.

Theorem 8.1. *For any $p > p_c$, there exists $c = c(p) < \infty$ so that for all n*

$$\mathrm{P}_p(n \leq |C| < \infty) \geq e^{-cn^{(d-1)/d}}.$$

Note that this rules out the possible exponential decay of the tail of the cluster size. (Why?) As can be seen by the proof, the reason for this decay rate is that it is a *surface area effect*. Here is the 2 line 'proof' of this theorem. By a law of large numbers type thing (and more precisely an ergodic theorem), the number of points belonging to the infinite cluster in a box with side length $n^{1/d}$ should be about $\theta(p)n$. However, with probability a fixed constant to the $n^{(d-1)/d}$, the inner boundary has all edges there while the outer boundary has no edges there which gives what we want.

Lemma 8.2. *Assume that $\theta(p) > 0$. Fix m and let X_m be the number of points in $B_m := [-m, m]^d$ which belong to the infinite cluster and F_m be the event that $\{X_m \geq |B_m|\theta(p)/2\}$. Then*

$$\mathrm{P}(F_m) \geq \theta(p)/2.$$

Proof. Clearly $E(X_m) = |B_m|\theta(p)$. We also have

$$E(X_m) = E[X_m|F_m]\mathrm{P}(F_m) + E[X_m|F_m^c]\mathrm{P}(F_m^c) \leq |B_m|\mathrm{P}(F_m) + |B_m|\theta(p)/2.$$

Now solve for $\mathrm{P}(F_m)$. □

Proof of Theorem 8.1.
Using the definition of B_m as in the previous lemma, we let F be the event that there are at least $|B_{n^{1/d}}|\theta(p)/2$ points in $[0, 2n^{1/d}]^d$ which reach the inner boundary. Let G be the event that all edges between points in the inner boundary are on and let H be the event that all edges between points in the inner boundary and points in the outer boundary are closed. Clearly these three events are independent. The probability of F has, by Lemma 8.2, a constant positive probability not depending on n, and G and H each have probabilty at least $e^{-cn^{(d-1)/d}}$ for some constant

$c < \infty$. The intersection of these three events implies that $|B_{n^{1/d}}|\theta(p)/2 \leq |C| < \infty$ yielding that

$$\mathrm{P}_p(|B_{n^{1/d}}|\theta(p)/2 \leq |C| < \infty) \geq e^{-cn^{(d-1)/d}}.$$

This easily yields the result. □

9. Conformal invariance and critical exponents

This section will touch on some very important recent developments in 2 dimensional percolation theory that has occurred in the last 10 years. I would not call this section even an overview since I will only touch on a few things. There is a lot to be said here and this section will be somewhat less precise than the earlier sections. See [36] for a thorough exposition of these topics.

9.1. Critical exponents

We begin by discussing the concept of critical exponents. We have mentioned in Section 8 that below the critical value, the probability that the origin is connected to distance n away decays exponentially in n, this being true for any dimension. It had been conjectured for some time that in 2 dimensions, at the critical value itself, the above probability decays like a power law with a certain power, called a *critical exponent*. While this is believed for $\mathbb{Z}^2$, it has only been proved for the hexagonal lattice. Recall that one does *site* percolation rather than bond (edge) percolation on this lattice; in addition, the critical value for this lattice is also 1/2 as Kesten also showed.

Let A_n be the event that there is a white path from the origin to distance n away.

Theorem 9.1. *(Lawler, Schramm and Werner,* [20]*) For the hexagonal lattice, we have*

$$\mathrm{P}_{1/2}(A_n) = n^{-5/48+o(1)}$$

where the $o(1)$ is a function of n going to 0 as $n \to \infty$.

Remarks: Critical values (such as 1/2 for the $\mathbb{Z}^2$ and the hexagonal lattice) are not considered *universal* since if you change the lattice in some way the critical value will typically change. However, a critical exponent such as 5/48 is believed to be universal as it is believed that if you take any 2-dimensional lattice and look at the critical value, then the above event will again decay as the $-\frac{5}{48}$th power of the distance.

Exercise 9.1. For $\mathbb{Z}^2$, show that there exists $c > 0$ so that for all n

$$\mathrm{P}_{1/2}(A_n) \leq n^{-c}.$$

Hint: Apply Lemma 7.3 to the dual graph.

