Homework 1: Due September 30

1. Your answers should be complete, clear and concise.

2. You are strongly encouraged to work together as this maximizes the amount of mathematics that you learn. However, after discussions with others, you should write up the solutions yourself.

3. You should write down whom you have discussed the problems with (whether they are in the class or not).

1. (a). Show that for every realization, the cluster of the origin is infinite if and if there exists an infinite self-avoiding path starting at the origin. (Since this is true for *every* realization, we say that such a thing is a *deterministic* fact; it has nothing to do with probability.)

Part (a) might seem completely obvious. To see that it actually requires a proof (although it is not so hard at all), there is the following part (b).

(b). Characterize those graphs for which the the cluster of some point is infinite if and if there exists an infinite self-avoiding path starting at that point.

2. (The first moment method).

The first moment method is using the (trivial) fact that for a nonnegative integer valued random variable X, $P(X > 0) \leq E[X]$. (We used this to show $p_c \geq 1/3$ for Z^2). Use this method to show the following.

Consider a graph which is an infinite tree T which has a root ρ which has a_0 "children", each of which has a_1 "children", etc. So, all people in generation k has a_k children (which might depend on k). Let A_n be the number of people in the nth generation (which is of course just $\prod_{i=0}^{n-1} a_i$). Show that

$$p_c(T) \geq 1/[\liminf_n A_n^{1/n}].$$

3. (The second moment method).

(a). Show that the "converse" of the first moment method is false by showing that for nonnegative integer valued random variables X, $E[X]$ can be arbitrarily large with $P(X > 0)$ arbitrarily small. (This shows that you will never be able to show that $P(X > 0)$ is of reasonable size based on knowledge of the first moment.)

(b). Show that for any nonnegative random variable X

$$P(X > 0) \geq \frac{E[X]^2}{E[X^2]}.$$

(This says that if the mean is large, then you can conclude that $P(X > 0)$ might be "reasonably" large **provided** you have a reasonably good upper bound on the second moment $E[X^2]$. Using the above inequality is called the second moment method and it is a very powerful tool in probability. Part (c) shows how it can be used.)

(c). With T being as in problem 2, show that in fact

$$p_c(T) \leq 1/[\liminf_n A_n^{1/n}]$$

(this gives an exact formula for the critical value for trees of this sort, an argument due to Russ Lyons.)

Hint: Let X_n be the number of people in the nth generation which can be

reached by an open path from the root ρ. Use the 2nd moment method for values of p larger than $1/[\liminf_n A_n^{1/n}]$ to show that for such values of p, $P(X_n > 0) > c$ for some $c > 0$ where c does not depend on n. Then use Fatou's lemma.

Advanced remarks: You can ignore these if you want. There is an interesting connection between the above and geometric measure theory. Consider an arbitrary tree where we don't require everyone on the kth level to have the same number of children. Consider the set of all infinite paths from the root ρ to infinity; this is called the boundary of the tree. The boundary can be metrized by defining the distance between two such paths as e^{-n} if they stick together until the nth level and then split off. (This yields a type of product topology). If you unravel the definitions, problems 2 and 3 essentially say that for trees where everyone on the kth level has the same number of children, the critical value for percolation is $1/e^M$ where M is the lower Minkowski dimension of the boundary of the tree in the above metric. For general trees, it turns out that the critical value for percolation is $1/e^H$ where H is the Hausdorff dimension of the boundary of the tree. So for trees where everyone at a given level have the same number of children, the Hausdorff and lower Minkowski dimensions agree.