Exercise 9.2. For $\mathbb{Z}^2$, show that for all n

$$\mathrm{P}_{1/2}(A_n) \geq (1/4)n^{-1}.$$

Hint: Use Lemma 7.1.

There are actually an infinite number of other known critical exponents. Let A_n^k be the event that there are k disjoint monochromatic paths starting from within distance (say) $2k$ of the origin all of which reach distance n from the origin and such that at least one of these monochromatic paths is white and at least one is black. This is referred to as the k-arm event. Let $A_R^{k,H}$ be the analogous event but where we restrict to the upper half plane and where the restriction of having at least one white path and one black path may be dropped. This is referred to as the half-plane k-arm event.

Theorem 9.2. *(Smirnov and Werner, [33]) For the hexagonal lattice, we have*
(i) For $k \geq 2$, $\mathrm{P}_{1/2}(A_n^k) = n^{-(k^2-1)/12+o(1)}$
(ii) For $k \geq 1$, $\mathrm{P}_{1/2}(A_n^{k,H}) = n^{-k(k+1)/6+o(1)}$ where again the $o(1)$ is a function of n going to 0 as $n \to \infty$.

The proofs of the above results crucially depended on "conformal invariance", a concept discussed in Section 9.3 and which was proved by Stas Smirnov.

9.2. "Elementary" critical exponents

Computing the exponents in Theorems 9.1 and 9.2 is highly nontrivial and relies on the important concept of conformal invariance to be discussed in the next subsection as well as an extremely important development called the Schramm-Löwner evolution. We will not do any of these two proofs; a few special cases of them are covered in [36]. (These cases are not the ones which can be done via the "elementary" methods of the present subsection.)

It turns out however that some exponents can be computed by "elementary" means. This section deals with a couple of them via some long exercises. Exercises 9.3 and 9.4 below are taken (with permission) (almost) completely verbatim from [36]. In order to do these exercises, we need another important inequality, which we managed so far without; this inequality, while arising quite often, is used only in this subsection as far as these notes are concerned. It is called the BK inequality named after van den Berg and Kesten; see [5]. The first ingredient is a somewhat subtle operation on events.

Definition 9.3. Given two events A and B depending on only finitely many edges, we define $A \circ B$ as the event that there are two disjoint sets of edges S and T such that the status of edges in S guarantee that A occurs (no matter what the status of edges outside of S are) and the status of edges in T guarantee that B occurs (no matter what the status of edges outside of T are).

Example: To understand this concept, consider a finite graph G where percolation is being performed, let x, y, z and w be four (not necessarily distinct) vertices, let A be the event that there is an open path from x to y, let B be the event that there is an open path from z to w and think about what $A \circ B$ is in this case. How should the probability $\mathrm{P}_p(A \circ B)$ compare with that of $\mathrm{P}_p(A)\mathrm{P}_p(B)$?

Theorem 9.4. *(van den Berg and Kesten, [5]) (BK inequality).*
If A and B are increasing events, then $\mathrm{P}_p(A \circ B) \le \mathrm{P}_p(A)\mathrm{P}_p(B)$.

While this might seem 'obvious' in some sense, the proof is certainly non-trivial. When it was proved, it was conjectured to hold for all events A and B. However, it took 12 more years (with false proofs by many people given on the way) before David Reimer proved the above inequality for general events A and B.

Exercise 9.3. **Two-arm exponent in the half-plane.** Let $p = 1/2$. Let Λ_n be the set of hexagons which are of distance at most n from the origin. We consider $H_n = \{z \in \Lambda_n \,:\, \Im(z) \ge 0\}$ where $\Im(z)$ is defined to be the imaginary part of the center of z. The boundary of H_n can be decomposed into two parts: a horizontal segment parallel to and close to the real axis and the "angular semi-circle" h_n. We say that a point x on the real line is n-good if there exist one open path originating from x and one closed path originating from $x + 1$, that both stay in $x + H_n$ and join these points to $x + h_n$ (these paths are called "arms"). Note that the probability w_n that a point x is n-good does not depend on x.

1) We consider percolation in H_{2n}.
a) Prove that with a probability that is bounded from below independently of n, there exists an open cluster O and a closed cluster C, that both intersect the segment $[-n/2, n/2]$ and h_{2n}, such that C is "to the right" of O.
b) Prove that in the above case, the right-most point of the intersection of O with the real line is n-good.
c) Deduce that for some absolute constant c, $w_n \ge c/n$.

2) We consider percolation in H_n.
a) Show that the probability that there exists at least k disjoint open paths joining h_n to $[-n/2, n/2]$ in H_n is bounded by λ^k for some constant λ that does not depend on n (hint: use the BK inequality). Show then that the number K of open clusters that join h_n to $[-n/2, n/2]$ satisfies $\mathrm{P}(K \ge k) \le \lambda^k$.
b) Show that each $2n$-good point in $[-n/2, n/2]$ is the right-most point of the intersection of one of these K clusters with the real line.
c) Deduce from this that for some absolute constant c', $(n + 1)w_{2n} \le E(K) \le c'$.

3) Conclude that for some positive absolute constants c_1 and c_2, $c_1/n \le w_n \le c_2/n$.

Exercise 9.4. **Three-arm exponent in the half-plane.** Let $p = 1/2$. We say that a point x is n-Good (mind the capital G) if it is the *unique* lowest point in $x + H_n$ of an open cluster C such that $C \not\subseteq x + H_n$. Note that the probability v_n that a point is n-Good does not depend on x.

1) Show that this event corresponds to the existence of three arms, having colors black, white and black, originating from the neighborhood of x in $x + H_n$.

2) Show that the expected number of clusters inside H_n that join $h_{n/2}$ to h_n is bounded. Compare this number of clusters with the number of $2n$-Good points in $H_{n/2}$ and deduce from this that for some constant c_1, $v_n \le c_1/n^2$.

3) Show that with probability bounded from below independently of n, there exists in $H_{n/2}$ an n-Good point (note that an argument is needed to show that with positive probability, there exists a cluster with a unique lowest point). Deduce that for some positive absolute constant c_2, $v_n \geq c_2/n^2$.

9.3. Conformal invariance

One of the key steps in proving the above theorems is to prove *conformal invariance* of percolation which is itself very important. Before even getting to this, we warm up with posing the following question, which could be thought of in either the context of $\mathbb{Z}^2$ or the hexagonal lattice.

Question: Letting a_n be the probability at criticality of having a crossing of $[0, 2n] \times [0, n]$, does $\lim_{n\to\infty} a_n$ exist?

Remark: We have seen in Section 7 that $a_n \leq 1/2$ and that $\liminf a_n > 0$.

The central limit theorem in probability is a sort of scaling limit. It says that if you add up many i.i.d. random variables and normalize them properly, you get a nice limiting object (which is the normal distribution). We would like to do something vaguely similar with percolation, where the lattice spacing goes to 0 and we ask if some limiting object emerges. In this regard, note that a_n is exactly the probability that there is a crossing of $[0, 2] \times [0, 1]$ on the lattice scaled down by $1/n$. In studying whether percolation performed on smaller and smaller lattices might have some limiting behavior, looking at a crossing of $[0, 2] \times [0, 1]$ is one particular (of many) global or macro-events that one may look at. If p is subcritical, then a_n goes exponentially to 0, which implies that subcritical percolation on the $1/n$ scaled down lattice has no interesting limiting behavior.

The conformal invariance conjecture contains 3 ingredients; (i) limits, such as the sequence a_n above exist (ii) their values depend only on the 'structure' of the domain and hence are conformally invariant and (iii) exact values, due to Cardy, of the values of these limits. We now make this precise.

Let Ω be a bounded simply connected domain of the plane and let A, B, C and D be 4 points on the boundary of Ω in clockwise order. Scale a 2-dimensional lattice, such as $\mathbb{Z}^2$ or the hexagonal lattice T, by $1/n$ and perform critical percolation on this scaled lattice. Let $\mathrm{P}(\Omega, A, B, C, D, n)$ denote that the probability that in the $1/n$ scaled hexagonal lattice, there is an open path in Ω going from the boundary of Ω between A and B to the boundary of Ω between C and D. (For $\mathbb{Z}^2$, an open path should be interpreted as a path of open edges while for T, it should be interpreted as a path of white hexagons.) The first half of the following conjecture is attributed to Michael Aizenman and the second half of the conjecture to John Cardy.

Conjecture 9.5. *(i) For all Ω and A, B, C and D as above,*

$$\mathrm{P}(\Omega, A, B, C, D, \infty) := \lim_{n\to\infty} \mathrm{P}(\Omega, A, B, C, D, n)$$

exists and is conformally invariant in the sense that if f is a conformal mapping, then $\mathrm{P}(\Omega, A, B, C, D, \infty) = \mathrm{P}(f(\Omega), f(A), f(B), f(C), f(D), \infty)$.

(ii) There is an explicit formula (which we do not state here) for $\mathrm{P}(\Omega, A, B, C, D, \infty)$*, called Cardy's formula, when* Ω *is a rectangle and* A, B, C *and* D *are the 4 corner points. (Since every* Ω*,* A, B, C *and* D *can be mapped to a unique such rectangle (with* A, B, C *and* D *going to the 4 corner points), this would specify the above limit in general assuming conformal invariance.)*

Cardy's formula was quite complicated involving hypergeometric functions but Lennart Carleson realized that assuming conformal invariance, there is a nicer set of "representing" domains with four specified points where the formula simplfies tremendously. Namely, if Ω is an equilateral triangle (with side lengths 1), A, B and C the three corner points and D (on the line between C and A) having distance x from C, then the above probability would just be x. Using Carleson's reformulation of Cardy's formula, Smirnov proved the above conjecture for the hexagonal lattice.

Theorem 9.6. [32] *For the hexagonal lattice, both (i) and (ii) of the above conjecture are true.*

This conjecture is also believed to hold on $\mathbb{Z}^2$ but is not (yet) proved in that case. An important related object is the interface between whites and blacks in the upper half plane when one places white hexagons on the positive x–axis and black hexagons on the negative x–axis; see Figure 9. In [29], Schramm described what this interface should be as the lattice spacing goes to 0, assuming conformal invariance. This paper is where the now famous SLE (for stochastic Löwner evolution and later called the Schramm-Löwner evolution) was introduced. Smirnov [32] proved the convergence for one interface and Camia and Newman [9] proved a "full scaling limit", which is a description of the behavior of all the interfaces together. The critical exponents described in Theorems 9.1 and 9.2 are proved by exploiting the SLE description of the interface. Theorem 9.1 and one case of Theorem 9.2 are described in [36].

In the summer school, we went through the proof of Theorem 9.6 in detail following precisely the argument in [2]. The argument can also be found on-line in [13] or in [36] as well as in a number of other places. The argument differed from [2] more or less only in presentation and so after some thought, I decided not to detail this argument here.

10. Noise sensitivity and dynamical percolation

This last section gives an *extremely* quick overview of some other interesting developments in percolation most of which are surveyed in [34].

Noise sensitivity: In [3], the concept of noise sensitivity is introduced. Here we only explain what this is in the context of percolation. Perform percolation with $p = 1/2$ and consider the event E_n that there is a L-R open crossing of an $(n+1) \times n$ box. Recall from Lemma 7.1 that $\mathrm{P}(E_n) = 1/2$ for each n. Now, fix $\epsilon > 0$ and flip the status of each edge (from open to closed or from closed to open) independently with probability ϵ. Let E_n^ϵ be the event that there is a L-R open crossing of the

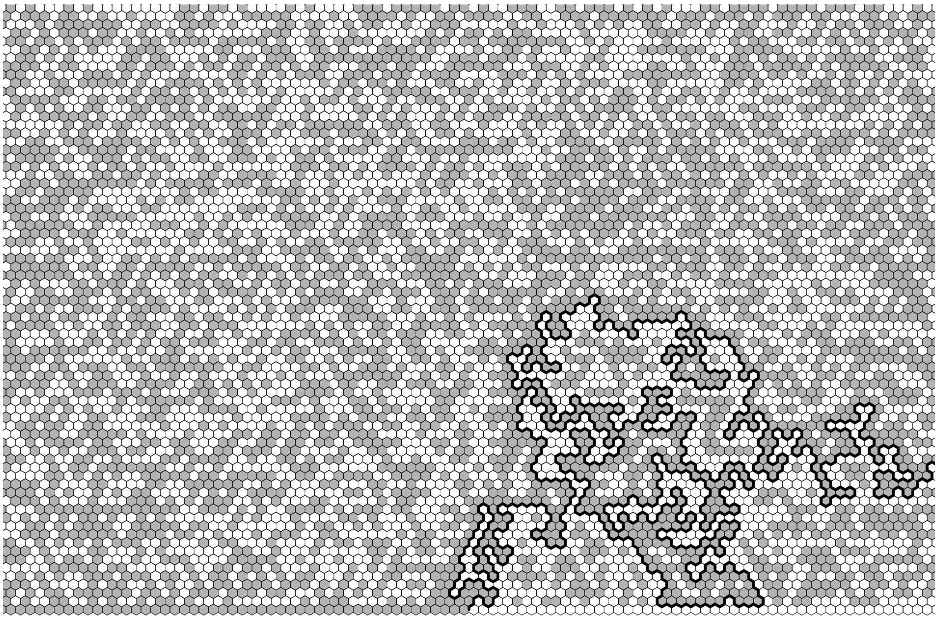

FIGURE 9. The percolation interface (picture by Oded Schramm and provided by Vincent Beffara)

same $(n+1)\times n$ box after the flipping procedure. Of course $\mathrm{P}(E_n^\epsilon)=1/2$ for each n. In [3], it was shown that for all $\epsilon>0$,

$$\lim_{n\to\infty}\mathrm{P}(E_n^\epsilon\cap E_n)=1/4.$$

This means that for any fixed ϵ, E_n and E_n^ϵ become, interestingly, asymptotically independent as n tends to ∞. One then says that crossing probabilities are *very sensitive to noise.* Later on, quantitative versions of this, where ϵ depends on n and goes to 0 with n were obtained. For the hexagonal lattice (where the critical exponents are known), it was shown in [30] that one still has asymptotic independence if $\epsilon_n=(1/n)^\gamma$ provided that $\gamma<1/8$. It was later shown in [11] that this asymptotic independence also holds for $\gamma<3/4$ and that this is sharp in that the correlation of E_n^ϵ and E_n goes to 1 if $\gamma>3/4$.

Dynamical percolation: Dynamical percolation, which was introduced in [14], is a model in which a time dynamics is introduced. In this model, given a fixed graph G and a p, the edges of G switch back and forth according to independent 2-state continuous time Markov chains where closed switches to open at rate p and open switches to closed at rate $1-p$. Clearly, P_p is the unique stationary distribution for this Markov process. The general question studied in dynamical percolation is whether, when we start with distribution P_p, there exist atypical times at which the percolation structure looks markedly different than that at a fixed time. In

almost all cases, the term “markedly different” refers to the existence or nonexistence of an infinite connected component. A number of results for this model were obtained in [14] of which we mention the following.
(i) For p less than the critical value, there are no times at which percolation occurs while for p larger than the critical value, percolation occurs at all times.
(ii) There exist graphs which do not percolate at criticality but for which there exist *exceptional times* (necessarily of Lebesgue measure 0) at which percolation occurs.
(iii) For large d, there are no exceptional times of percolation for dynamical percolation run at criticality on $\mathbb{Z}^d$. (Recall that we have seen, without proof, that ordinary percolation on $\mathbb{Z}^d$ for large d does not percolate.) The key property exploited for this result is that the percolation function has a ’finite derivative’ at the critical value.

It was left open what occurs at $\mathbb{Z}^2$ where it is known that the percolation function has an ’infinite derivative’ at the critical value. Using the known critical exponents for the hexagonal lattice, it was proved in [30] that there are in fact exceptional times of percolation for this lattice when run at criticality and that the Hausdorff dimension of such times is in $[1/6, 31/36]$ with the upper bound being conjectured. In [11], it was then shown that the upper bound of $31/36$ is correct and that even $\mathbb{Z}^2$ has exceptional times of percolation at criticality and also that this set has positive Hausdorff dimension.

While the relationship between noise sensitivity and dynamical percolation might not be clear at first sight, it turns out that there is a strong relation between them because the ‘2nd moment calculations’ necessary to carry out a second moment argument for the dynamical percolation model involve terms which are closely related to the correlations considered in the definition of noise sensitivity. See [34] for many more details.

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